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# Ideal Two-Dimensional Electron Gas in a Magnetic Field and at Non-Zero Temperatures

## An Alternative Approach

By

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A new exact expression is derived for the free energy of an ideal two-dimensional electron gas (2DEG) in a uniform magnetic field and at low, but finite, temperatures. The approach eliminates a 2D-peculiarity that neither the weak nor the strong magnetic field limit can be easily taken in the result derived from the standard Sondheimer-Wilson treatment. In the strong magnetic field limit, the result reduces to an existing one, for which a misunderstanding needs to be clarified. In the weak field limit, it agrees with a result obtainable through Euler's summation formula. The conditions are discussed under which the nondegenerate situation may occur.

Es wird ein neuer exakter Ausdruck für die freie Energie eines idealen zweidimensionalen Elektronengases (2DEG) im gleichförmigen Magnetfeld und bei niedrigen, jedoch endlichen Temperaturen abgeleitet. Das Verfahren eliminiert eine 2D-Besonderheit, so daß weder der Grenzfall des schwachen noch des starken Magnetfelds sich leicht für das Ergebnis verwenden läßt, das aus dem Sondheimer-Wilson-Standardverfahren abgeleitet wird. Im Grenzfall des starken Magnetfeldes reduziert sich das Ergebnis auf ein bisher existierendes, für das ein Mißverständnis geklärt werden muß. Im Grenzfall des schwachen Feldes stimmt es mit einem Ergebnis überein, das mit der Eulerschen Summationsformel erhältlich ist. Die Bedingungen, unter denen der nichtentartete Fall auftreten kann, werden ebenfalls diskutiert.

## 1. Introduction

Two-dimensional electron systems, which exist in semiconductor inversion layers and GaAs–GaAlAs superlattices, have been subject to intensive investigations, both theoretically and experimentally [1]. The model of an ideal two-dimensional electron gas (2DEG) at zero-temperature was first employed by Peierls [2] in 1933 to qualitatively explain the de Haas-van Alphen effect of a three-dimensional metal. Because of the experimental realizations of a 2DEG, the development of the general theory for a 2DEG has been greatly accelerated.

It is common knowledge that the density of states of an *ideal* 2DEG in a magnetic field consists of a series of discrete Landau levels. In a real 2DEG found in various semiconductor devices, these Landau levels are broadened by a number of scattering mechanisms [1, 3]. Nevertheless, the model of an ideal 2DEG has the advantage of often leading to a clear physical picture in practice (see e.g., [4 to 6]). From a pure theoretical point of view, improving the physical description of an ideal 2DEG also has its intrinsic values.

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Recently Wang and O'Connell [6] investigated the magnetization of an ideal 2DEG with a Landé factor $g = 2$, in the strong magnetic field limit and for non-zero temperatures. Comparison between their results and those in the literature were made [6]. In the present paper, we wish to discuss the thermodynamics of the system in a more comprehensive fashion. We shall concentrate on the case without spin splitting ($g = 0$) [3 to 5].

Similar to the 3D case, the usual expression for the free energy of a 2DEG can be written as a sum of an infinite Fourier series [7 to 9]. The advantage of this expression is that it shows clearly the oscillatory pattern of the de Haas-van Alphen effect. Also it enables one to easily include various damping effects [7 to 9] caused by finite temperatures and Landau level broadenings. But the drawback of this expression is that neither the weak nor the strong magnetic field limit can be taken in a straightforward way, which is unique for the 2DEG [10]. We shall in this paper derive a new expression for the free energy of an ideal 2DEG in a magnetic field. It overcomes the above drawback. In the strong magnetic field limit, after further approximations are taken, it reduces to a form obtained by Vagner et al. [4, 5]. We shall clarify a misunderstanding in [4] and [5], which is critical to correct calculations of the Fermi energy, magnetic suscepticility and other physical properties of the system. In the weak field limit, our result reduced to the one obtainable by using Euler's summation formula.

The standard expression has already been obtained through a propagator method [7]. In order to see the essential difference between the 2D and the 3D cases, we shall derive, in the next section, this result by using the original Sondheimer-Wilson technique, through which the famous result for a three-dimensional electron gas was obtained [11, 12]. In Section 3, we shall first derive our result for the free energy and then proceed to discuss various limiting cases. In the light of our results, we shall also evaluate the condition for the non-degenerate situation [13] to occur. Finally, a summary will be presented in Section 4.

## 2. The Sondheimer-Wilson Treatment

For a 2DEG with a spin-splitting factor $g$ and unbroadened Landau levels, the energy spectrum is given by

$$E_{\nu\sigma} = 2\mu_B H\left[\left(\nu + \frac{1}{2}\right) + \frac{g\sigma}{4}\right], \qquad \nu = 0, 1, 2, \ldots, \quad \sigma = \pm 1\,. \tag{2.1}$$

Here $\mu_B = e\hbar/2m$ is the effective Bohr magneton, $m$ is the effective mass of the electron, $H$ is the magnetic field and $g$ is the Landé factor multiplied by $(m/m_0)$, where $m_0$ is the electron mass in vacuum. Also, it is obvious that $\hbar\omega_c = 2\mu_B H$, where $\omega_c = (eH)/mc$ is the cyclotron frequency. The degeneracy per unit area for each Landau level is

$$D = \frac{eH}{2\pi\hbar}\,. \tag{2.2}$$

From henceforth, we shall always consider a unit area; a generalization to an arbitrary area is trivial.

Hence the one-electron Boltzmann partition function is [11]

$$Z(\beta) = \sum_{\substack{\nu=0\\ \sigma=\pm 1}}^{\infty} D \exp(-\beta E_{\nu\sigma}) = \left(\frac{m}{2\pi\hbar^2}\right)\frac{(\mu_B H)\cosh(\beta\mu_B H g/2)}{\sinh(\beta\mu_B H)}\,, \tag{2.3}$$

where $\beta = 1/kT$ is the inverse temperature. To evaluate the free energy with Fermi statistics, we shall use the Sondheimer-Wilson (SW) technique which was originally invented to solve exactly the same problem in the 3D case [12].

