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NOTES AND DISCUSSION

Calculation of the General Relativistic Perihelion Shift Using Isotropic Coordinates

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This is a pedagogical note, our purpose being to show that the familiar expression for the precession of perihelion, $\Delta\phi$ say, in Einstein's general theory of relativity may be derived in a simpler and more transparent manner if one makes use of isotropic coordinates instead of the usual,[1] standard coordinate system. The essence of our method is based on the fact that the use of isotropic coordinates enables us to replace the usual cubic equation in the general relativistic orbit calculation by a quadratic equation. We also wish to demonstrate explicitly a point often ignored, or at best glossed over, by many textbooks. This is the fact that the derivation of $\Delta\phi$ to 1st (lowest) order in m/P [where $m=GM/C^2$ and[2] $P=a(1-e^2)$, with M denoting the mass of the matter distribution, a the semimajor axis of the orbit, and e the eccentricity of the orbit] requires knowledge of the inverse of the coefficient of dt^2 in the line element to 2nd order in m/r (in other words, nonlinear terms in the field equations are required, and of course this is why the precession of the perihelion of mercury is such a crucial test for Einstein's theory).

For a spherically symmetric matter distribution, the exterior solution of Einstein's field equations leads to the familiar line element of Schwarzschild (in standard coordinates),

$$ds^2=[1-(2m/r)]c^2dt^2-[1-(2m/r)]^{-1}dr^2-r^2[d\theta^2+\sin^2\theta d\phi^2]. \quad (1)$$

The equations of motion of a body in this gravitational field lead to the orbit equation[3]

$$(du/d\phi)^2=2mu^3-u^2+(2m/h^2)u+[(c^2l^2-1)/h^2], \quad (2)$$

where $u=1/r$ and where h and l are constants of the motion. Equation (2) is the usual starting point for the derivation of $\Delta\phi$; the procedure is straightforward but unnecessarily long compared to the method we now present.

The Schwarzschild line element in isotropic coordinates may be written[1]

$$ds^2=e^{\nu}c^2dt^2-e^{-\lambda}(d\rho^2+\rho^2d\theta^2+\rho^2\sin^2\theta d\phi^2), \quad (3)$$

where

$$r=\rho[1+(m/2\rho)]^2, \quad (4)$$

$$e^{\nu}=[1-(m/2\rho)]^2[1+(m/2\rho)]^{-2}, \quad (5)$$

and

$$e^{-\lambda}=[1+(m/2\rho)]^4. \quad (6)$$

Analogous to the derivation of Eq. (2), we write down the equations of motion and hence derive the following orbit equation

$$(du/d\phi)^2=-u^2+(e^{-\lambda}/h^2)(c^2l^2e^{-\nu}-1), \quad (7)$$

where now $u=1/\rho$ and where h and l again denote constants of the motion. It should be noted that calculation of $\Delta\phi$ to lowest order in m/r is equivalent, from Eq. (4), to calculation of $\Delta\phi$ to lowest order in m/ρ. Now it is well known[1,2] that, to lowest order, $cl=1$ and $h^2=mP$. It thus follows, if we expand in powers of m/ρ, that the expression $(c^2l^2e^{-\nu}-1)$ of Eq. (7) contains no term of order unity and therefore it is necessary to keep terms in the expansion of $e^{-\nu}$ to order $(m/\rho)^2$ but that only terms up to order m/ρ are required in the expansion of $e^{-\lambda}$. Thus, to the order required, Eq. (7) may be written

$$(du/d\phi)^2=-[1-6(m^2/h^2)]u^2+(2m/h^2)(2c^2l^2-1)u+[(c^2l^2-1)/h^2]. \quad (8)$$

Note that the rhs, (right-hand side) of Eq. (8) is a *quadratic* equation in u, whereas the rhs of Eq. (2) is a *cubic* equation in u; this makes it simpler to derive $\Delta\phi$ from Eq. (8) than from Eq. (2). Now the path of a planet in the field of the sun in elementary Newtonian theory is an ellipse and the equation of the path may be written as[2]

$$uP=(1+e\cos\phi). \quad (9)$$

Both at perihelion and aphelion we have $du/d\phi=0$ corresponding to $u=(1+e)/P$ and $u=(1-e)/P$, respectively.

