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ON TOLMAN'S MASS-ENERGY RELATION AND A NEW TOLMAN-TYPE RELATION

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Tolman derived the mass-energy relation $\mathscr{E} = \int(\mathfrak{T}_0{}^0 + \mathfrak{T}_i{}^i)dV = mc^2$ using a particular choice of coordinates, viz. the Schwarzschild solution for the metric tensor in isotropic coordinates for a body of mass m at rest at the origin. Here we show that this relation retains the same form for the case of a very general coordinate system. The latter includes the Schwarzschild and harmonic coordinates as special cases. In addition, we give a new Tolman-type relation $\int(\mathfrak{t}_0{}^0 + \mathfrak{t}_i{}^i)dV = 0$. The quantities $\mathfrak{T}_\mu{}^\nu$ and $\mathfrak{t}_\mu{}^\nu$ are the energy-momentum densities for matter and the gravitational field, respectively.

1. Introduction

In 1930, Tolman[1] gave the mass-energy relation

$$\mathscr{E} = \int(\mathfrak{T}_0{}^0 + \xi_0{}^0)dV = \int(\mathfrak{T}_0{}^0 + \mathfrak{T}_i{}^i)dV = mc^2, \tag{1.1}$$

and the momentum relation

$$P_i = \int \frac{1}{c}(\mathfrak{T}_i{}^0 + \xi_i{}^0)dV = 0. \tag{1.2}$$

There is also a similar additional relation

$$\int(\mathfrak{T}_i{}^j + \xi_i{}^j)dV = 0, \tag{1.3}$$

that does not appear in Tolman's[1] paper or book. In the above, $\mathfrak{T}_\mu{}^\nu$ is the energy-

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momentum density for matter, and $\xi_\mu{}^\nu$ is the *canonical* energy-momentum density for the gravitational field. We will use $\mathfrak{t}_\mu{}^\nu$ to designate the energy-momentum density for the gravitational field (note that many authors, including Tolman, use $\mathfrak{t}_\mu{}^\nu$ to designate the *canonical* energy-momentum density for the gravitational field). In deriving Eqs. (1.1) and (1.2), Tolman used the Schwarzschild solution for the metric tensor in isotropic coordinates for a body of mass m at rest at the origin. Equation (1.3) can be similarly derived. Throughout this paper, we will use the metric with signature $(++++)$ so that $x^\mu = (x^i, x^4)$ where $x^4 \equiv ix^0 \equiv ict$. We also define $\mathfrak{T}_0{}^0 \equiv (-i)^2\mathfrak{T}_4{}^4 = -\mathfrak{T}_4{}^4$ and $\mathfrak{T}_i{}^0 \equiv -i\mathfrak{T}_i{}^4$, etc. Because of the signature of our metric, we have a $+\,\mathfrak{T}_i{}^i$ in Eq. (1.1) rather than a $-\mathfrak{T}_i{}^i$ as one would have with the signature used by Tolman.[1]

In Sec. 2 of this paper, we discuss the field equations of general relativity in the Papapetrou[2] form. In Secs. 3 and 4, we derive the relations of Eqs. (1.1)–(1.3) using a very general coordinate system $\mathbf{r}$, t related to harmonic[3] coordinates $\mathbf{r}_{(h)}$, $t_{(h)}$ by the coordinate transformation

$$\mathbf{r}_{(h)} = \mathbf{r}\left(1 - \alpha\frac{Gm}{c^2 r}\right), \qquad t_{(h)} = t, \tag{1.4}$$

where α is an arbitrary dimensionless parameter, $\mathbf{r} = (x^1, x^2, x^3)$, G is Newton's gravitational constant, and c is the speed of light. For $\alpha = 0$, $\mathbf{r}$ is in harmonic coordinates[3] while for $\alpha = 1$, $\mathbf{r}$ is in Schwarzschild coordinates.[4,5] The exact Schwarzschild solution for the metric tensor in harmonic coordinates for a (nonrotating) body of mass m at rest at the origin is given by Fock.[3]

