Radiation reaction in electrodynamics and the elimination of runaway solutions

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The familiar Abraham–Lorentz theory of radiation reaction in classical non-relativistic electrodynamics exhibits many problems such as "runaway solutions" and violation of causality. As shown by many authors, such problems can be alleviated by dropping the assumption of a point electron. We also drop this assumption (by introducing a form-factor with a large cutoff frequency Ω) but we present a new approach based on the use of the generalized *quantum* Langevin equation. For an electric dipole interaction, an exact treatment is possible and we obtain a new equation of motion which, in spite of being third order, does not lead to runaway solutions or solutions which violate causality (the sole proviso being that Ω cannot exceed an upper limit of $3Mc^3/2e^2=1.60\times10^{23}\ \mathrm{s}^{-1}$). Furthermore, Ω appears in the third-derivative term but we show that, to a very good approximation, this term may be dropped so that we end up with a simple second-order equation which does not contain Ω and whose solutions are well-behaved.

The Abraham–Lorentz (AL) equation represents a well-known method for incorporating radiation reaction effects into the non-relativistic equation of motion for a classical radiating electron [1–4]. Deficiencies in this equation manifest themselves in the existence of "runaway solutions". Attempts to improve the AL approach by the use of integrodifferential equations encounter problems associated with the violation of causality (the acceleration at a particular time being determined by the value of the field at future times). The same problems are reflected in Dirac's relativistic extension. There is now a consensus that such problems can be alleviated by dropping the assumption of a point electron [2,3]. This is particularly exemplified in the work of Moniz and Sharp [2]. The latter authors do not employ the dipole approximation, as a result of which the Compton wavelength comes into play in the role of a size parameter for the extended electron. By contrast, our considerations will be restricted to the dipole interaction with the goal of carrying out the analysis in an exact manner and analyzing carefully the role of the cutoff frequency, Ω, (which enters into the electron form-factor and, in essence, defines the type of extended model used for the electron) in the equation of motion.

The most general quantum equation obtained in our analysis, eq. (5) below, is a new result and it reduces in the classical limit to (7). Both equations contain Ω. We point out that a particular choice for Ω (viz. $\Omega\to\infty$, corresponding to a point electron) leads to the AL equation of motion but we argue on physical grounds why this choice should be discarded. We then go on to deduce that an equation independent of Ω may be achieved by working to or-

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der $\omega\tau_e$ (a very good approximation, since τ_e is 2/3 of the familiar photon propagation time across the classical electron radius and has a value of 6×10^{-24} s, whereas ω is a typical frequency associated with an external field) and, furthermore, that the correct equation to this order is given by (8) since it leads to solutions which are well-behaved.

Our approach is based on recent work which we have carried out on dissipative problems. In ref. [5], we presented an exact solution to the problem of a quantum oscillator-dipole, with charge e, interacting with a blackbody radiation field. In this work, a form-factor was given to the electron and thus we are dealing with an extended electron model. Writing down the well-known Hamiltonian of non-relativistic quantum mechanics leads us, with the help of the Heisenberg equations of motion, to a generalized quantum Langevin equation. This equation contains a fluctuating force which does not vanish at temperature $T=0$ and, concomitantly, a damping term which reflects the fact that our equation incorporates the fluctuation–dissipation theorem. Thus, even in the absence of an external force, we are able to deduce the form of the radiation reaction damping force!

In ref. [6], we included an external field and also simplified the results. Next, in ref. [7], we made it clear that the analysis also extended to the case of an electron in an *arbitrary* potential $V(\boldsymbol{r})$, interacting with the radiation field in the dipole approximation and subject to an externally imposed time-dependent c-number field $f(t)$. Because of the novel nature of many of the results obtained, we were also motivated to verify them using more conventional, albeit longer, derivations [8].

