for b and $b^\dagger$ we find

$$\mu^{(\pm)}(t)=1+\int|\{[\alpha-\bar{\alpha}(t)]e^{i\omega_a t}\pm i[\beta^*-\bar{\beta}^*(t)]e^{-i\omega_b t}\}|^2 \times P(\alpha,\beta,t)d^2\alpha d^2\beta. \quad (8.13)$$

It is clear that if $P(\alpha,\beta,t)$ is to be non-negative then we must have $\mu^{(\pm)}(t)\geq 1$. By virtue of the values (8.10) for the time-dependent correlation functions we see then that $P(\alpha,\beta,t)$ can remain non-negative only within the time interval specified by

$$\frac{1}{\mu^{(+)}(0)}\leq e^{2\kappa t}\leq \mu^{(-)}(0). \quad (8.14)$$

We have not of course proved in general that no P representation exists outside the interval defined by the condition (8.14), though that is indeed the case for the examples considered in the last section. For example in the case of the initially chaotic mixture with equal mean quantum numbers $\langle n\rangle$ for each mode, we have $\mu^{(\pm)}(0)=1+2\langle n\rangle$, and the time interval defined by Eq. (8.14) is thus $-\tau\leq t\leq\tau$, with τ defined by Eq. (7.8); we have seen that for this example a P representation ceases to exist outside the stated interval.

It is worth emphasizing that the condition (8.14) is merely a necessary one for the existence of a non-negative P function. We should not be surprised, therefore, if a P representation fails to exist inside the indicated interval of time. This is indeed the case for an initially chaotic mixture when the two mean occupation numbers are unequal. In that case the bounds in Eq. (8.14) are specified by $\mu^{(\pm)}(0)=1+\langle n\rangle+\langle m\rangle$, but the interval within which the P representation exists is the one given by Eq. (7.24). It is in general a smaller interval.

PHYSICAL REVIEW VOLUME 160, NUMBER 5 25 AUGUST 1967

Radiation of Gravitational Waves in Brans-Dicke General-Relativity Theory

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(Received 28 March 1967; revised manuscript received 17 May 1967)

The rate of radiation of gravitational waves, $P_{\rm BD}\equiv P_T+P_S$, from a system of binary stars is calculated using the general-relativity theory of Brans and Dicke. It is shown that, for a circular orbit, the radiation arises purely from the tensor-field contribution P_T, and $P_{\rm BD}$ is smaller than the corresponding Einstein result $P_{\rm E}$, by a factor $(2\omega+3)/(2\omega+4)$, where ω is the usual Brans-Dicke dimensionless constant ($\omega\simeq 6$). In the case of an elliptic orbit, there is also a contribution P_S from the scalar field. However, we show that the contribution of P_S to $P_{\rm BD}$ is always negligible compared to the contribution P_T. Thus we are able to conclude that, for all values of the eccentricity e, the Brans-Dicke theory predicts that the rate of gravitational radiation from a system of binary stars is always smaller, by a factor of the order of $\frac{15}{16}$, than the rate predicted by Einstein's theory.

A SCALAR as well as a tensor field has been included in Brans and Dicke's[1] modification of Einstein's general theory of relativity. The effect of the scalar field is to make the gravitational constant G dependent on the mass distribution in the universe. Subsequently, Dicke[2] showed that the BD theory could be put into a form in which the rest masses of all particles vary with position, being functions of ϕ, the scalar field. For our purposes the latter representation of the theory will be the more useful; the reasons for this choice are discussed in the paper of Morganstern and Chiu[3] on scalar gravitational fields in pulsating stars. The BD theory has gained some support recently as a result of the work of Dicke and Goldenberg[4] on the contribution of solar oblateness to the precession of the perihelion of Mercury. However, it is still not possible to decide between the Einstein and BD theories on the basis of present experimental data. It is the purpose of

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[1] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961) (hereafter referred to as BD).
[2] R. H. Dicke, Phys. Rev. **125**, 2163 (1962).
[3] R. E. Morganstern and Hong-Yee Chiu, Phys. Rev. **157**, 1228 (1967).
[4] R. H. Dicke and H. M. Goldenberg, Phys. Rev. Letters **18**, 313 (1967).

this paper to examine the radiation of gravitational waves[5] in the BD theory in the hope of providing another arena of confrontation between the rival theories. Specifically, we wish to consider the radiation from a system of binary stars.