Making the coordinate transformation of Eq. (1.4) and then keeping only terms to order $1/r$ (because we will only need the results at large distances from the origin), we obtain the following results for the metric tensor in the $\mathbf{r}$, t coordinate system:

$$g_{ij} = \delta_{ij} + \frac{2Gm}{c^2 r}\left[(1-\alpha)\delta_{ij} + \alpha\frac{x^i x^j}{r^2}\right], \qquad g_{44} = 1 - \frac{2Gm}{c^2 r}, \tag{1.5}$$

$$g^{ij} = \delta_{ij} - \frac{2Gm}{c^2 r}\left[(1-\alpha)\delta_{ij} + \alpha\frac{x^i x^j}{r^2}\right], \qquad g^{44} = 1 + \frac{2Gm}{c^2 r}, \tag{1.6}$$

$$\mathfrak{g}^{ij} = \delta_{ij} - \frac{2Gm}{c^2 r}\left[\alpha\frac{x^i x^j}{r^2}\right], \qquad \mathfrak{g}^{44} = 1 + \frac{2Gm}{c^2 r}(2-\alpha), \tag{1.7}$$

$$g_{i4} = g^{i4} = \mathfrak{g}^{i4} = 0, \qquad g^{1/2} = 1 + \frac{2Gm}{c^2 r}(1-\alpha), \tag{1.8}$$

where $g \equiv |g_{\mu\nu}|$ and $\mathfrak{g}^{\mu\nu} \equiv g^{1/2}g^{\mu\nu}$. To the approximation made in Eqs. (1.5)–(1.8), harmonic and isotropic coordinates are the same and correspond to $\alpha = 0$. Thus, the coordinates $\mathbf{r}$, t defined by Eq. (1.4) represents our generalization of the coordinates used by Tolman. If we consider the case of a rotating body (with spin angular momentum s) at rest at the origin there will be off diagonal terms[2,4] in the metric tensor $g_{4j} = 2Gi(\mathbf{s} \times \mathbf{r})_j/(cr)^3$ which to order $1/r$ are zero. There will also be terms of order $1/r^3$ in the metric tensor due to the rotating body's quadrupole moment[4] which to order $1/r$ are zero. Thus, Eqs. (1.5)–(1.8) are unaltered for the case of a rotating body.

Our conclusion is given in Sec. 4, where we also comment on the connection of the Tolman-type relations with other energy relations found in the literature. In addition, we include an Appendix A where we show that $\int \mathfrak{t}_\mu{}^\nu \, dV = \int \xi_\mu{}^\nu \, dV$.

2. Field Equations

The field equations of general relativity can be written as

$$g^{1/2} G_\mu{}^\nu = -\frac{1}{2}\kappa^2 \mathfrak{T}_\mu{}^\nu, \tag{2.1}$$

where $\mathfrak{T}_\mu{}^\nu = g^{1/2} T_\mu{}^\nu$ and $\kappa^2 = 16\pi G/c^4$. From the results of Papapetrou,[2] it can be shown that $g^{1/2} G_\mu{}^\nu$ can be written as the identity

$$g^{1/2} G_\mu{}^\nu \equiv -\frac{1}{2}\left[\frac{\partial}{\partial x^\alpha}(\mathfrak{g}_\alpha^{\mu\nu} - \mathfrak{g}_\nu^{\mu\alpha} - \mathfrak{g}_\mu^{\nu\alpha} + \delta_\mu^\nu \mathfrak{g}_\beta^{\alpha\beta}) - \kappa^2 \mathfrak{t}_\mu{}^\nu\right], \tag{2.2}$$

where $\mathfrak{g}_\alpha^{\mu\nu} \equiv \partial \mathfrak{g}^{\mu\nu}/\partial x^\alpha$, $g_{\mu\nu,\alpha} \equiv \partial g_{\mu\nu}/\partial x^\alpha$ and

$$\mathfrak{t}_\mu{}^\nu \equiv \mathfrak{s}_\mu{}^\nu + \frac{1}{2}\frac{\partial}{\partial x^\lambda}(\mathfrak{f}_{\mu,\lambda\nu} + \mathfrak{f}_{\lambda,\mu\nu} + \mathfrak{f}_{\nu,\lambda\mu}), \tag{2.3}$$

