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Inconsistency of the rotating wave approximation with the Ehrenfest theorem

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Abstract

We point out that the equation of motion derived from the well-known RWA (rotating-wave-approximation) Hamilitonian is inconsistent with the Ehrenfest theorem. By contrast, we find that the very general independent-oscillator model for a dissipative system [Ford et al., Phys. Rev. A 37 (1988) 4419] is consistent with the Ehrenfest theorem.

The rotating-wave-approximation (RWA) is an integral part of the foundations of quantum optics and is discussed in both the old and the more modern textbooks [1,2]. It concerns the interaction of a reservoir, consisting of an infinite number of oscillators, with either a two-level atom or a cavity mode or a charged harmonic oscillator. Here we concentrate on the case of a harmonic oscillator of frequency ω_0 but our observations will hold in general.

The "Langevin equation" derived from the RWA Hamilitonian has the form [1,2]

$$\dot{a} = \mathrm{i}\omega_0 a + \tfrac{1}{2}\gamma a = B(t), \tag{1}$$

where $B(t)$ is a random force operator with mean zero, a is the usual annihilation operator associated with the oscillator, and the dot denotes time derivative. Forming the mean, or expected value of this equation we find

$$\langle \dot{a} \rangle + \left(\mathrm{i}\omega_0 + \tfrac{1}{2}\gamma\right)\langle a \rangle = 0. \tag{2}$$

The solution of this equation is

$$\langle a(t) \rangle = \mathrm{e}^{-(\mathrm{i}\omega_0 + \gamma/2)t} \langle a(0) \rangle. \tag{3}$$

Now, the coordinate and momentum operators are real (Hermitian) operators related to the complex annihilation operator through the relations

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}\,(a + a^+), \quad p = \sqrt{\frac{m\hbar\omega}{2}}\,\frac{a - a^+}{\mathrm{i}}. \tag{4}$$

Thus, the solution (3) is equivalent with the solutions

$$\langle x(t) \rangle = \mathrm{e}^{-\gamma t/2}\left(\langle x(0) \rangle \cos\,\omega_0 t + \frac{\langle p(0) \rangle}{m\omega_0} \sin\,\omega_0 t\right). \tag{5a}$$

$$\langle p(t) \rangle = \mathrm{e}^{-\gamma t/2}\left[-m\omega_0 \langle x(0) \rangle \sin\,\omega_0 t + \langle p(0) \rangle \cos\,\omega_0 t\right]. \tag{5b}$$

The Ehrenfest theorem states that for the oscillator the mean values satisfy the classical equations of

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motion [3,4]. The classical equation of motion for the damped oscillator are

$$m\dot{x} = p, \tag{6a}$$

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0. \tag{6b}$$

With the solution (5), we see that

$$m\langle \dot{x}\rangle = \langle p\rangle - \tfrac{1}{2}m\gamma\langle x\rangle, \tag{7a}$$

$$m\langle \ddot{x}\rangle + m\gamma\langle \dot{x}\rangle + m\omega_0^2\langle x\rangle = -\frac{\gamma^2}{4\omega_0}e^{-\gamma t/2}\left[m\omega_0\langle x(0)\rangle \cos\,\omega_0 t + \langle p(0)\rangle \sin\,\omega_0 t\right]. \tag{7b}$$

Alternatively, using (5a) in (7b), we may write

$$m\langle \ddot{x}\rangle + m\gamma\langle \dot{x}\rangle + \left(m\omega_0^2 + \tfrac{1}{4}m\gamma^2\right)\langle x\rangle = 0. \tag{8}$$

Hence, from either (7) or (8), we see, by comparison with (6), that the Ehrenfest theorem is not satisfied. However, in the weak coupling approximation ($\gamma \ll \omega_0$), which is generally the realm of application of the RWA, the error term is small in both (7) and (8). Nevertheless, it is non-zero. In fact, in the same order of weak coupling in which one would neglect the fourth (unwanted) term on the left-side of (8), one should also neglect the second term. Thus, what in reality is then being considered is the free oscillator, in which the coupling to the random environment is neglected.

By contrast, we will next examine a very general model for a dissipative system, viz. the IO (independent-oscillator) model [5]. A particular case of this model is that of a charged oscillator interacting with the radiation field with a Hamiltonian which can be approximated to give the RWA Hamiltonian [5]. We found that the IO model leads to an equation of motion which takes the form of a generalized quantum Langevin equation,

$$m\ddot{x} + \int_{-\infty}^{t} dt'\ \mu(t-t')\,\dot{x}(t') + V'(x) = F(t), \tag{9}$$

where x is the position operator, the dot denotes the derivative with respect to t and where $V'(x) = dV(x)/dx$ is the negative of the time-independent external force and $\mu(t)$ is the so-called memory function. In addition, $F(t)$ is the random (fluctuation) force operator. Explicit expressions for $\mu(t)$ and $F(t)$ were written down in terms of the heat bath parameters [5]. Forming the mean value of this equation, we obtain

$$m\langle \ddot{x}\rangle + \int_{-\infty}^{\infty} dt'\ m(t-t')\langle \dot{x}(t)\rangle + \langle V'\langle x\rangle\rangle = 0. \tag{10}$$

In other words, the mean values satisfy a generalization (to include memory terms and an arbitrary potential) of the classical equation of motion. In addition, (6a) is satisfied [5]. In other words, the very general independent-oscillator model is consistent with the Ehrenfest theorem, in contrast to the case of the RWA model.

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