

## Brief Reports

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### Energy balance for a dissipative system

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In general, dissipative problems involve both frictional and fluctuation forces. As it turns out, solutions of many of these problems are encompassed by the generalized quantum Langevin equation. Here, we use this equation to obtain a general expression for the work done by the fluctuation force on a quantum particle. At equilibrium, this work is necessary to balance the energy lost by the particle due to the frictional force.

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Brownian motion is the epitome of dissipative problems and it is described most elegantly by Langevin's classical stochastic differential equation [1]. In recent years, there has been widespread interest in dissipative problems arising in a variety of areas in physics. As it turns out, solutions of many of these problems are encompassed by a generalization of the Langevin equation to encompass quantum, memory, and non-Markovian effects, as well as arbitrary temperature and the presence of an external potential  $V(x)$ . We refer to this as the generalized quantum Langevin equation (GLE):

$$m\ddot{x} + \int_{-\infty}^t dt' \mu(t-t') \dot{x}(t') + V'(x) = F(t), \quad (1)$$

where  $m$  and  $x$  denote the mass and coordinate of the particle, respectively, and the dot denotes differentiation with respect to time. Also  $V'(x) = dV(x)/dx$  is the negative of the time-independent external force. In addition, the frictional (dissipative) term on the left side of the equation is characterized by the memory function  $\mu(t)$  while on the right side is the random (fluctuation or noise) term  $F(t)$ .

A detailed discussion of Eq. (1) appears in Ref. [2]. In particular, it was pointed out that the GLE corresponds to a macroscopic description of a quantum system interacting with a quantum-mechanical heat bath and that this description can be precisely formulated, using such general principles as causality and the second law of thermodynamics. We also stressed that this is a model-independent description. However, the GLE can be realized with a simple and convenient model, viz., the

independent-oscillator (IO) model [2].

The effect of an applied  $c$ -number force  $f(t)$  can be taken into account simply by adding  $f(t)$  to the right side of Eq. (1). However, we take  $f(t)$  to be zero for the present analysis since in Ref. [2] (Sec. III) we already calculated the work done by  $f(t)$ . Here, we concentrate on the work done by the fluctuation force on the quantum particle which is necessary to maintain equilibrium by balancing the energy lost by the particle due to the frictional force. In fact, very few discussion in the literature discuss this aspect of the fluctuation force (the main emphasis being on the effect of the external force) and when they do, the discussion is mainly within the framework of the Markovian approximation. The latter implies the absence of "memory effects" or, equivalently, the choice of  $\mu(t)$  to be proportional to  $\delta(t)$  so that the GLE given in (1) reduces to the more usual Langevin equation. It is our purpose here to present a very general approach which will incorporate (a) non-Markovian effects, (b) quantum effects, and (c) arbitrary temperatures. As we will demonstrate, the GLE is an ideal way to tackle such a problem for the simple reason that, by its nature, fluctuation and dissipative effects are readily separated from each other. In addition, we have at our disposal the very useful knowledge and results (such as the fluctuation-dissipation theorem) already developed in this general area. This will enable us to obtain a very general result, Eq. (12) below, for the work done by the fluctuation force. This result is then applied both to the familiar case of the Ohmic (Markovian) heat bath and to the case of a blackbody radiation heat bath (which manifests both

non-Markovian and quantum effects).

We turn now to the calculation of the expectation value of the instantaneous power supplied by the fluctuation force,  $P_F$ , say. Keeping in mind that we are working with operators, we use a symmetrized form, viz.,

$$P_F = \frac{1}{2} \langle v(t)F(t) + F(t)v(t) \rangle, \quad (2)$$

where  $v(t) = \dot{x}(t)$  is the particle velocity operator. It immediately follows from Eq. (1) that

$$P_F = \frac{d}{dt} \langle \frac{1}{2} m \dot{x}^2 + V(x) \rangle + \int_{-\infty}^t dt' \mu(t-t') \frac{1}{2} \langle \dot{x}(t)\dot{x}(t') + \dot{x}(t')\dot{x}(t) \rangle. \quad (3)$$