The radiation field is treated as a heat bath and central to our description is the requirement that the heat bath be passive (which implies that there is no means by which energy can be removed from the heat bath by a cyclic process). In essence, this requirement demands: causality, the second law of thermodynamics for irreversible processes, and a real response (i.e. x is a Hermitean operator).

Thus, confining our attention to one dimension, since an extension to three dimensions is trivial [7], our result for the equation of motion of a quantum particle of bare mass m moving in a potential $V(x)$ and linearly coupled to a passive heat bath at temperature T is in the form of a generalized quantum Langevin equation:

$$m\ddot{x}(t)+\int_{-\infty}^{t}\mathrm{d}t'\,\mu(t-t')\dot{x}(t')+V'(x)$$
$$=F(t)+f(t)\,, \qquad (1)$$

where $F(t)$ is the operator-valued random (fluctuating) force, $f(t)$ is the external c-number force, $\mu(t)$ is the memory function and where the dot and prime denote, respectively, the derivative with respect to t and x. As we have discussed in detail, in ref. [7], the coupling to the heat bath is characterized by the Fourier transform of the memory function, $\tilde{\mu}(\omega)$, with $\mathrm{Im}\,\omega>0$, and, based on the general principles enunciated above, we showed that $\tilde{\mu}(\omega)$ is restricted to being a positive real function, i.e. it is analytic in the upper half of the ω plane, it has a positive real part and it obeys the reality condition $\tilde{\mu}(\omega)=\tilde{\mu}(-\omega)^*$.

Eq. (1) provides the foundation for a general treatment of dissipative problems in many branches of physics. For the *particular* case of the blackbody radiation heat bath, we chose an electron form-factor [5], $\Omega^2/(\Omega^2+\omega^2)$, à la Feynman [9], with a shape convenient for calculation but arbitrary in the sense that it depends on the choice of Ω, a large cut-off frequency. Mass renormalization is also necessary, leading to the key result (see refs. [5-7] for details)

$$\mu(t)=M\Omega^2\tau_e[2\delta(t)-\Omega\exp(-\Omega t)]\,, \qquad (2)$$

where e and M are the charge and renormalized (observed) mass of the particle, respectively, $\tau_e=2e^2/3Mc^3=6\times10^{-24}$ s. In addition [5], the renormalized mass M is given in terms of the bare mass m by the relation

$$M=m+2e^2\Omega/3c^3=m+\tau_e\Omega M\,. \qquad (3)$$

The *non-Markovian* nature of the motion, which is made manifest by the second term on the right-side of (2), is an essential feature. It follows, using (1)–(3), that

$$m\ddot{x}(t)+M\Omega^2\tau_e\dot{x}(t)$$
$$-M\Omega^3\tau_e\int_{-\infty}^{t}dt'\,\exp[-\Omega(t-t')]\dot{x}(t')+V'(x)$$
$$=F(t)+f(t)\ . \qquad (4)$$

Next, we multiply across by $\exp(-\Omega t)(d/dt)\times\exp(\Omega t)$, and use (3), to obtain

$$(m/\Omega)\dddot{x}(t)M\ddot{x}(t)+V'_{\rm eff}(x)$$
$$=F_{\rm eff}(t)+f_{\rm eff}(t)\ , \qquad (5)$$

where

$$f_{\rm eff}(t)\equiv f(t)+\Omega^{-1}\dot{f}(t)\ , \qquad (6)$$

and similarly for the other "effective" quantities. This is an exact result within the framework of our mode. Also, (5) is an operator equation. However, for the purpose of making contact with the AL and other classical equations, we will take mean values and also set $V(x)=0$. Thus, the fluctuating force $F(t)$ is eliminated and all quantities are now to be interpreted as classical quantities. Hence, we obtain

$$(m/\Omega)\dddot{x}(t)+M\ddot{x}(t)=M(\Omega^{-1}-\tau_e)\dddot{x}(t)+M\ddot{x}(t)$$
$$=f(t)+\Omega^{-1}\dot{f}(t)\ . \qquad (7)$$