The essential point of the SW technique is the following. Suppose the one-particle Boltzmann partition function is $Z(\beta)$ and a function $q(E)$ is defined by the Laplace-inverse of $Z(\beta)/\beta^2$:

$$q(E) \equiv L_E^{-1}\left[\frac{Z(\beta)}{\beta^2}\right] = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{Z(\beta)}{\beta^2}\, e^{\beta E}\, d\beta\,, \tag{2.4}$$

then the free energy of a non-interacting $n$-particle system with Fermi statistics can be determined by

$$F = n\mu - \int\limits_0^\infty q(E)\,\frac{\partial f}{\partial E}\, dE\,, \tag{2.5}$$

where $\mu$ is the Fermi energy and $f$ is the Fermi-Dirac distribution function:

$$f(E) = \frac{1}{\exp[\beta(E-\mu)] + 1}\,. \tag{2.6}$$

In the present case, we shall substitute (2.3) into (2.4) and carry out the Laplace inverse. To evaluate the complex integral, it is convenient to employ an integration contour. Fig. 1 shows the original SW contour [12]. In the present case, it is possible to choose a simpler one, e.g., a continuous semicircle, since the real axis is no longer a branch cut, in contrast to the corresponding 3D case. But we choose to use the SW contour (Fig. 1), in order to see the difference between the 2D and the 3D cases clearly. Thus the integration in (2.4) can be rewritten as follows:

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} = \sum_{\nu=\pm 1}^{\pm\infty} \gamma_\nu + \frac{1}{2\pi i}\int\limits_{\substack{R_c \\ R_c\to+\infty}} d\beta + \frac{1}{2\pi i}\int\limits_{\substack{\delta \\ \delta\to 0}} d\beta\,. \tag{2.7}$$

Here $\gamma_\nu$ is the residue of the integrand at $\beta_\nu = i\nu\pi/(\mu_B H)$, that is

$$\gamma_\nu = \frac{2(-1)^{\nu-1}}{(\nu\pi)^2}\exp\left(\frac{\nu\pi E}{\mu_B H}\, i\right)(\mu H_B)\cos\left(\frac{g\nu\pi}{2}\right). \tag{2.8}$$

Hence the first term in (2.7) gives

$$\sum_{\nu=\pm 1}^{\pm\infty} \gamma_\nu = \mu_B H \sum_{\nu=1}^{\infty} \frac{2(-1)^{\nu-1}\cos\left(\dfrac{\nu\pi E}{\mu_B H}\right)\cos\left(\dfrac{g\nu\pi}{2}\right)}{(\nu\pi)^2}. \tag{2.9}$$

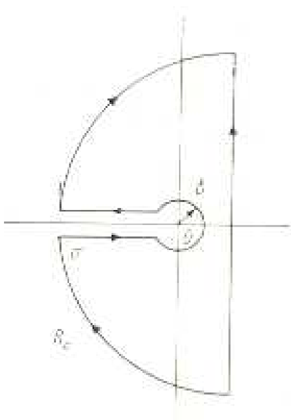


Fig. 1. The closed contour used in the evaluation of the integral appearing in (2.4)

As a matter of fact, this is the term that gives rise to the de Haas-van Alphen effect. It is easy to show from Jordan's lemma that

$$\frac{1}{2\pi i}\int_{\substack{R_0 \\ R_0\to-\infty}} d\beta = 0\,. \tag{2.10}$$

To evaluate the third term in (2.7), we observe that

$$P(\chi) \equiv -\frac{\cosh(\chi g/2)}{\chi^2\sinh\chi} + \frac{1}{\chi^3} + \left(\frac{g^2}{8}-\frac{1}{6}\right)\frac{1}{\chi} = 0(\chi)\,,\qquad \chi\to 0\,, \tag{2.11}$$

and

$$\frac{1}{2\pi i}\int_\sigma \frac{e^{E\beta}\,d\beta}{\beta^2\sinh(\mu_B H\beta)} = \frac{(\mu_B H)^2}{2\pi i}\int_\sigma\left[\frac{1}{(\mu_B H\beta)^3} + \left(\frac{g^2}{8}-\frac{1}{6}\right)\frac{1}{\mu_B H\beta}\right]e^{E\beta}\,d\beta - \frac{(\mu_B H)^2}{2\pi i}\int_\sigma P(\mu_B H\beta)\,e^{E\beta}\,d\beta\,. \tag{2.12}$$

The first integral in (2.12) is readily obtained by means of Hankel's formula for the gamma function:

$$\frac{1}{\Gamma(Z)} = -\frac{1}{2\pi i}\int_\sigma t^{-Z}\,e^t\,dt\,. \tag{2.13}$$

The second integral in (2.12) again consists of two parts: integration along the small circle around origin ($I_1$) and that along the negative $x$-axis ($I_2$). $I_1$ vanishes as the radius of the circle $\delta\to 0$ because of (2.11). Also $I_2 = 0$ since the integrations along opposite directions cancel. In contrast to the 2DEG, this term ($I_2$) does contribute in the 3D case, where $\beta^2$ in the denominator of the left hand side of (2.12) should be replaced by $\beta^{5/2}$. The phase difference between the integrations along opposite directions gives rise to a non-zero "field variation of the steady diamagnetism" [12].

We combine the above results to obtain

$$\varphi(E) = \left(\frac{m}{\pi\hbar^2}\right)\left[\frac{E^2}{2} + \left(\frac{g^2}{8}-\frac{1}{6}\right)(\mu_B H)^2 - 2(\mu_B H)^2\sum_{\nu=1}^{\infty}\frac{(-1)^\nu\cos\left(\frac{\nu\pi E}{\mu_B H}\right)\cos\left(\frac{g\nu\pi}{2}\right)}{(\nu\pi)^2}\right]. \tag{2.14}$$

It should be emphasized (2.14) is an exact result. Putting (2.14) into (2.5), we can find the free energy. The integration with the Fermi-Dirac distribution function, in general, has to be performed numerically. But in most experimental situations, the temperatures are so low that $kT \ll \mu$, which enables us to get an analytic expression for the free energy. We wish to comment that $kT \ll \mu$ is fulfilled for the 3DEG (bulk metals) even at room-temperature. Because of the relatively small effective electron mass in the 2DEG, this condition is true only at very low temperatures ($\approx 1$ K).