First, we observe that the first term on the right side of Eq. (3) must be zero because, for a stationary process, expectation values of time-dependent quantities must be invariant under time translation or, in other words, they are constant. This may be seen mathematically by considering an arbitrary time-dependent operator  $A(t)$ , whose expectation value is given by

$$\begin{aligned} \langle A(t) \rangle &= \frac{\text{Tr} \{ e^{-\beta H} e^{iHt/\hbar} A(0) e^{-iHt/\hbar} \}}{\text{Tr} e^{-\beta H}} \\ &= \frac{\text{Tr} \{ A(0) \}}{\text{Tr} e^{-\beta H}}, \end{aligned} \quad (4)$$

where  $\beta = (kT)^{-1}$ ,  $T$  being the temperature, and where we have used the cyclic property of a product of operators under the trace. From Eq. (4), it is clear that  $\langle A(t) \rangle$  is time independent for a canonical ensemble, which proves our assertion above.

Secondly, to simplify the second term on the right side of Eq. (3), we need to calculate the velocity-autocorrelation function. This requires a knowledge of the generalized susceptibility  $\alpha(\omega)$  which is equivalent to knowing the solution to the GLE. This solution is readily obtained when  $V(x) = 0$ , corresponding to the original Brownian motion problem. As shown by Ford, Lewis, and O'Connell [3,4], a solution is also possible in the case of an oscillator. Taking  $V(x) = \frac{1}{2} m \omega_0^2 x^2$ , these authors obtained [see Eqs. (1)–(3) of Ref. [3]]

$$\bar{x}(\omega) = \alpha(\omega) \bar{F}(\omega), \quad (5)$$

where

$$\alpha(\omega) = [-m\omega^2 + m\omega_0^2 - i\omega\bar{\mu}(\omega)]^{-1}, \quad (6)$$

and the superposed tilde is used to denote the Fourier transform. Thus  $\bar{x}(\omega)$  is the Fourier transform of the operator  $x(t)$ :

$$\bar{x}(\omega) = \int_{-\infty}^{\infty} dt x(t) e^{i\omega t}. \quad (7)$$

Also, since  $\mu(t)$  is 0 for negative  $t$ , we have [see Ref. [2], Eqs. (4.9) and (2.4)]

$$\bar{\mu}(\omega) = \int_0^{\infty} dt \mu(t) e^{i\omega t}, \quad \text{Im}\omega > 0. \quad (8)$$

For the same reason, the upper limit in the integral in Eq. (1) may be replaced by  $+\infty$ .

Next, we use the fluctuation-dissipation theorem [5–7]

$$\begin{aligned} C_{xx}(t-t') &\equiv \frac{1}{2} \langle x(t)x(t') + x(t')x(t) \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \bar{C}_{xx}(\omega) e^{-i\omega(t-t')}, \end{aligned} \quad (9)$$

where

$$\bar{C}_{xx}(\omega) = \hbar \text{Im}\alpha(\omega) \coth(\hbar\omega/2kT). \quad (10)$$

It readily follows that the Fourier transform of the velocity-autocorrelation functions is  $\omega^2 \bar{C}_{xx}(\omega)$ . Hence, putting all this together, we obtain

$$\begin{aligned} P_F &= \int_{-\infty}^{\infty} dt' \mu(t-t') \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \omega^2 \bar{C}_{xx}(\omega) e^{-i\omega(t-t')} \\ &= \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} d\omega \omega^2 \text{Im}\alpha(\omega) \bar{\mu}(-\omega) \coth(\hbar\omega/2kT). \end{aligned} \quad (11)$$

Noting that only the even part of the integrand in Eq. (11) contributes and also noting [2,3] that  $\bar{\mu}(-\omega)^* = \bar{\mu}(\omega)$  and  $\alpha(-\omega)^* = \alpha(\omega)$ , so that  $\text{Im}\alpha(\omega)$  is an odd function of  $\omega$  whereas  $\text{Re}\bar{\mu}(\omega)$  is an even function of  $\omega$ , it follows that we can write  $P_F(t)$  in the following convenient forms:

$$\begin{aligned} P_F &= \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} d\omega \omega^2 \alpha(\omega) \text{Re}\bar{\mu}(\omega) \coth \left[ \frac{\hbar\omega}{2kT} \right] \\ &= \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} d\omega \omega^2 \text{Im}\alpha(\omega) \text{Re}\bar{\mu}(\omega) \coth \left[ \frac{\hbar\omega}{2kT} \right]. \end{aligned} \quad (12)$$