$$\mathfrak{s}_\mu{}^\nu \equiv -\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\nu^{\alpha\beta}} \mathfrak{g}_\mu^{\alpha\beta} + \delta_\mu^\nu \mathfrak{L}, \tag{2.4}$$

$$\mathfrak{f}_{\lambda,\mu\nu} \equiv 2\left(\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\lambda^{\mu\alpha}} \mathfrak{g}^{\alpha\nu} - \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\lambda^{\nu\alpha}} \mathfrak{g}^{\alpha\mu}\right) = -\mathfrak{f}_{\lambda,\nu\mu}, \tag{2.5}$$

$$\mathfrak{L} \equiv \kappa^{-2} \mathfrak{g}^{\mu\nu}\left(\left\{{\alpha \atop \mu\beta}\right\}\left\{{\beta \atop \nu\alpha}\right\} - \left\{{\alpha \atop \mu\nu}\right\}\left\{{\beta \atop \alpha\beta}\right\}\right), \tag{2.6}$$

$$\left\{{\lambda \atop \mu\nu}\right\} \equiv \frac{1}{2} g^{\lambda\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}). \tag{2.7}$$

The Lagrangian density $\mathfrak{L}$ for the gravitational field is discussed in detail by Tolman[1] and Eddington.[6] The symmetrization process of Belinfante[7] was used by Papapetrou[2] in obtaining $\mathfrak{t}_\mu{}^\nu$.

Using Eq. (2.2) in (2.1), the field equations can be expressed as[2]

$$\frac{\partial}{\partial x^\alpha}(\mathfrak{g}_\alpha^{\mu\nu} - \mathfrak{g}_\nu^{\mu\alpha} - \mathfrak{g}_\mu^{\nu\alpha} + \delta_\mu^\nu \mathfrak{g}_\beta^{\alpha\beta}) = \kappa^2 \theta_{\mu\nu}, \tag{2.8}$$

where $\theta_{\mu\nu} \equiv \mathfrak{T}_\mu{}^\nu + t_\mu{}^\nu$ is the total (symmetrical) energy-momentum density. The left-hand side of Eq. (2.8) remains unchanged under the interchange of μ and ν and, thus $\theta_{\mu\nu} = \theta_{\nu\mu}$. However, $\mathfrak{T}_\mu{}^\nu$ and $t_\mu{}^\nu$ do not *individually* remain unchanged under interchange of μ and ν. The left-hand side of Eq. (2.8) becomes identically zero when differentiated with respect to x^ν and, thus, $\partial\theta_{\mu\nu}/\partial x^\nu = 0$. From this conservation law, it follows that energy and momentum are given, respectively, by[8]

$$\mathscr{E} = \int \theta_{00}\, dV, \qquad P_i = \int \frac{1}{c}\theta_{i0}\, dV. \tag{2.9}$$

If one examines Eqs. (2.2), (2.3), (2.5), and (2.8) carefully, the placement of some upper and lower indices *may seem* to be wrong. These equations are, however, correct. We could have, for example, written Eq. (2.8) in the form

$$\frac{\partial}{\partial x^\alpha}(\delta^{\alpha\beta}\mathfrak{g}_\beta^{\mu\nu} - \delta^{\nu\beta}\mathfrak{g}_\beta^{\mu\alpha} - \delta^{\mu\beta}\mathfrak{g}_\beta^{\nu\alpha} + \delta^{\mu\nu}\mathfrak{g}_\beta^{\alpha\beta}) = \kappa^2\delta^{\mu\beta}(\mathfrak{T}_\beta{}^\nu + t_\beta{}^\nu), \tag{2.10a}$$

or

$$\frac{\partial^2}{\partial x^\alpha \partial x^\beta}(\delta^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \delta^{\nu\beta}\mathfrak{g}^{\mu\alpha} - \delta^{\mu\beta}\mathfrak{g}^{\nu\alpha} + \delta^{\mu\nu}\mathfrak{g}^{\alpha\beta}) = \kappa^2\delta^{\mu\beta}(\mathfrak{T}_\beta{}^\nu + t_\beta{}^\nu). \tag{2.10b}$$

The placement of indices in Eqs. (2.10a) and (2.10b) now seems to be correct. However, Eqs. (2.8), (2.10a), and (2.10b) are exactly the same and are equally correct. Compare our Eq. (2.10b) with Papapetrou's[2] Eq. (3.10). Papapetrou uses a flat space metric $\gamma_{\mu\nu}$ with signature $(- - - +)$ which we are using a flat space metric $\delta_{\mu\nu}$ with signature $(+ + + +)$ and, thus, Eq. (2.8), and also Eqs. (2.2), (2.3), and (2.5), are correct in our notation, but would not be correct in Papapetrou's notation.