The integral involving the first two terms in (2.14) can be carried out using the following standard formula [11]:

$$\int_0^\infty \theta(E)\,\frac{\partial f}{\partial E}\,dE = -\theta(\mu) - \frac{\pi^2}{6}(kT)^2\,\theta''(\mu) + 0[(kT)^4]\,. \tag{2.15}$$

Since the last term in (2.14) oscillates very rapidly, we shall deal with it more carefully:

$$2\int_0^\infty \cos\left(\frac{\nu\pi E}{\mu_B H}\right)\frac{\partial f}{\partial E}\,dE = R\left\{e^{i(\nu\pi\mu/\mu_B H)}\int_{-\beta\mu}^{\infty}\frac{e^{2\pi i \nu Z/\beta\mu_B H}}{\cosh^2 Z}\,dZ\right\}. \tag{2.16}$$

Here R{ } means the real part of { }. If we replace $-\beta\mu$ by $-\infty$, the result of the integral in (2.16) is $-(2\pi^2\nu/\beta\mu_B H)/\sinh(\pi^2\nu/\beta\mu_B H)$. Finally, we obtain the free energy for an ideal 2DEG in the *low-temperature limit*:

$$F = n\mu - \left(\frac{m}{\pi\hbar^2}\right)\left\{\frac{\mu^2}{2} + \frac{\pi^2}{6}(kT)^2 + \left(\frac{g^2}{8}-\frac{1}{6}\right)(\mu H_B)^2 - \right.$$
$$\left. - 2(\mu_B H)^2\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}\left(\frac{kT}{\mu_B H}\right)\frac{\cos(\nu\pi\mu/\mu_B H)\cos(g\nu\pi/2)}{\sinh(\pi^2\nu kT/\mu_B H)}\right\}. \tag{2.17}$$

The second term in the bracket of (2.17) gives rise to the steady part of the specific heat of the 2DEG:

$$C_V^{(st)} = \frac{m\pi k^2 T}{3\hbar^2}. \tag{2.18}$$

Similarly, the third term gives the steady magnetic moment

$$M^{(st)} = \left(\frac{g^2}{4}-\frac{1}{3}\right)\left(\frac{m\mu_B^2 H}{\pi\hbar^2}\right), \tag{2.19}$$

which consists of the spin paramagnetic moment and the Landau diamagnetic moment, their ratio being the same as in the 3D case [14]:

$$|M_p^{(st)}/M_d^{(st)}| = \tfrac{3}{4}\,g^2. \tag{2.20}$$

We note that (2.17) agrees with the result derived earlier by Isihara et al. [7], via propagator techniques. The oscillatory part in (2.17) was derived by Shoenberg [8] but in the steady part, the spin and temperature effects, which lead to (2.18) and (2.19), were missing in Shoenberg's formalism [9].

### 3. An Alternative Formulation

As pointed out in Sec. 1, there are some weak points associated with the expression for the free energy derived in the previous section ((2.17)) viz. neither the weak nor the strong magnetic field limit can be easily taken. It is the purpose of the present section to derive an alternative formula for the free energy. The different approach that we shall follow is to sum up the oscillatory Fourier series in (2.14) before carrying out the integration with the Fermi-Dirac distribution function. In this section we consider spinless electrons ($g = 0$) [3].

The oscillatory part of the function $\varphi(E)$ is

$$\varphi_1(E) \equiv -2\left(\frac{m}{\pi\hbar^2}\right)(\mu_B H)^2\sum_{\nu=1}^{\infty}\frac{(-1)^\nu\cos\left(\frac{\nu\pi E}{\mu_B H}\right)}{(\nu\pi)^2} =$$
$$= \left(\frac{m}{\pi\hbar^2}\right)\left\{\frac{(\mu_B H)^2}{6} - \frac{1}{2}[E - 2(\nu_E + 1)\,\mu_B H]^2\right\}. \tag{3.1}$$

Here $\nu_E$ is defined as the index of the highest Landau level with energy less than or equal to $E$ if $E \geqq E_0$, where $E_0$ is the energy of the lowest Landau level, and for $0 \leqq E < E_0$, $\nu_E = -1$. It is easy to verify that the second line in (3.1) is a periodic function of $E$, with a period $2\mu_B H$. The right hand side of the first line in (3.1) is in fact the Fourier expansion of the second line.

Substituting (3.1) into (2.5) and integrating by parts, we have

$$\begin{aligned} I_1 &\equiv \int_0^\infty \varphi_1(E)\,\frac{\partial f}{\partial E}\,\mathrm{d}E \\ &= \varphi_1(E)\,f(E)\,\Big|_0^\infty - \int_0^\infty \varphi_1'(E)\,f\,\mathrm{d}E \\ &= -\left(\frac{m}{\pi\hbar^2}\right)\frac{(\mu_B H)^2}{6} + I_2\,, \end{aligned} \tag{3.2}$$

where

$$I_2 \equiv -\int_0^\infty \varphi_1'(E)\,f\,\mathrm{d}E\,, \tag{3.3}$$

and

$$\varphi_1'(E) = -\left(\frac{m}{\pi\hbar^2}\right)[E - 2(\nu_E + 1)\,\mu_B H]\,. \tag{3.4}$$

We set $s = \beta(E - \mu)$ in (3.3) to obtain

$$\begin{aligned} I_2 &= -\frac{1}{\beta}\int_{-\beta\mu}^{\infty} \varphi_1'\,\frac{\mathrm{d}s}{e^s + 1} \\ &= -\frac{1}{\beta}\int_0^\infty \varphi_1'\,\frac{e^{-s}\,\mathrm{d}s}{e^{-s} + 1} - \frac{1}{\beta}\int_{-\beta\mu}^{0} \varphi_1'\,\frac{\mathrm{d}s}{e^s + 1} \\ &\equiv I_3 + I_4\,. \end{aligned} \tag{3.5}$$

Here $I_3$ and $I_4$ can be evaluated by expanding the Fermi-Dirac distribution function appropriately, which is straightforward but lengthy. The detailed derivations appear in Appendix A. Our new expression for the free energy is the following:

$$\begin{aligned} F = n\mu - \left(\frac{m}{\pi\hbar^2}\right)\Bigg\{&\frac{\mu^2}{2} - \frac{1}{2}[\mu - 2(l+1)\,\mu_B H]^2 + \\ &+ 2(kT)(\mu_B H)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu+1)}\left[\frac{\cosh(\nu+1)X}{\sinh(\nu+1)b} + \frac{e^{-(\nu+1)\beta\mu}}{(\nu+1)b}\right]\Bigg\}. \end{aligned} \tag{3.6}$$

EXACT!

with

$$b = \beta\mu_B H\,, \qquad X = \beta[\mu - 2(l+1)\,\mu_B H]\,. \tag{3.7}$$

Here $l$ *is the index of the highest Landau level with energy less than or equal to* $\mu$, if $\mu \geqq E_0$; and $l = -1$ if the Fermi energy is below the lowest Landau level ($\mu < E_0$).

We emphasize that our new expression for the free energy given by (3.6) is *exact*. No approximations have been made in the derivation (see Appendix A). By contrast, the standard expression for the free energy, which is given by (2.17), involves an approximation in the oscillatory part, where an error is introduced through replacing $(-\mu/kT)$ by $-\infty$ in a Fermi integration (see (2.16)).

