

## A UNIFIED APPROACH TO QUANTUM DYNAMICAL MAPS AND GAUSSIAN WIGNER DISTRIBUTIONS

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The KLM conditions are conditions that are necessary and sufficient for a phase-space function to be a Wigner distribution function (WDF). We apply them here to discuss three questions that have arisen recently: (1) For which WDFs  $P_0$  will the map  $P \rightarrow P_0 * P$  be a quantum dynamical map – i.e. a map that takes WDFs to WDFs? (2) What are necessary and sufficient conditions for a phase-space gaussian to be a WDF? (3) Are there non-gaussian, non-negative WDFs?

Our purpose in writing this paper is to extend, simplify, and unify the results of three recent papers [1–3] that deal with Wigner distribution functions. The results that we get will come as an application of what in a previous paper [4] we termed the KLM [5] conditions; these conditions are necessary and sufficient for a phase-space function to be a WDF. Some related results are discussed in another work by one of us [6].

Before we describe our results, we need to summarize the results found in the three papers mentioned above. The first paper [1] concerns the question of when the convolution of two WDFs yields a WDF. In the language used there, the question is this: For what WDFs  $P_0$  will the linear transformation  $P \rightarrow P_0 * P$  ( $*$   $\equiv$  convolution) be a quantum dynamical map? (A *quantum dynamical map* is a linear transformation that takes quantum mechanical states into quantum mechanical states. In the phase-space formulation, the states are the WDFs; hence, such a map will be a linear transformation that takes phase-space functions to phase-space functions, and in particular takes WDFs to WDFs. For a complete discussion, see ref. [1] and the references cited there.)

Here are the main results of ref. [1]: (1) The map  $P \rightarrow P_0 * P$  will be a quantum dynamical map whenever  $P_0$  is a (pointwise) non-negative WDF. (2) There is a WDF  $P_0$ , which does take on negative values, for which  $P \rightarrow P_0 * P$  is *not* a quantum dynamical map. We mention that the authors also prove that whenever  $P$  and  $P_0$  are WDFs – whether or not either of them is non-negative – that the convolution  $P * P_0$  will always be a non-negative function, although it will not in general be a WDF. This result was, however, established earlier by O'Connell and Rajagopal [7], and was used, in a way that is related to viewing  $P \rightarrow P_0 * P$  as a dynamical map, by O'Connell and Walls [8]. It was also proved by Bertrand et al. [9].

The results of the second paper are easier to describe, although technically much harder to write down. What Simon et al. do in ref. [2] is give necessary and sufficient conditions for a gaussian phase-space function to be a WDF. Since gaussians come up in a variety of places, this result is important. For instance, in his review article on the evolution of wave packets, Littlejohn [10] essentially asked the question answered by Simon et al. [2]; he had in

mind applications to propagating gaussian wave packets.

In the third paper [3], Gracia-Bondía and Várilly, using techniques involving symplectic Fourier transforms [4], independently arrive at the characterization of gaussian WDFs given by Simon et al. [2]. They also obtain a result implicit in the work of Jagannathan et al. [1]: *The convolution of a WDF with a suitably normalized positive function is still a WDF.* Using this result they give a method for generating non-negative, non-gaussian WDFs. Such WDFs cannot represent pure states, for the Hudson-Soto-Claverie theorem [11,12] guarantees that every non-negative WDF representing a pure state is a gaussian. Being able to generate non-gaussian, non-negative WDFs is important because it demonstrates the existence of a large class of phase-space functions which are both quantum states (WDFs) and classical states (non-negative functions with unit integral over phase-space).

We shall adopt the notation that we used in ref. [4]:

- (i)  $z = (\mathbf{q}, \mathbf{p})$  is a point in phase-space.
- (ii)  $\sigma(z_1, z_2) = \mathbf{q}_2 \cdot \mathbf{p}_1 - \mathbf{q}_1 \cdot \mathbf{p}_2$  is a symplectic form; that is,  $\sigma$  is bilinear and antisymmetric.
- (iii)  $\tilde{g}(a) = \int g(z) e^{i\sigma(a,z)} dz$ , where  $dz = d^n q d^n p$ , is the symplectic Fourier transform of  $g$ .

Comments: First, in ref. [1]  $z$  denotes a slightly different quantity than  $(\mathbf{q}, \mathbf{p})$ . Second, on those occasions where we have to use matrix multiplication, we will regard  $z$  as a  $2n$  dimensional column vector, and  $z^T$  ( $=z$  transpose) as the corresponding row vector. We will apply a similar convention for the variable  $a$  that is dual to  $z$ . Third, following Simon et al. [2], we let

$$\beta = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}, \tag{1}$$

where  $0_{n \times n}$  and  $I_{n \times n}$  are the  $n \times n$  zero and identity matrices, respectively. Using  $\beta$ , one has

$$\sigma(z_1, z_2) = z_2^T \beta z_1. \tag{2}$$

We say that a function  $F(a)$ , which is defined and continuous on the dual of phase-space, is of  $\eta$ -positive type [4-6] if, for every finite set of points  $\{a_1, \dots, a_m\}$ , the matrix

$$M_{jk} = F(a_j - a_k) \exp\left[\frac{1}{2}i\eta\sigma(a_k, a_j)\right] \tag{3}$$

is non-negative (in the quadratic form sense).