Two comments are in order at this stage. First of all, we note the generality of the result in that we have not yet specified the cutoff frequency Ω, which, of course, determines the form-factor. Secondly, we note the presence of an $\dddot{x}(t)$ term as in the AL equation, so one might immediately wonder how runaway solutions can be avoided. The latter can be checked out by choosing, say, $f(t)=\exp(-\Omega t)f(0)$ so that the right-hand-side of (7) is zero. Then (7) has two possible solutions, viz. $\ddot{x}(t)=0$ or $\ddot{x}(t)=\exp[-(M/m)\Omega t]\ddot{x}(0)$. It is clear that the runaway solutions occur only in the case where the bare mass, m, is negative. But, as we will now demonstrate, physical considerations demand that $m\geqslant 0$ and thus our equation does not lead to runaway solutions. Our argument is simply based on the fact, mentioned earlier, that $\tilde{\mu}(\omega)$ is a positive function (which follows from the second law of thermodynamics). This is equivalent to the demand (see ref. [5]) that all the poles of $\alpha(\omega)$, the generalized susceptibility, must be in the lower half of the complex plane. The latter conclusion also follows from the principle of causality [10], which demands that the response due to an external force cannot precede the force.

Hence, if we consider eq. (19) of ref. [5], the demand that the poles of $\alpha(\omega)$ lie in the lower half-plane leads to the conclusion that the Ω' appearing therein cannot be negative and furthermore, from eqs. (20) and (23) of the same reference, we have that $\Omega^{-1}=\Omega'^{-1}+\tau_e$. Thus, demanding that $\Omega'\geqslant 0$ is tantamount to demanding that $\Omega^{-1}>\tau_e$ or, equivalently, that Ω be no larger than $\Omega_e\equiv\tau_e^{-1}=1.60\times10^{23}$ s^{-1} and, from (3), we see that this in turn implies that $m\geqslant 0$. The fact that causality arguments impose an upper limit on Ω also rules out the possibility of a point-electron model (which requires $\Omega\to\infty$ and a negative value for m) and thus it becomes clear why the AL equation (which is based on a point-electron model) leads to acausal solutions. In fact, if we (erroneously) take $\Omega\to\infty$ in (7) we obtain the AL equation.

Given that $m\geqslant 0$ and $\Omega_e\geqslant\Omega\gg\omega$, where ω denotes typical frequencies of physical interest, the question arises as to the sensitivity of the motion to the choice of Ω, subject to the restriction that Ω is of the order of magnitude of Ω_e but not greater. The answer, not unsurprisingly, is that it is very insensitive. To lowest order, we have the basic equation $M\ddot{x}(t)=f(t)$ and, in the case of an external field oscillating at a frequency ω, it is clear that the other three terms in (7) are of relative order $\omega\tau_e$. In fact, substituting $M\dddot{x}(t)=\dot{f}(t)$ from the lowest-order result into (7), we obtain a result which is correct to first order in τ_e, viz.

$$M\ddot{x}(t)=f(t)+\tau_e\dot{f}(t)\ . \qquad (8)$$

This is a rather striking result in that it is only a second-order equation, it is correct to first order in τ_e and it is independent of the cutoff frequency Ω. It is also of interest to note that it follows exactly from (7) in the case where one selects for Ω its largest permissible value, viz. τ_e^{-1} (corresponding to letting $m\to 0$, which is also equivalent to choosing the closest approach to a point electron consistent with causality).

If one chooses to continue the iteration process to order τ_e^2 then the result obtained is the same as (8) except that on the right-hand side there is an additional term, viz. $\tau_e(\tau_e-\Omega^{-1})\ddot{f}$, which demonstrates explicitly the dependence on the cutoff frequency in this order. However, we emphasize again the relative smallness of the τ_e^2 terms and the fact that they actually vanish if Ω is selected to be τ_e^{-1}.