Now we are ready to discuss the limiting cases. As we shall see, our new expression for the free energy [(3.6)] has the another advantage over the standard one [(2.17)] viz. both the weak and strong magnetic field limits can be easily taken. In the strong magnetic field limit, (3.6) leads to the result obtained earlier by Vagner et al [4, 5], provided that a misunderstanding in [4] is corrected. In the weak field case, (3.6) can be cast into a form obtainable via Euler's summation formula. By contrast, it is not straightforward to take either of these limits in the standard expression given by (2.17).

### *3.1 Strong magnetic field limit: $b \gg 1$ ($\mu_B H \gg kT$)*

Since $b \gg 1$ in the strong field limit, we expand $[\sinh (r+1)\, b]^{-1}$ in (3.6), keeping the first two terms only. We obtain the following:

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(r+1)} \frac{\cosh (r+1)\, X}{\sinh (r+1)\, b} \approx \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+1)} \{e^{(r+1)(X-b)} + e^{-(r+1)(X+b)}\} =$$
$$= \ln [1 + e^{(X-b)}] + \ln [1 + e^{-(X+b)}] . \qquad (3.8)$$

Combination of (3.6) and (3.8) gives the following approximation for the free energy in the strong magnetic field limit:

$$F = n\mu - \left(\frac{2m}{\pi\hbar^2}\right) \{\mu(l+1)\, \mu_B H - (\mu_B H)^2 (l+1)^2 +$$
$$+ (kT)(\mu_B H) \ln [1 + e^{(X-b)}] + (kT)(\mu_B H) \ln [1 + e^{-(X+b)}]\} . \qquad (3.9)$$

We now turn to a determination of the Fermi energy and the index $l$. In general, the Fermi energy can be evaluated from the free energy by letting $\partial F/\partial\mu = 0$, which is (see (3.9)):

$$n = \frac{2m}{\pi\hbar^2} \left\{(l+1)\, \mu_B H + (\mu_B H) \left[\frac{e^{X-b}}{1 + e^{X-b}} - \frac{e^{-X-b}}{1 + e^{-X-b}}\right]\right\} . \qquad (3.10)$$

Solving (3.10) for $e^X$ we obtain

$$(e^X)_{\pm} = \frac{-A(c^2+1) \pm [A^2(c^2+1)^2 - 4c^2(A^2-1)]^{1/2}}{2c(A-1)} , \text{ for } A \neq 1 , \qquad (3.11)$$

where

$$c = \exp (b) , \qquad (3.12)$$

$$A = \frac{\mu_0^{(0)}}{2\mu_B H} - (l-1) ,$$
$$= \frac{n}{D_s} - (l+1) , \qquad (3.13)$$

where $\mu_0^{(0)} = \pi\hbar^2 n/m$ is the Fermi energy at zero magnetic field and zero temperature, $D_s = 2D$ is the degeneracy of each Landau level, including spin degeneracy.

We now examine our solution (3.11) further. First we consider $(e^X)_-$, whose positivity requires that

$$-1 < A < 1 , \qquad (3.14)$$

which is the first constraint on $A$.

We now can rewrite (3.11) by keeping $(e^X)_-$ only:

$$\mu = (E_l + \mu_B H) - kT \ln\left\{\frac{\Delta(c^2+1) + [\Delta^2(c^2+1)^2 - 4c^2(\Delta^2-1)]^{1/2}}{2c(1-\Delta)}\right\}. \tag{3.15}$$

It is straightforward to verify that (3.15) agrees with the result obtained by Vagner et al. [4], provided that $l = n_F$, where $n_F$ is the index of the highest filled Landau level defined in [4][2]). As we shall soon see, it is not always true, depending on the filling situation of the highest filled Landau level. One sees from (3.9) and (3.15) that a correct determination of $l$ is obviously critical to correct calculations of the free energy and other thermodynamical properties.

Now we shall try to find $l$, which we recall is the index of the highest Landau level with energy less than or equal to the Fermi energy. The procedure described below is rather lengthy. But we believe that it is the simpliest and yet rigorous. First of all, we make the following observation. When $\Delta = 0$, the highest level is completely filled and (3.15) reduces to $\mu = E_l + \mu_B H$, which means the Fermi energy stays exactly at the mid-point of $l$-th and $(l+1)$-th Landau levels when the ratio $n/D_s$ is an integer. In this case the index $l$ is simply $(n/D_s - 1)$. In other words, the Fermi energy is above the highest filled Landau level by $\mu_B H$ if all filled levels are completely filled.

A second conclusion that we can draw immediately from (3.15) is the following: in the strong magnetic field limit $\mu < E_l + \mu_B H$ if $\Delta < 0$, and $\mu > E_l + \mu_B H$ if $\Delta > 0$. The proof is given in Appendix B.

One can make a third observation. Since the average number of electrons on a Landau level is

$$\langle n_l \rangle = \frac{D_s}{\exp[\beta(E_l - \mu)] + 1}, \tag{3.16}$$

this Landau level will be half filled if and only if $\mu = E_l$. In this case one has $\Delta = -\frac{1}{2}$.

Keeping the three points mentioned above in mind, we now describe a possible experiment. Starting with $\Delta = 0$, $l = l'$ say, $\mu = E_{l'} + \mu_B H$, and keeping the field constant, we slowly reduce the electron number density $n$, by reducing the gate voltage in a MOSFET say. Thus the number of electrons on the highest filled Landau level $E_{l'}$, which is initially completely filled, begins to decrease, and so do $\Delta$ and the Fermi energy. When the $l'$-th level is only half filled, $\Delta = -\frac{1}{2}$, and we have $\mu = E_{l'}$.

Now we reverse the procedure by starting with $\Delta = 0$ and $l = l'$ but increasing the number of electrons. Hence the Fermi energy and $\Delta$ also increases. The $(l'-1)$-th Landau level begins to be filled. When $\Delta$ is close to but less than $\frac{1}{2}$, i.e. the $(l'-1)$-the Landau level is close to but is less than half filled, the Fermi energy is close to but below $E_{l'-1}$. As the number of electrons still increases, the $(l'-1)$-th Landau level is half filled, and we have $\mu = E_{l'-1}$. Instead of $\frac{1}{2}$, $\Delta$ takes the value $-\frac{1}{2}$ again, since the index $l$, which is the highest Landau level with energy less than or equal to the Fermi energy, now changes from $l'$ to $(l'+1)$.