In addition, we note that Eq. (6) implies that

$$\text{Im}\alpha(\omega) = \omega |\alpha(\omega)|^2 \text{Re}\bar{\mu}(\omega), \quad (13)$$

so that it is immediately clear that  $P_F > 0$  always.

For example, in the case of classical Brownian motion in an Ohmic heat bath at high temperature, for which  $\omega_0 = 0$ ,  $\text{Re}\bar{\mu}(\omega) = m\gamma$ , and  $kT \gg \hbar\omega$ , we obtain, from Eqs. (12) and (6),

$$P_F = \frac{kT}{\pi} m\gamma \int_{-\infty}^{\infty} d\omega \omega \text{Im}\alpha(\omega), \quad (14)$$

with  $\alpha(\omega) = [-m\omega(\omega + i\gamma)]^{-1}$ . It follows that

$$P_F = kT\gamma. \quad (15)$$

In other words, the rate of work being done by the fluctuation force is proportional to the dissipation. This is a manifestation of the general principle that, at equilibrium, the energy lost by a particle due to dissipation is compensated by the energy received from the fluctuation force.

Next, we consider an example of a non-Markovian interaction, viz., the case of a blackbody radiation heat bath. This is more physical than the Ohmic case and it has the merit that a universally accepted Hamiltonian may be written down. In fact, we have shown, by use of a series of unitary transformations, that the nonrelativistic Hamiltonian of quantum electrodynamics in the dipole approximation can be transformed into the IO Hamiltonian, with an associated GLE containing explicit and exact values for the memory function and the random force. In the case of the oscillator potential, this led, via Eq. (6) and mass renormalization, to an expression [3,4]

for the susceptibility  $\alpha(\omega)$ . Explicitly, in the case where  $\omega_0=0$ , we have [3,4]

$$\text{Im}\alpha(\omega)=\tau_e/M\omega \quad (16)$$

and

$$\text{Re}\tilde{\mu}(\omega)=M\omega^2/\tau_e(\omega^2+\tau_e^{-2}), \quad (17)$$

where  $M$  is the renormalized (observable) electron mass and  $\tau_e=2e^2/3mc^3=6.24\times 10^{-24}\text{s}$ . It follows, using Eqs. (12), (16), and (17), that

$$P_F=\frac{\hbar}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{\omega^3}{\omega^2+\tau_e^{-2}}\coth\left[\frac{\hbar\omega}{2kT}\right]. \quad (18)$$

This result is divergent. This may appear at first to be surprising in view of the accuracy with which the calculation was carried out. The explanation lies in the use of the dipole approximation. Such an approximation is valid for many calculations involving the interaction of a nonrelativistic electron with the electromagnetic field but, as in the case of Bethe's nonrelativistic calculation of

the Lamb shift [8], it is not adequate when the calculation involves the emission and reabsorption of virtual photons (which is implicitly involved in the calculation of  $P_F$ ). As with Bethe [8], if we introduce a high-frequency cutoff,  $\omega_{\text{max}}$  say, then, for example, in the high-temperature limit, we obtain

$$P_F=\frac{2kT}{\pi}\int_0^{\omega_{\text{max}}}d\omega\frac{\omega^2}{\omega^2+\tau_e^{-2}} \\ =\frac{2kT}{\pi}\{\omega_{\text{max}}-\tau_e^{-1}\tan^{-1}(\omega_{\text{max}}\tau_e)\}. \quad (19)$$

We conclude that, just as Au and Feinberg [9] went beyond Bethe's calculation [8] to include retardation effects, the incorporation of retardation effects in the blackbody radiation heat-bath problem is a desirable next step.

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