3. Derivation of Relations in r, *t* System

Equation (2.8), for the case when $g_{i4} = 0$ and the metric tensor does not depend on time, can be put in the form

$$\frac{\partial}{\partial x^l}[\delta_4^\mu\delta_4^\nu\mathfrak{g}_l^{44} + \delta_m^\mu\delta_k^\nu\mathfrak{g}_l^{mk} - (\delta_\nu^m\delta_k^\mu + \delta_\mu^m\delta_k^\nu)\mathfrak{g}_m^{kl} + \delta_\mu^\nu\mathfrak{g}_m^{lm}] = \kappa^2\theta_{\mu\nu}, \tag{3.1}$$

from which it follows that $\theta_{i4} = \theta_{4i} = 0$ and, thus, from Eq. (2.9) one obtains $P_i = 0$. For the same case, it is easy to show that $\mathfrak{T}_i{}^0 = \mathfrak{T}_0{}^i = 0$, $t_i{}^0 = t_0{}^i = 0$, and $\xi_i{}^0 = \xi_0{}^i = 0$ follow from Eqs. (2.1), (2.3), and (2.4), respectively.

However, for the rest of this paper, we shall be interested in the case where Eqs. (1.5)–(1.8) hold at large distances from the origin, and the metric tensor does not depend on time. This case allows for the possibility of a *rotating* body (that has a *time-*

independent metric tensor associated with it) being at rest at the origin. For this case, Eq. (2.8) can be put in the form

$$\frac{\partial}{\partial x^l}(\mathfrak{g}_l^{\mu\nu} - \mathfrak{g}_\nu^{\mu l} - \mathfrak{g}_\mu^{\nu l} + \delta_\mu^\nu g_\beta^{l\beta}) = \kappa^2 \theta_{\mu\nu}. \tag{3.2}$$

Integrating Eq. (3.2) over volume and using the divergence theorem gives us

$$\mathscr{E} = -\int \theta_{44}\, dV = -\kappa^{-2} \int_S (\mathfrak{g}_l^{44} + \mathfrak{g}_m^{lm})\, dS_l, \tag{3.3}$$

$$P_i = -\int \frac{i}{c}\theta_{i4}\, dV = -\kappa^{-2} \int_S \frac{i}{c}(0)_{il}\, dS_l = 0, \tag{3.4}$$

$$\int \theta_{ij}\, dV = \kappa^{-2} \int_S (\mathfrak{g}_l^{ij} - \mathfrak{g}_j^{il} - \mathfrak{g}_i^{jl} + \delta_i^j \mathfrak{g}_m^{lm})\, dS_l, \tag{3.5}$$

where S is a spherical surface approaching infinity. Because the surface S is at a large distance from the origin, we have set $g_{i4} = 0$ in the surface integrals in Eqs. (3.3)–(3.5). Thus, these equations give exactly the same final results as one would have obtained if one had used Eq. (3.1) instead of Eq. (3.2). Using Eq. (1.7), we find that

$$\mathfrak{g}_l^{44} + \mathfrak{g}_m^{lm} = -\frac{4Gm}{c^2 r^3} x^l, \tag{3.6}$$

$$\mathfrak{g}_l^{ij} - \mathfrak{g}_j^{il} - \mathfrak{g}_i^{jl} + \delta_i^j \mathfrak{g}_m^{lm} = \frac{2\alpha G m}{c^2 r^3}(\delta_{ij} x^l - 3x^i x^j x^l / r^2). \tag{3.7}$$

Using Eq. (3.6) in (3.3) and Eq. (3.7) in (3.5), we get

$$\mathscr{E} = -\int \theta_{44}\, dV = \frac{4Gm}{\kappa^2 c^2} \int_S \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = mc^2, \tag{3.8}$$

$$\int \theta_{ij}\, dV = \frac{2\alpha G m}{\kappa^2 c^2} \int_S \left(\delta_{ij} - \frac{3x^i x^j}{r^2}\right) \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = 0. \tag{3.9}$$