From the above analysis, we conclude that for a given number of electrons $n$ and a magnetic field $H$, the index $l$ can be determined by the following constraint:

$$-\frac{1}{2} \leq \Delta = \frac{\mu_0^{(0)}}{2\mu_B H} - (l+1) < \frac{1}{2}, \tag{3.17}$$

[2]) In (3) of [4] and thereafter, the Landau levels are defined to be $E_n = (n - 1/2)(\hbar\omega_c)$, where $n = 1, 2, \ldots$. In our paper we follow the conventional definition: $E_n = (n + 1/2)(\hbar\omega_c)$, where $n = 0, 1, 2, \ldots$. Hence our $n_F$ is equal to that used in [4] minus one. This will not effect our discussions on the physical difference between $n_F$ and $l$.

which is equivalent to

$$l = \left[ \frac{\mu_0^{(0)}}{2\mu_B H} - \frac{1}{2} \right]. \tag{3,18}$$

Here $[\chi]$ is the largest integer that does not exceed $\chi$. If $-1 < \chi < 0$, we define $[\chi] = -1$.

Once $l$ is found, the free energy and the Fermi energy are readily calculated from (3,9) and (3.15).

As we have clearly seen, $l$ and $n_F$, the later being the index of the highest filled Landau level, are not always the same. This subtle point was overlooked in [4]. Precisely, one has

$$n_F = l\,, \quad \text{when} \quad -\tfrac{1}{2} \leq \Delta \leq 0\,, \tag{3.19}$$

or when the $n_F$-th level is more than half filled. But

$$n_F = l + 1\,, \quad \text{when} \quad 0 < \Delta < \tfrac{1}{2}\,, \tag{3,20}$$

or when $n_F$-th level is less than half filled. In other words, in contrast to (3,18), one has

$$n_F = \left[ \frac{\mu_0^{(0)}}{2\mu_B H} \right]. \tag{3,21}$$

A specially interesting case is the following. When the magnetic field is so strong that even the lowest Landau level is less than half filled, $l = -1$ and $n_F = 0$ (see (3,18) and (3.21)), while the Fermi energy is pushed below the lowest Landau level and the electron statistics may become non-degenerate [13]. In the light of what we shall derive below, we shall discuss the condition of non-degeneracy. Here we emphasize again that *it is crucial to determine the index $l$ correctly before one can calculate any physical properties of the system*, e.g., the Fermi energy and the magnetic susceptibility.

Here we recall that the other solution of (3,11), i.e., $(e^X)_+$, has not been considered yet. It is easy to prove that $(e^X)_- > 0$ only if $\Delta > 1$, which is not allowed by (3,17). Hence one should disgard this solution.

When the external magnetic field is very strong, $c$ [see (3.12)] is exponentially large. As either the number of electrons or the magnetic field (or both) varies, $\Delta$ defined by (3.13) takes values in the region $(-\frac{1}{2}, \frac{1}{2})$. In most cases $|\Delta|$ is not extremely small, so that $|\Delta c|$ is still a very large quantity. As we shall show below, every physical property, except for the specific heat, is approximately equal to the corresponding zero temperature value, the difference being the order of $(kT/\mu_B H)$. The specific heat vanishes identically in this case. When $|\Delta c| \approx 1$, the region being exponentially small, very large effects due to temperature will appear in some quantities. In the following discussions, we shall consider these two kinds of regions separately.

### *3.2 Strong magnetic field limit: $b \gg 1$ and $|\Delta c| \gg 1$*

Equation (3,15) allows further simplification when $\Delta$ is not very close to zero so that

$$|\Delta c| \gg 1\,. \tag{3,22}$$

First we consider the case where $0 < \Delta < \frac{1}{2}$. Expansion of the temperature-dependent term in (3,15) gives

$$\mu = E_{l+1} - kT \ln\left(\frac{1}{\Delta} - 1\right). \tag{3,23}$$

Similarly we have in the case where $-\frac{1}{2} \leqq \Delta < 0$:

$$\mu = E_l - kT \ln\left(\frac{1}{\Delta + 1} - 1\right). \tag{3.24}$$

By taking (3.19) and (3.20) into account, we can combine (3.23) and (3.24) to obtain the following compact result:

$$\mu = E_{\nu_F} - kT \ln\left(\frac{1}{\Delta_F} - 1\right), \tag{3.25}$$

where $\Delta_F$ is obtained by substituting $n_F$ for $l+1$ in (3.13):

$$\Delta_F = \frac{n}{D_s} - n_F = \frac{\mu_0^{(0)}}{2\mu_B H} - n_F \tag{3.26}$$

One sees that $\Delta_F$ is always positive, whereas $-1/2 \leqq \Delta < 1/2$.

Equation (3.25) tells us that the Fermi energy is very close to the highest filled Landau level, the difference being the order of $kT$, when this level is not so completely filled or empty that (3.22) holds.

Similarly, if (3.22) is true, the free energy takes the following form:

$$F = n\mu - D_s n_F \mu + D_s n_F^2 (\mu_B H) + kT\, D_s \ln(1 - \Delta_F). \tag{3.27}$$

When we substitute (3.25) into (3.27), we obtain

$$F = [(2n_F + 1)\, n - D_s(n_F + 1)\, n_F]\, \mu_B H + \\ + kT\, D_s[(1 - \Delta_F) \ln(1 - \Delta_F) + \Delta_F \ln \Delta_F]. \tag{3.28}$$

It is interesting to see that the temperature-dependent term in the free energy [(3.28)] is linear in temperature. Hence the internal energy $U$, which can be calculated from the free energy by $\partial(\beta F)/\partial\beta$, where $\beta = 1/kT$, is temperature-independent:

$$U \equiv U_0 = \\ = [(2n_F + 1)\, n - D_s(n_F + 1)\, n_F]\, \mu_B H. \tag{3.29}$$

Here the subscript "0" denotes the zero temperature [2]. One can immediately conclude that the specific heat vanishes in this case:

$$C_V = \frac{\partial U}{\partial T} = \\ = 0. \tag{3.30}$$

Now we turn to a consideration of the magnetic susceptibility. The magnetic moment can be easily derived from (3.28) by taking the derivative with respect to the magnetic field,

$$M = -\frac{\partial F}{\partial H} = \\ = M_0 + M_T, \tag{3.31}$$

where $M_0$ is the magnetic moment at zero temperature [2], given by

$$M_0 = n\mu_B[2n_F(n_F + 1)\, x - (2n_F + 1)], \quad \text{if} \quad \frac{1}{n_F + 1} < x < \frac{1}{n_F}, \quad n_F = 1, 2, \ldots;$$

and

$$= -n\mu_B, \quad \text{if} \quad x > 1, \quad \text{or} \quad n_F = 0, \tag{3.32}$$

with $x = H/H_0$, $H_0 = \pi\hbar n/e = 2.07 \times 10^{-11}$ T cm² $n$, and $M_T$ is the temperature-dependent contribution:

$$M_T = -2n\mu_B \left(\frac{kT}{\mu_0^{(0)}}\right)[(n_F + 1) \ln (1 - \Delta_F) - n_F \ln \Delta_F] . \tag{3.33}$$