Returning again to the general relativistic case, we consider orbits with perihelion and aphelion distances given by $P/(1+e)$ and $P/(1-e)$, respectively (where of course the quantities e, a, and P now refer to the relativistic orbit). Thus the rhs of Eq. (8) must contain the factor

$$\left[\left(u-\frac{1+e}{P}\right)\left(u-\frac{1-e}{P}\right)\right].$$

Therefore, the orbit equation may be written

$$\left(\frac{du}{d\phi}\right)^2=-\left(1-6\frac{m^2}{h^2}\right)\left(u^2-\frac{2}{P}u+\frac{1-e^2}{P^2}\right). \quad (10)$$

By analogy with Eq. (9), we write the general relativistic solution of the orbit equation in the form[4]

$$uP=(1+e\cos x), \quad (11)$$

this equation being in essence a definition of the angle x. From Eq. (11) it immediately follows that

$$\left(u^2-\frac{2}{P}u+\frac{1-e^2}{P^2}\right)=-\frac{e^2}{P^2}\sin^2x=-\left(\frac{du}{dx}\right)^2. \quad (12)$$

Thus substituting Eq. (11) into Eq. (10) we find, to lowest order in m,

$$d\phi=[1+3(m^2/h^2)]dx. \quad (13)$$

Now from Eq. (11) we see that the perihelion of the orbit occurs for $x=0,\ 2\pi,\ \cdots$ and, from Eq. (13), this corresponds to values of ϕ equal to $0,\ 2\pi[1+3(m^2/h^2)],\ \cdots$. Thus,

$$\Delta\phi=6\pi(m^2/h^2)=6\pi(m/P), \tag{14}$$

which is the familiar result of Einstein for the precession of perihelion.

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[1] See, for example, R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Book Co., New York, 1965).

[2] L. D. Landau and E. M. Lifshitz. *Mechanics* (Pergamon Press, Inc., New York, 1960), p. 36.

[3] Ref. 1, p. 181.

[4] C. Darwin, Proc. Roy. Soc. (London) **A263**, 39 (1961).

Comment on "Central Force Motion without Calculus"

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In a recent article,[1] Parke and Bergmann obtain orbital equations for inverse-square central force particle motion by assuming a constant total mechanical energy with the potential energy inversely proportional to the focal radius from the center of force to the particle. This assumption does, of course, guarantee a correct form for the law of force and the associated orbital equations but it implies prior knowledge of the form of the potential energy.

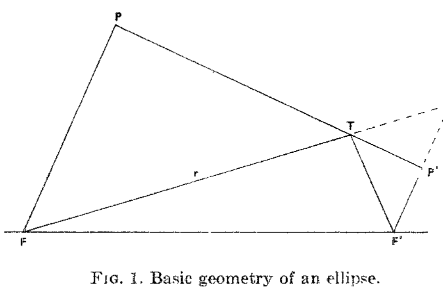

FIG. 1. Basic geometry of an ellipse.

Specifically, the determination of the potential energy from the law of force requires the use of operational techniques of the calculus.

The precise form of the potential energy as well as the expression for the total energy is determined directly from fundamental geometric and dynamic parameters of the problem. The dynamic and geometric relationships necessary to deduce the expression for the total energy (as well as the law of force) are already developed by Parke and Bergmann, and a direct extension of their results leads to the desired equation.

For the case of elliptic central force motion with the force directed toward a focus, the required dynamical parameter is provided by Kepler's second law in the form of Eq. (2) of the cited article,

$$L=mv(FP). \tag{1}$$

The essential geometric property is provided by Eq. 6) of the Parke and Bergmann article which is (Fig. 1),[2]

$$(FP)(F'P')=b^2. \tag{2}$$

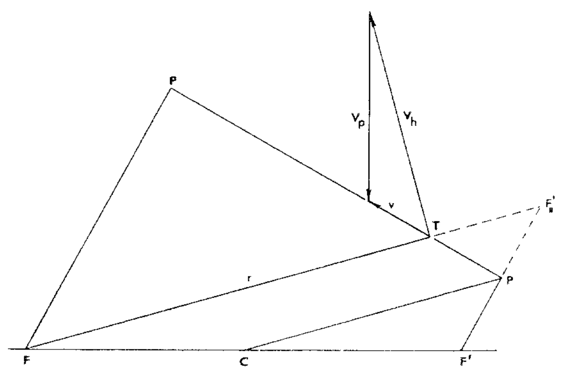

FIG. 2. The invariant velocity components.

Equations (1) and (2) combine to give the magnitude of the linear velocity at any point T of the elliptical path,

$$v=(L/mb^2)(F'P'). \tag{3}$$

The geometric center of the ellipse bisects the line segment FF' and P' is the midpoint of FF_R' so that CP' is parallel to FT (Fig. 2). If the linear velocity of the particle at position T is resolved into invariant components, V_h, normal to the focal radius FT, and, V_p, normal to the major axis of the ellipse, the velocity triangle, $v,\ V_h,\ V_p$, is similar to the triangle $F'P'C$. It follows that the velocity components, V_h and V_p, have constant magnitudes at all positions in the elliptical path.[3]

The law of cosines,

$$v^2=V_h^2+V_p^2-2V_hV_p\cos\theta, \tag{4}$$

and Varignon's theorem of moments for vector quantities,

$$v(FP)=V_h(FT)-V_p(FT)\cos\theta, \tag{5}$$

combine with Eq. (1) to give,

$$v^2=V_p^2-V_h^2+\frac{2(LV_h/m)}{(FT)}. \tag{6}$$