Next let us express $\mathscr{E}$ in the form[1]

$$\mathscr{E} = -\int (\mathfrak{T}_4{}^4 + \mathfrak{t}_4{}^4)\, dV, \tag{3.10}$$

which is consistent (see Appendix A) with Eq. (2.9). Using Eq. (2.4) and $R_\rho{}^\rho = -G_\rho{}^\rho = \frac{1}{2}\kappa^2 T_\rho{}^\rho$ along with[1,6]

$$\mathfrak{L} = -\kappa^{-2} g^{1/2} R_\rho{}^\rho + \frac{\partial}{\partial x^\alpha}\left(\mathfrak{g}^{\mu\nu} \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\alpha^{\mu\nu}}\right), \tag{3.11}$$

in Eq. (3.10) one obtains[1]

$$\mathscr{E} = -\int \left[\frac{1}{2}(\mathfrak{T}_4{}^4 - \mathfrak{T}_i{}^i) + \frac{\partial}{\partial x^l}\left(\mathfrak{g}^{\mu\nu}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_l^{\mu\nu}}\right) + \mathfrak{g}^{\mu\nu}\frac{\partial}{\partial x^4}\left(\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_4^{\mu\nu}}\right)\right] dV. \tag{3.12}$$

Using the divergence theorem and considering only the case where $g_{\mu\nu}$ does *not* depend on time, Eq. (3.12) can be put in the form[1]

$$\mathscr{E} = \frac{1}{2}\int (\mathfrak{T}_0{}^0 + \mathfrak{T}_i{}^i)\, dV - \int_S \mathfrak{g}^{\mu\nu}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_l^{\mu\nu}}\, dS_l. \tag{3.13}$$

We will evaluate the second term in Eq. (3.13) by using the (exact) result

$$\kappa^2 \mathfrak{g}^{\mu\nu}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\alpha^{\mu\nu}} = \mathfrak{g}_\rho^{\alpha\rho} + \frac{1}{2} g^{\alpha\rho} g_{\mu\nu} \mathfrak{g}_\rho^{\mu\nu}, \tag{3.14}$$

which is not given in either Ref. 1 or 6. Using Eqs. (1.5)–(1.8), we find, to order $1/r^2$, that

$$\mathfrak{g}_\rho^{l\rho} + \frac{1}{2} g^{l\rho} g_{\mu\nu} \mathfrak{g}_\rho^{\mu\nu} = \mathfrak{g}_m^{lm} + \frac{1}{2}(\mathfrak{g}_l^{44} + \mathfrak{g}_l^{mm}) = -\frac{2Gm}{c^2 r^3} x^l, \tag{3.15}$$

and, thus, using Eqs. (3.14), (3.15), and (3.8), we obtain

$$-\int_S \mathfrak{g}^{\mu\nu}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_l^{\mu\nu}}\, dS_l = \frac{2Gm}{\kappa^2 c^2}\int_S \frac{\mathbf{r}\cdot d\mathbf{S}}{r^3} = \frac{1}{2} mc^2 = \frac{1}{2}\mathscr{E}. \tag{3.16}$$

Equations (3.13) and (3.16) yield the final result

$$\mathscr{E} = \int (\mathfrak{T}_0{}^0 + \mathfrak{T}_i{}^i)\, dV = mc^2. \tag{3.17}$$

4. Conclusion

From the results of Sec. 3 (for a rotating or nonrotating body at rest at the origin and a time-independent metric tensor), it follows that

$$\mathscr{E} = \int (\mathfrak{T}_0{}^0 + \mathfrak{t}_0{}^0)\, dV = \int (\mathfrak{T}_0{}^0 + \mathfrak{T}_i{}^i)\, dV = mc^2, \tag{4.1}$$

$$P_i = \int \frac{1}{c}(\mathfrak{T}_i{}^0 + \mathfrak{t}_i{}^0)\, dV = 0, \tag{4.2}$$