The magnetic susceptibility is readily evaluated in the usual manner. The temperature-independent part of the result is

$$\begin{aligned} \chi_0 &= \frac{\partial M_0}{\partial H} = \\ &= 2\left(\frac{n\mu_B}{H_0}\right)(n_F + 1)\, n_F\,, \quad \text{if} \quad \frac{1}{n_F + 1} < x < \frac{1}{n_F}\,, \\ &= -\infty\,, \qquad \text{if} \quad x = 1/n_F\,, \quad n_F = 1, 2, \ldots\,, \\ &= 0\,, \qquad \text{if} \quad x_F > 1\,, \quad \text{or} \quad n_F = 0\,. \end{aligned} \tag{3.34}$$

Similarly we obtain

$$\begin{aligned} \chi_T &= \frac{\partial M_T}{\partial H} = \\ &= -2\left(\frac{n\mu_B}{H_0}\right)\left(\frac{kT}{\mu_0^{(0)}}\right)\left(\frac{n_F + 1}{1 - \Delta_F} - \frac{n_F}{\Delta_F}\right). \end{aligned} \tag{3.35}$$

### *3.3 Strong magnetic field limit: $b \gg 1$ and $|\Delta c| \ll 1$*

In some cases where the highest filled Landau level is so close to being empty or completely filled, which means $|\Delta| \to 0$ [see (3.13)], that

$$|\Delta c| \ll 1\,, \tag{3.36}$$

despite the fact that $c$ is very large in the strong magnetic field limit. In this case, the temperature effects on various physical quantities, such as the Fermi energy and the magnetic susceptibility, are no longer negligible [4, 5].

The Fermi energy in this limit can be expressed as follows [from (3.15)]:

$$\mu = 2(l + 1)\, \mu_B H + \frac{\Delta c\, kT}{2}\,. \tag{3.37}$$

Recalling the relation between indices $l$ and $n_F$, we know that (3.37) is different from the result by Vagner et al. (Equation (7a) in [5]).

From (3.7) and (3.37), we obtain

$$X = \frac{\Delta c}{2}\,, \quad \text{with} \quad |X| \ll b\,. \tag{3.38}$$

Substituting (3.38) into (3.9), we obtain the result for the free energy in this limit

$$\begin{aligned} F = m\mu - \frac{m}{\pi\hbar^2}\,(&[\mu(l + 1)\, \mu_B H - \tfrac{1}{2}\,(\mu_B H)^2\,(l - 1)^2] - \\ &+ 4kT\,(\mu_B H) \exp{(-b\,)}\}\,. \end{aligned} \tag{3.39}$$

Thermodynamical properties can be evaluated from (3.39) in a straightforward fashion, as illustrated in part 3.2 of this section. One finds that the specific heat is exponentially small in this region. But the temperature effect on the magnetic moment and the Fermi energy are comparable with their zero-temperature values. Precisely, the singularities in these quantities are rounded out by non-zero temperature effects.

### 3.4 Quantizing magnetic field and non-degenerate limits

Now we return to the interesting case where the Fermi energy is pushed below the lowest Landau level by a so-called quantizing magnetic field. In this situation we recall that $l = -1$ and $n_T = 0$, hence $\Delta = \mu_0^{(0)}/2\mu_B H$ from (3.13). Here $\mu_0^{(0)}$ is the zero-temperature-zero-field Fermi energy which depends on the electron number density only. Hence

$$|\Delta e| = \frac{\mu_0^{(0)}}{2\mu_B H}\exp\left(\frac{\mu_B H}{kT}\right). \tag{3.40}$$

We observe that (3.36) is never true in this quantizing magnetic field limit, i.e., one always has $|\Delta e| \gg 1$ in this case. Hence the discussions in Part 3.2 of this section applies as the magnetic field varies from this quantizing field to infinity.

For instance, the Fermi energy, which is given by (3.25), is

$$\mu = E_0 - kT\ln\left[\frac{2\mu_B H}{\mu_0^{(0)}} - 1\right]. \tag{3.41}$$

As we can see from (3.41), when the magnetic field gets so strong that $(\mu_B H/\mu_0^{(0)}) \gg 1$, the difference $(\mu - E_0)$ depends logarithmically on the magnetic field. As first argued by Zawadzki [13], the electron statistics will be non-degenerate if the magnetic field is strong enough. We now ask the question how strong the magnetic field has to be in order to achieve such a situation. Consider, for instance, the GaAs/AlGaAs sample used in the magnetic susceptibility measurement by Eisenstein et al. [3c]. The electron number density $n = 5.4 \times 10^{11}$ cm$^{-2}$, and the electron effective mass $m = 0.0665 m_0$. It is easy to verify that a magnetic field as strong as 300 Tesla is needed in order to push the Fermi energy below the lowest Landau level by $5kT$, where the statistics barely becomes non-degenerate. But on the other hand, $E_0 - \mu \gtrsim 5kT$ can be achieved by requiring that $n \lesssim 6.5 \times 10^9$ cm$^{-2}$ with a magnetic field $B \gtrsim 20$ T.

### 3.5 Weak magnetic field limit: $b \ll 1$ ($\mu_B H \ll kT$)

When the magnetic field is weak, by which we mean that

$$\mu_B H \ll kT, \quad \text{or} \quad b = \mu_B H/kT \ll 1, \tag{3.42}$$

$$\sin(\nu+1)\,b \approx (\nu+1)\,b, \qquad \cosh(\nu+1)\,b \approx 1,$$

$$\sum_{\nu=0}^{\infty}(-1)^\nu(\nu+1)^{-2} = \pi^2/12,$$

the free energy given by (3.6) takes the following form:

$$F = n\mu - \left(\frac{m}{\pi\hbar^2}\right)\left\{\frac{\mu^2}{2} - \frac{1}{2}[\mu - 2(l+1)\,\mu_B H]^2 + \frac{\pi^2}{6}(kT)^2\right\}. \tag{3.43}$$

We shall show that (3.43) agrees completely with the free energy derived via Euler's summation formula. One recalls that the direct summation method via Euler's formula was first employed by Landau and Dingle in discussions on the 3D weak field magnetism [15].