In the transformed BD theory,[2] the field equations are

$$\bar{R}_{ij}-\tfrac{1}{2}\bar{g}_{ij}\bar{R}=(8\pi/c^4)\bar{\phi}^{-1}\bar{T}_{ij}+\tfrac{1}{2}(2\omega+3)\bar{\Omega}_{ij}, \quad (1)$$

$$\bar{\Box}_{\bar{g}}(\ln\lambda)=(-\bar{g})^{-1/2}[(-\bar{g})^{-1/2}\bar{g}^{ij}(\ln\lambda)_{,i}]_{,j}$$

$$=\frac{8\pi}{(2\omega+3)}\frac{\bar{\phi}^{-1}}{c^4}\bar{T}, \quad (2)$$

$$\bar{\Omega}_{ij}=\lambda^{-2}[\lambda_{,i}\lambda_{,j}-\tfrac{1}{2}\bar{g}_{ij}\bar{g}^{lm}\lambda_{,l}\lambda_{,m}], \quad (3)$$

where the scale factor $\lambda(x^i)$ is a dynamical variable in the transformed (barred-quantities) theory,[2] analogous to the scalar field ϕ in the original theory,[1] and ω is the usual dimensionless constant for the λ field ($\omega\simeq 6$). In addition, $ds^2=g_{ij}dx^idx^j$, $d\bar{s}^2=\bar{g}_{ij}dx^idx^j (i, j=1, \cdots, 4)$ (note that the generalized coordinates dx^i are held constant under the transformation), and $\phi=\lambda\bar{\phi}$, where $\bar{\phi}$ is a constant. Using the conservation law[2]

$$\left[(-\bar{g})\left(\bar{T}_i{}^j+\frac{2\omega+3}{2}\frac{c^4\bar{\phi}}{8\pi}\bar{\Omega}_i{}^j+\bar{t}_i{}^j\right)\right]_{,j}=0, \quad (4)$$

(which defines[6] the pseudotensor $\bar{t}_i{}^j$), the gravitational radiation rate in BD theory is found to be[3] the sum of contributions from the tensor and scalar fields, viz.,

$$P_{\mathrm{BD}}=\frac{\partial}{\partial x^0}\int(-\bar{g})\bar{T}_0{}^0d^3x$$

$$=P_T+P_S, \quad (5)$$

where

$$P_T=c\int_{\Sigma(R)}(-g)\bar{t}_0{}^rdS \quad (6)$$

and

$$P_S=\frac{2\omega+3}{16\pi}c^5\bar{\phi}\int_{\Sigma(R)}(-\bar{g})\bar{\Omega}_0{}^rdS. \quad (7)$$

The integration is over the bounding surface S of a sphere $\Sigma(R)$ of radius R ($R\rightarrow\infty$) centered at the center of gravity of the binary stars.

The energy-momentum pseudotensor $\bar{t}_j{}^i$ is defined by the same functional of the metric as in the Einstein theory (with a different gravitational constant), in which the metric is no longer a solution of Einstein's equation but, instead, a solution of Dicke's barred equations.

We can make the weak-field solutions in the barred system similar to the Einstein equations by making a choice of $\bar{\phi}$. We take[7]

$$\bar{\phi}=\lim_{R\rightarrow\infty}\phi(R,t)=G^{-1}\left(\frac{2\omega+4}{2\omega+3}\right). \quad (8)$$

With this choice of $\bar{\phi}$ we have $\lim_{R\rightarrow\infty}\lambda=1$. Let h_{ij} and ξ be the first-order corrections to g_{ij} and ϕ, as given by BD. So, we have $g_{ij}=\eta_{ij}+h_{ij}$ and $\lambda=1+\xi\bar{\phi}^{-1}$. Then, using $\bar{g}_{ij}=\lambda g_{ij}$, we can write $\bar{g}_{ij}=\eta_{ij}+\bar{h}_{ij}$ and find $\bar{h}_{ij}$. It is then found that the solutions in the barred system are exactly those of Einstein (with a different gravitational constant), in the weak-field approximation. Finally, we obtain

$$P_T=\left(\frac{2\omega+3}{2\omega+4}\right)P_{\mathrm{E}}. \quad (9)$$

Here P_{E} is the gravitational rate in the pure Einstein theory and is given by[8]

$$P_{\mathrm{E}}=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left\{1+\frac{73}{24}e^2+\frac{37}{96}e^4\right\}, \quad (10)$$

where a is the semimajor axis of the ellipse and e is the eccentricity.

The contribution of the scalar field P_S can be rewritten in the weak-field case as

$$P_S=\lim_{R\rightarrow\infty}\frac{2\omega+3}{4}c^5\bar{\phi}R^2\lambda_{,0}\lambda^{,r}. \quad (11)$$

To solve Eq. (2) for λ, we transform back into the old (unbarred) units and get[1]

$$\Box\lambda=\frac{8\pi\bar{\phi}^{-1}}{2\omega+3}\frac{T}{c^4}. \quad (12)$$

Thus

$$\lambda=1-\frac{2\bar{\phi}^{-1}}{2\omega+3}\frac{1}{c^4}\int T_{(t-r'/c)}\frac{d^3x'}{r'}, \quad (13)$$

where T is the usual energy-momentum-contracted tensor, evaluated at the retarded time. For the purpose of applying this formula to the binary-star system, it is convenient to treat the latter as a one-body problem with reduced mass μ equal to $m_1m_2/(m_1+m_2)$. Then r' refers to the distance from μ to the observer, the latter being a distance R from the center of the ellipse traced

[5] For a critical review of the gravitational radiation problem see, for example, F. A. E. Pirani, in *Gravitation: an Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 6.

[6] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959).