We conclude that, to a high degree of accuracy, (8) is the correct equation of motion for the radiating electron. It is in fact simpler than the AL equation (being second order whereas the latter is third order) and it does not lead to either runaway solutions or acausal behaviour. The presence in this equation of the time dependence of the external field is particularly striking and, since the external field is a given quantity, such a term is easy to handle. An extension to the relativistic domain is also possible, details of which will be forthcoming. It is also of interest to note that Eliezer [11] also proposed eq. (8). However, he did not derive it from first principles as we have done. In addition, we have derived a more general equation, viz. (5), which not only contains the fluctuating force and the external potential in addition to the external force but it is also a quantum-mechanical equation. (From our previous discussion, we note that, to first order in τ_e, we could drop the $\dddot{x}(t)$ term in (5) which leaves us with a simple second-order equation and which does not contain Ω.) For example, if we take V to be the Coulumb potential due to a proton in (the three-dimensional version of) eq. (5), then our quantum equation describes the motion of the hydrogen atom with inclusion of Lamb shift fluctuation forces (arising from the $F(t)$).

Some comments on (8) are now in order. First of all, we note that, in the case of a zero external field, $x(t)=x(0)$ is the only solution, in contrast to what occurs in the case of the AL equation (where a runaway solution is a possibility). Secondly, in the case of a constant external field there is an apparent paradox in that (8) becomes strictly a Newtonian equation (since $\dot{f}(t)=0$). Nevertheless, there is still emission of radiation because its existence depends only on $f(t)$ and not $\dot{f}(t)$. More insight can be gained by solving (8) in the case of an arbitrary $f(t)$ subject only to the condition that it is switched on the distant past and off in the distance future, i.e. $f(\pm\infty)=0$. The result is, not unsurprisingly, that the work done by the force $f(t)$ on the electron is equal to the power radiated plus the gain in the electron's kinetic energy. However, the final velocity is exactly the same as it would be without radiation being present but this is not as surprising as it might seem at first glance since dipole radiation is symmetric about the particle displacement, as a result of which there is no net momentum exerted by the field on the particle.

Finally, a comment about the memory function $\mu(t)$ appearing in (2). The fact that it is not solely a delta function (except in the unphysical AL case where $\Omega\to\infty$) is a reflection of the fact that the electromagnetic field has a time dependence. In other words, not only does the radiated field affect the electron but the electron affects the field so that the field seen by the electron at time t depends on prior history which is why (4) has an integral from $-\infty$ to t.

In conclusion, we note that the techniques employed transcend the present problem in the sense that they are applicable to the analysis of a variety of problems in physics where dissipation plays a role.

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References

[1] J.D. Jackson, Classical electrodynamics, 2nd Ed. (Wiley, New York, 1975) ch. 17.
[2] E.J. Moniz and D.H. Sharp, Phys. Rev. D 15 (1977) 2850.
[3] S. Coleman, in: Electromagnetism: paths to research, ed. D. Teplitz (Plenum, New York, 1982);
P. Pearle, in: Electromagnetism: paths to research, ed. D. Teplitz (Plenum, New York, 1982).
[4] N.P. Klepikov, Sov. Phys. Usp. 26 (1985) 506.
[5] G.W. Ford, J.T. Lewis and R.F. O'Connell, Phys. Rev. Lett. 55 (1985) 2273.
[6] G.W. Ford, J.T. Lewis and R.F. O'Connell, Phys. Rev. A 36 (1987) 1466.
[7] G.W. Ford, J.T. Lewis and R.F. O'Connell, Phys. Rev. A 37 (1988) 4419.
[8] G.W. Ford, J.T. Lewis and R.F. O'Connell, Ann. Phys. (NY) 185 (1988) 270; J. Stat. Phys. 53 (1988) 439.
[9] R.P. Feynman, Phys. Rev. 76 (1949) 769.
[10] J.S. Toll, Phys. Rev. 104 (1956) 1760.
[11] C.J. Eliezer, Proc. R. Soc. A 194 (1948) 543.