The values  $\eta = \hbar$  and  $\eta = 0$  have important physical interpretations. A phase-space function  $P(z)$  is a WDF if and only if its symplectic Fourier transform  $\tilde{P}(a)$  satisfies the KLM conditions [4,5]: (i)  $\tilde{P}(a)$  is continuous and of  $\hbar$ -positive type; (ii)  $\tilde{P}(0) = 1$ . If  $\tilde{P}(a)$  is of 0-positive-type and if  $\tilde{P}(0) = 1$ , then  $\tilde{P}(a)$  is the symplectic Fourier transform of a classical state - a non-negative probability measure defined on phase-space [4]. If, in addition, such a  $\tilde{P}(a)$  turns out to be the symplectic Fourier transform of a function  $P(z)$ , then  $P(z)$  will be pointwise non-negative. (We should point out that functions of 0-positive type are actually functions classified by Bochner as being of positive-type: see refs. [4] and [6] for details.)

A continuous  $F(a)$  may be of  $\eta$ -positive type for several values of  $\eta$ . For example, the symplectic Fourier transform  $\tilde{P}(a)$  of a non-negative WDF  $P$  is of  $\eta$ -positive type for  $\eta = 0$  and  $\eta = \hbar$ , because  $P$  is both a classical and a quantal state. Let us collect the  $\eta$ 's for which  $F$  is of  $\eta$ -positive type into a set,

$$W(F) \equiv \{\eta: F \text{ is of } \eta\text{-positive type}\}, \tag{4}$$

which we shall call the *Wigner spectrum* [6] of  $F$ .

The Wigner spectrum of  $F$  has a number of interesting properties [6]:

- (i)  $F$  is the symplectic Fourier transform of a WDF if and only if  $F(0) = 1$  and  $\hbar \in W(F)$ . (KLM conditions.)
- (ii)  $F$  is the symplectic Fourier transform of a classical state (a probability measure or a non-negative function with integral over phase-space being 1) if and only if  $F(0) = 1$  and  $0 \in W(F)$ . (Bochner conditions.)
- (iii)  $W(F)$  is invariant under linear canonical transformations of  $a$ .
- (iv) If  $\eta \in W(F)$ , then  $-\eta \in W(F)$ .
- (v)  $W(FG) \supseteq W(F) + W(G)$ . (By this we mean that  $W(FG)$  contains all sums  $\eta_1 + \eta_2$  with  $\eta_1 \in W(F)$  and  $\eta_2 \in W(G)$ .)
- (vi) If  $\lambda \neq 0$  is a real number, then  $W(F(\lambda a)) = \lambda^2 W(F(a))$ .

All of these properties, and a few more, are discussed and established in ref. [6].

The reason for introducing them here is that they provide a convenient way to get the results of ref. [1], and to extend them. The map  $P \rightarrow P_0 * P$  goes over

to the map  $\tilde{P} \rightarrow \tilde{P}_0(a)\tilde{P}(a)$  when symplectic Fourier transforms are taken. Assuming that  $P_0$  and  $P$  are WDFs, we see by (i) – the KLM conditions – and by (iv) that  $\pm\hbar$  is in both  $W(\tilde{P}_0)$  and  $W(\tilde{P})$ . By property (v), we have that  $0 = \hbar - \hbar \in W(\tilde{P}_0\tilde{P})$ . Since  $\tilde{P}_0\tilde{P} = \tilde{P}_0 * P$ , we see by (ii) that  $P_0 * P$  must be non-negative. If  $P_0$  is a non-negative WDF, then, by (i), (ii), and (iv), both 0 and  $\pm\hbar$  are in  $W(\tilde{P}_0)$ . Hence, by (v), both  $0 + \hbar = \hbar$  and  $-\hbar + \hbar = 0$  are in  $W(\tilde{P}_0\tilde{P}) = W(\tilde{P}_0 * P)$ , so  $P_0 * P$  is a *non-negative* WDF, by (i) and (ii).

At this point, we have recovered one of the results from ref. [1]; viz.,  $P \rightarrow P_0 * P$  is a quantum dynamical map if  $P_0$  is a non-negative WDF. On the other hand, if  $0 \notin W(\tilde{P}_0)$  but  $\pm\hbar$  and  $\pm 2\hbar$  do, then, again by (v),  $2\hbar - \hbar \in W(\tilde{P}_0\tilde{P})$ , so  $P_0 * P$  is a WDF and  $P \rightarrow P_0 * P$  is thus a quantum dynamical map in this case too.