[7] By "infinity" we mean distance large compared to the orbit of the binary stars.

[8] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).

out by μ. Thus, as $R\rightarrow\infty$, we can take $r'=R$ and

$$\lambda=1-\frac{2\bar{\phi}^{-1}}{(2\omega+3)}\frac{1}{c^4R}\int T_{(t-r'/c)}\,d^3x'. \tag{14}$$

Since in the weak-field limit the Lorentz transformation law is operative, we can write

$$\int T_{(t-r'/c)}\,d^3x'=-\mu c^2\left(1-\frac{v^2}{2c^2}\right), \tag{15}$$

where v is the velocity of the reduced mass. Thus

$$\lambda=\frac{\bar{\phi}^{-1}}{(2\omega+3)}\frac{\mu}{c^4R}\frac{v^2}{2c^2}+\text{constant term}. \tag{16}$$

Now

$$\frac{d}{dt}(v^2)=\frac{d^3}{dt^3}(l^2). \tag{17}$$

Thus, substituting for λ in Eq. (11) and noting that $\delta/\partial r=-\partial/\partial x^0=-(1/c)\partial/\partial t$, we obtain

$$P_S=\frac{\bar{\phi}^{-1}}{4c^5}\frac{\mu^2}{(2\omega+3)}\left[\frac{d^3(l^2)}{dt^3}\right]^2. \tag{18}$$

Now[8]

$$\left[\frac{d^3(\mu l^2)}{dt^3}\right]^2=\frac{4G^3m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^5}e^2(1+e\cos\psi)^4\sin^2\psi, \tag{19}$$

where ψ is the true anomaly of the ellipse. Averaging over one period of the elliptic motion leads to

$$P_S=\frac{G^4}{2c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\frac{1}{(2\omega+4)}e^2(1+\tfrac{1}{4}e^2). \tag{20}$$

We see immediately that, for a circular orbit, P_S is zero; thus the BD theory predicts that only the tensor field contributes to the gravitational theory, just as in the Einstein case [with the important difference, however, that the rate is decreased by a factor $(2\omega+3)/(2\omega+4)$]. The null contribution from the scalar field can be understood physically from the fact that, for a circular orbit, the scalar field at the positions of m_1 and m_2 is unchanged. This will not be true, of course, for a noncircular orbit.

We see that P_S is always less than P_T, particularly for the values of e close to zero. However, as $e\rightarrow 1$ (very eccentric orbits), then, because of the factor $(1-e^2)^{-7/2}$, both P_T and P_S increase sharply. Using Eqs. (5), (9), (10), and (20), we can write

$$P_{\rm BD}=P_T[1+g(e)], \tag{21}$$

where

$$g(e)=\left(\frac{0.00521e^2+0.00130e^4}{1+3.042e^2+0.385e^4}\right). \tag{22}$$

In the evaluation of $g(e)$, we selected $\omega=6$; this is the minimum allowed value[1] of ω, and thus the value of $g(e)$ given by Eq. (22) is the maximum allowed value. Now, it is clear that $g(e)<6\times10^{-3}$ for all values of e. This means that *the contribution of P_S to $P_{\rm BD}$ is always negligible compared to the contribution of P_T*. Thus, to a very good approximation, we can write

$$P_{\rm BD}=P_T=\frac{2\omega+3}{2\omega+4}P_{\rm E}\simeq\frac{15}{16}P_{\rm E}, \tag{23}$$

where $P_{\rm E}$ is given by Eq. (10). In other words, *for all values of the eccentricity e*, the Brans-Dicke theory predicts that the rate of gravitational radiation from a system of binary stars is always smaller, by a factor of the order of $\frac{15}{16}$, than the rate predicted by Einstein's theory.

It is convenient to write

$$P_{\rm E}=P_\odot f(e), \tag{24}$$

where

$$P_\odot=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5} \tag{25}$$

is the average power radiated from a circular orbit of radius a, and the enhancement factor $f(e)$ is given by

$$f(e)=(1+3.042e^2+0.3853e^4)(1-e^2)^{-7/2}. \tag{26}$$

A plot of $f(e)$ against e is given by Peters and Mathews.[8] From Eqs. (25) and (26) it is clear that the most promising sources of this gravitational radiation are binary stars of large mass, short period, and very eccentric orbits. For a discussion of these sources and for an over-all review of the prospect of the experimental detection of gravitational waves we refer the reader to the recent review paper of Braginskii.[9] The latter paper also contains estimates of $P_\odot$. We would also like to point out that an indirect method of proving the existence of gravitational waves is by observing the predicted increase in the period of revolution of binary stars.[6,9]

ACKNOWLEDGMENTS

This research was accomplished while one of the authors (RFOC) held a National Research Council Senior Research Associateship supported by the National Aeronautics and Space Administration, and both authors would like to thank Dr. Robert Jastrow for his hospitality at the Institute for Space Studies.

[9] V. B. Braginskii, Usp. Fiz. Nauk **86**, 433 (1965) [English transl.: Soviet Phys.—Usp. **8**, 513 (1966)].