$$\int (\mathfrak{T}_i{}^i + \mathfrak{t}_i{}^i)\, dV = 0, \tag{4.3}$$

hold in the very general **r**, t coordinate system defined by Eq. (1.4). The correctness of Eqs. (1.1)–(1.3) in the **r**, t coordinate system follows from Eqs. (4.1)–(4.3) and Appendix A. In other words, the Tolman relations and Eq. (1.3) retain the same form in this general coordinate system and, in particular, α does not appear in these equations. From Eqs. (4.1) and (4.3), we find that

$$\int (\mathfrak{t}_0{}^0 + \mathfrak{t}_i{}^i)\, dV = 0. \tag{4.4}$$

Equation (4.4) is a new Tolman-type relation and is similar to Eq. (3.17) in that the integral in Eq. (4.4) replaces the matter quantities in the integral of Eq. (3.17) by the same type of gravitational field quantities.

Finally, we wish to comment on the connection of the Tolman-type relations with other energy relations found in the literature. Much of the recent work on conservation laws has been devoted to finding coordinate independent expressions for the energy and momentum of the gravitational field[9] or to defining the notion of a coordinate-independent quasi-local energy through twistor theory.[10] Such approaches, while in many cases very sophisticated, are generally very complex and difficult to interpret physically. Our approach has been to tackle the problem from the opposite direction i.e., start with results (Tolman's and the generalizations thereof) which are true for a particular choice of coordinates and then obtain the corresponding results for the case of a very general coordinate system. Regarding the latter, since we only need results for the metric tensor at large distances from the origin, Eq. (1.4)—with only one arbitrary parameter—is indeed quite general. Thus, if, instead of Eq. (1.4), we had started with an equation with two or more arbitrary parameters, we would still have only one arbitrary parameter remaining in Eqs. (1.5) and (1.8), which hold at large distances from the origin. Thus, in essence, we have a very general coordinate system while keeping our results in a form which conveys the optimum physical enlightenment.

Appendix A

In this appendix, we will show that $\int \mathfrak{t}_\mu{}^\nu\, dV = \int \mathfrak{s}_\mu{}^\nu\, dV$ where Eqs. (1.5)–(1.8) hold at large distances from the origin and the metric tensor is independent of time. Equation (2.5) can be put in the form

$$\mathfrak{f}_{\lambda,\mu\nu} = 2\left[\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\lambda^{\mu\alpha}}(\mathfrak{g}^{\alpha\nu} - \delta_{\alpha\nu}) - \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\lambda^{\nu\alpha}}(\mathfrak{g}^{\alpha\mu} - \delta_{\alpha\mu})\right], \tag{A1}$$

because $\partial \mathfrak{L}/\partial \mathfrak{g}_\lambda^{\mu\nu} = \partial \mathfrak{L}/\partial \mathfrak{g}_\lambda^{\nu\mu}$. We also have[1,6]

$$\kappa^2 \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\lambda^{\rho\alpha}} = -\left\{ {\lambda \atop \rho\alpha} \right\} + \frac{1}{2}\delta_\rho^\lambda \left\{ {\beta \atop \alpha\beta} \right\} + \frac{1}{2}\delta_\alpha^\lambda \left\{ {\beta \atop \rho\beta} \right\}. \tag{A2}$$

From Eqs. (1.5)–(1.8), it is clear that $g^{\mu\nu} - \delta_{\mu\nu}$ is of order $1/r$, $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ is of order $1/r^2$, and, thus, $\mathfrak{f}_{\lambda,\mu\nu}$ is of order $1/r^3$. Using the divergence theorem and noting that the metric tensor is independent of time, we obtain from Eq. (2.3)

$$\int \mathfrak{t}_\mu{}^\nu dV = \int \mathfrak{s}_\mu{}^\nu dV + \int_S \frac{1}{2}(\mathfrak{f}_{\mu,l\nu} + \mathfrak{f}_{l,\mu\nu} + \mathfrak{f}_{\nu,l\mu})\, dS_l$$

$$= \int \mathfrak{s}_\mu{}^\nu dV, \tag{A3}$$

because $\mathfrak{f}_{\lambda,\mu\nu}$ is of order $1/r^3$ and the spherical surface S approaches infinity.

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