By definition, the free energy of a 2DEG is

$$F = n\mu - \frac{1}{\beta} \sum_{\nu=0}^{\infty} D_{\nu} \ln \{1 + \exp[-\beta(E_{\nu} - \mu)]\}, \tag{3.44}$$

where $D_{\nu} = m\mu_{\mathrm{B}}H/\pi\hbar^2$ is the degeneracy of each Landau level. Euler's summation formula states that

$$\sum_{\nu=0}^{N} \theta(\nu) = \int_0^N \theta(\chi)\,\mathrm{d}\chi + \frac{\theta(0) + \theta(N)}{2} \int_0^N \xi(\chi)\,\theta''(\chi)\,\mathrm{d}\chi, \tag{3.45}$$

where $\xi(\chi)$ is a periodic function with a period 1 and

$$\xi(\chi) = \frac{\chi(\chi - 1)}{2}, \quad \text{if} \quad 0 \leqq \chi \leqq 1. \tag{3.46}$$

In our case, $N = +\infty$, and

$$\theta(\chi) = -\frac{m\mu_{\mathrm{B}}H}{\pi\beta\hbar^2} \ln \{1 + \exp[\beta(\mu - 2\mu_{\mathrm{B}}H(\chi + \tfrac{1}{2}))]\}. \tag{3.47}$$

We observe that the first derivative of $\theta(\chi)$ is proportional to the Fermi-Dirac distribution function. Hence the second derivation is very close to a δ-function, i.e.,

$$\theta''(\chi) = -\frac{4m(\mu_{\mathrm{B}}H)^2}{\pi\hbar^2}\,\delta\left(\chi + \frac{1}{2} - \frac{\mu}{\mu_{\mathrm{B}}H}\right). \tag{3.48}$$

Then the first term in (3.45) can be evaluated by integrating by parts twice and using (3.48), to obtain

$$\int_0^{\infty} \theta(\chi)\,\mathrm{d}\chi = -\left(\frac{m}{\pi\hbar^2}\right)\left[\frac{(\mu - \mu_{\mathrm{B}}H)^2}{2} + \frac{\pi^2(kT)^2}{6}\right]. \tag{3.49}$$

The second term in (3.45) is just

$$\frac{\theta(0) + \theta(+\infty)}{2} = -\left(\frac{m}{\pi\hbar^2}\right)\mu_{\mathrm{B}}H(\mu - \mu_{\mathrm{B}}H). \tag{3.50}$$

Again using (3.48), we obtain the third term:

$$-\int_0^{+\infty} \xi(\chi)\,\theta''(\chi)\,\mathrm{d}\chi = \frac{m(\mu_{\mathrm{B}}H)^2}{\pi\hbar^2}\,\xi\left(\frac{\mu}{\mu_{\mathrm{B}}H}\right). \tag{3.51}$$

Recalling that $\xi(\chi)$ is periodic, one can rewrite (3.51) as follows:

$$-\int_0^{\infty} \xi(\chi)\,\theta''(\chi)\,\mathrm{d}\chi = \frac{m(\mu_{\mathrm{B}}H)^2}{2\pi\hbar^2}\left[\frac{\mu}{\mu_{\mathrm{B}}H} - (2j - 1)\right]\left[\frac{\mu}{\mu_{\mathrm{B}}H} - (2j + 3)\right],$$

$$\text{for} \quad (2j - 1)\,\mu_{\mathrm{B}}H \leqq \mu < (2j + 3)\,\mu_{\mathrm{B}}H. \tag{3.52}$$

Combining (3.49), (3.50), and (3.52) we obtain exactly (3.43) again, which provides a check on our new expression for the free energy given by (3.6).

## 4. Summary

In this paper we have derived a new expression for the free energy of an ideal two-dimensional electron gas in a uniform magnetic field. Compared to the standard formulation, the present form enables one to obtain results in both the weak and the strong magnetic field limit in a straightforward fashion. In the weak magnetic field limit, it agrees with the result derived via Euler's summation formula. In the strong field regime, it reduces to a form obtained by Vagner et al [4, 5], provided that a subtlety is clarified. We have presented discussions on the Fermi energy, the specific heat and the magnetic moment. The possibility of the non-degenerate situation first proposed by Zawadzki [13] has also been discussed in the light of our results.

## Appendix A

### *Evaluations of $I_3$ and $I_4$ in (3.5)*

Letting $y = \beta E$, one can rewrite $I_3$ as follows:

$$I_3 = -\frac{1}{\beta}\int_0^\infty \varphi_1 \sum_{r=0}^\infty (-1)^r \mathrm{e}^{-(r+1)z}\,\mathrm{d}z$$

$$= -\frac{1}{\beta}\sum_{r=0}^\infty \mathrm{e}^{\beta\mu(r+1)}(-1)^r \int_{\beta\mu}^\infty \varphi_1\,\mathrm{e}^{-(r+1)y}\,\mathrm{d}y\,. \qquad \text{(A1)}$$

Substituting (3.4) into (A1) one obtains

$$I_3 = \left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\sum_{r=0}^\infty (-1)^r \mathrm{e}^{\beta\mu(r+1)}\int_{\beta\mu}^\infty [y - 2(\nu_E + 1)\,b]\,\mathrm{e}^{-(r+1)y}\,\mathrm{d}y$$

$$\equiv \left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\sum_{r=0}^\infty (-1)^r \mathrm{e}^{\beta\mu(r+1)}\,(I_5 - bI_6)\,, \qquad \text{(A2)}$$

where $I_5$ can be easily derived:

$$I_5 = \int_{\beta\mu}^\infty y\,\mathrm{e}^{-(r+1)y}\,\mathrm{d}y$$

$$= \left[\frac{\beta\mu}{r+1} - \frac{1}{(r+1)^2}\right]\mathrm{e}^{-(r+1)\beta\mu}\,. \qquad \text{(A3)}$$

But the evaluation of $I_6$ in (A2) requires some work. We write:

$$I_6 = 2(l+1)\int_{\beta\mu}^{(2l+3)b}\mathrm{d}y\,\mathrm{e}^{-(r+1)y} + \sum_{n=l+2}^\infty 2n\int_{(2n-1)b}^{(2n+1)b}\mathrm{d}y\,\mathrm{e}^{-(r+1)y}\,, \qquad \text{(A4)}$$

where we recall that $l$ is the index of the highest Landau level with energy less than or equal to the Fermi energy. It follows that

$$I_6 = 2(l+1)\left.\frac{e^{-(\nu+1)y}}{(\nu+1)}\right|_{(2l+3)b}^{\beta\mu} + \sum_{n=l+2}^{\infty} 2n \left.\frac{e^{-(\nu+1)y}}{(\nu+1)}\right|_{(2n+1)b}^{(2n-1)b}$$

$$= \frac{1}{(\nu+1)}\{2(l+1)[e^{-(\nu+1)\beta\mu} - e^{-(\nu+1)(2l+3)b}] +$$

$$+ 2(l+2)[e^{-(\nu+1)(2l+3)b} - e^{-(\nu+1)(2l+5)b}] +$$

$$+ 2(l+3)[e^{-(\nu+1)(2l+5)b} - e^{-(\nu+1)(2l+7)b}] +$$

$$+ \ldots\}$$

$$= \frac{2(l+1)}{\nu+1} e^{-(\nu+1)\beta\mu} - \frac{1}{\nu+1}\{2\,e^{-(\nu+1)(2l+3)b} + 2\,e^{-(\nu+1)(2l+5)b} +$$

$$+ 2\,e^{-(\nu+1)(2l+7)b} +$$

$$+ \ldots\}$$

$$= \frac{2(l+1)}{(\nu+1)} e^{-(\nu+1)\beta\mu} - \frac{2}{(\nu+1)} \frac{e^{-(\nu+1)(2l+3)b}}{1 - e^{-2(\nu+1)b}}. \tag{A5}$$

Substituting (A3) and (A5) into (A2) one obtains

$$I_5 = \left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\sum_{\nu=0}^{\infty}(-1)^{\nu}\left\{\frac{\beta\mu}{\nu+1} - \frac{1}{(\nu+1)^2} - \frac{2b(l+1)}{\nu+1} -\right.$$

$$\left. - \frac{2b}{(\nu+1)}\frac{e^{-(\nu+1)[(2l+3)b-\beta\mu]}}{1 - e^{-2(\nu+1)b}}\right\}. \tag{A6}$$

The quantity $I_4$ shall be evaluated in a similar fashion:

$$I_4 = \left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\int_0^{\beta\mu}\frac{dy}{1 - e^{-\beta\mu}}[y - 2(\nu_E + 1)b]$$

$$= \left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\sum_{\nu=0}^{\infty}(-1)^{\nu} e^{-\nu\beta\mu}(I_7 - bI_8), \tag{A7}$$

where

$$I_7 = \int_0^{\beta\mu} e^{\nu y}\, y\, dy$$

$$= e^{\nu\beta\mu}\left(\frac{\beta\mu}{\nu} - \frac{1}{\nu^2}\right) + \frac{1}{\nu^2}, \quad \text{for} \quad \nu \neq 0; \tag{A8}$$

$$I_7 = \tfrac{1}{2}(\beta\mu)^2, \quad \text{for} \quad \nu = 0. \tag{A9}$$

and

$$I_8 = 2\int_0^{\beta\mu} e^{\nu y}(\nu_E + 1)\,dy$$

$$= 2\sum_{\nu_E=-1}^{l-1}\int_{(2\nu_E+1)b}^{(2\nu_E+3)b} e^{\nu y}(\nu_E + 1)\,dy - 2\int_{(2l+1)b}^{\beta\mu} e^{\nu y}(l+1)\,dy$$

$$= \frac{1}{\nu}\{2(e^{3b\nu} - e^{b\nu}) + 4(e^{5b\nu} - e^{3b\nu}) + \ldots +$$

$$+ 2l[e^{(2l+1)b\nu} - e^{(2l-1)b\nu}] + 2(l+1)[e^{\beta\mu\nu} - e^{(2l+1)b\nu}]\}$$

$$= \frac{1}{\nu}\left\{2(l+1)\,e^{\nu\beta\mu} - \frac{2[e^{(2l+1)b\nu} - e^{-b\nu}]}{1 - e^{-2b\nu}}\right\}, \quad \text{for} \quad \nu \neq 0; \tag{A10}$$

$$I_8 = \frac{b[4(l+1/2)^2 - 1]}{2} + 2(l+1)[\beta\mu - (2l+1)b], \quad \text{for} \quad \nu = 0. \tag{A11}$$

Combination of (A7) to (A11) gives

$$I_4 = \left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\left\{\frac{1}{2}(\beta\mu)^2 - \frac{b^2}{2}[(2l+1)^2 - 1] - 2b(l+1)[\beta\mu - (2l+1)b]\right\} -$$

$$-\left(\frac{m}{\pi\hbar^2}\right)\frac{1}{\beta^2}\sum_{\nu=0}^{\infty}(-1)^\nu\left\{\frac{\beta\mu}{\nu+1} - \frac{1}{(\nu+1)^2} - \frac{2b(l+1)}{\nu+1} +\right.$$

$$\left. + \frac{2b}{(\nu+1)}\,\frac{e^{[(2l+3)b-\beta\mu](\nu+1)}}{1 - e^{-2b(\nu+1)}} + \frac{e^{-(\nu+1)\beta\mu}}{(\nu+1)^2}\right\}. \tag{A12}$$

Finally, we combine (A6) and (A12) to obtain the free energy [(3.6)].

## Appendix B

### *More discussions on (3.16)*

In this appendix, we shall prove that [see (3.15)]

$$\mu \gtreqqless E_l + \mu_B H, \quad \text{if} \quad \Delta \gtreqqless 0. \tag{B1}$$

Denoting

$$G = \frac{\Delta(c^2+1) + [\Delta^2(c^2+1)^2 + 4c^2(\Delta^2 - 1)]^{1/2}}{2c(1-\Delta)}, \tag{B2}$$

we can rewrite the Fermi energy given by (3.15) as follows:

$$\mu = (E_l + \mu_B H) + kT \ln G. \tag{B3}$$

Hence (B1) is equivalent to the following:

$$G \gtreqqless 1, \quad \text{if} \quad \Delta \gtreqqless 0. \tag{B4}$$

We shall prove (B4) instead.

When $\Delta < 0$, we have

$$\Delta(c+1)^2 < 0. \tag{B5}$$

Multiplying (B5) by $4c(\Delta - 1)$, which is negative since $|\Delta| \leq 1/2$, we obtain

$$8c^2\Delta^2 - 8c^2\Delta - 4c\Delta^2(c^2+1) - 4c\Delta(1+c^2) > 0. \tag{B6}$$

It is not difficult to show that (B5) is exactly the same as $G < 1$. Similarly, by starting with $\Delta(c+1)^2 \geq 0$ when $\Delta \geq 0$, one can easily deduce that $G \geq 1$ if $\Delta \geq 0$.

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