Evidently the function

$$P_H(z) = \frac{1}{\pi\hbar} (2|z|^2/\hbar - 1) e^{-|z|^2/\hbar}, \quad (5)$$

which is the WDF that represents the first excited state of the one-dimensional harmonic oscillator with unit mass and unit frequency, is such that  $\pm 2\hbar \notin W(\tilde{P}_H)$ , for Jagannathan et al. [1] show that  $P \rightarrow P_H * P$  is *not* a quantum dynamical map. Using a combination of (i) through (vi) one can produce [6] a function,

$$P_0(z) \equiv \frac{1}{3\pi\hbar} (3|z|^2/2\hbar - 1) e^{-|z|^2/2\hbar}, \quad (6)$$

for which  $0 \notin W(\tilde{P}_0)$  but  $\pm 2\hbar \in W(\tilde{P}_0)$ . This function, though it takes on negative values, is such that  $P \rightarrow P_0 * P$  is a quantum dynamical map.

The existence of (6) shows that an obvious conjecture that one might like to make – that  $P \rightarrow P_0 * P$  is a quantum dynamical map if and only if  $P_0$  is a non-negative WDF – is false. In its place, we conjecture that  $P \rightarrow P_0 * P$  is a quantum dynamical map if and only if either 0 or  $\pm 2\hbar$  are in  $W(\tilde{P}_0)$ . (Of course, since we assume that  $P_0$  is a WDF,  $\pm\hbar \in W(\tilde{P}_0)$  automatically.)

Let us now turn our attention to the question of which phase-space Gaussians are WDFs, a question asked by Littlejohn [10] and addressed and answered in refs. [2] and [3]. A properly normalized phase-space Gaussian that is, for the sake of convenience, centered at the origin has the form

$$P_A(z) = \pi^{-n} \sqrt{\det A} \exp(-z^T A z), \quad (7)$$

where  $A$  is a real, symmetric, positive definite  $2n \times 2n$  matrix. (The normalization for  $A$  and for WDFs in general differs from that used by Littlejohn [10].) Using standard manipulations with Gaussians, one has that

$$\tilde{P}_A(a) = \exp(-\frac{1}{4} a^T B a), \quad B = \beta^T A^{-1} \beta; \quad (8)$$

$B$  is also positive definite, real, and symmetric, for  $\beta$  is a real, orthogonal matrix.

Because  $\tilde{P}_A$  is the symplectic Fourier transform of a non-negative function (classical state), one has that  $0 \in W(\tilde{P}_A)$ . Are there any other points in its Wigner spectrum? To answer this, we note that with a little algebra, one can show that if  $\tilde{P}_A$  is put in (3) the non-negativity of the resulting  $M_{jk}$  is equivalent to the non-negativity of the matrix with entries

$$N_{jk} = \exp[\frac{1}{2} a_j^T (B + i\eta\beta) a_k]. \quad (9)$$

It turns out [6] that a necessary and sufficient condition for the matrix with entries  $N_{jk}$  to be non-negative is that  $B + i\eta\beta$  be non-negative as a *complex*,  $2n \times 2n$  matrix.

Since  $\beta$  is a real, orthogonal  $2n \times 2n$  matrix, it is also a unitary matrix. The condition that  $B + i\eta\beta = \beta^T A^{-1} \beta + i\eta\beta$  be non-negative is therefore equivalent to

$$\beta(B + i\eta\beta)\beta^T = A^{-1} + i\eta\beta \quad (10)$$

being non-negative. Thus what we have found is that  $\eta \in W(\tilde{P}_A)$  if and only if  $A^{-1} + i\eta\beta$  is non-negative as a complex,  $2n \times 2n$  matrix. In particular, this means that  $P_A$  is a WDF if and only if  $A^{-1} + i\hbar\beta$  is non-negative – a condition that is substantially simpler to work with than the conditions given in refs. [2] and [3].

In addition to ref. [2], Simon et al. have written another paper [13] that deals not only with Gaussian WDFs, but also with their optical counterparts, Gaussian Wolf functions. With minor modifications, our results should also apply to them.

Using the properties of the Wigner spectrum, we can recover and strengthen the results obtained by Gracia-Bondía and Várilly [3]. To begin, if  $P$  is a WDF and  $\mu$  is a probability measure on phase-space (that is,  $\mu$  is an *arbitrary* classical state, not just one corresponding to a phase-space function), then by

properties (i) and (ii) we have that  $\hbar \in W(\tilde{P})$  and  $0 \in W(\tilde{\mu})$ . Applying (v), we see that  $0 + \hbar = \hbar \in W(\tilde{\mu}\tilde{P})$ , and so by (i) and the convolution theorem we find that  $\mu * P$  is a WDF. We have thus proved that *the convolution of a WDF with a probability measure is still a WDF*. In other words, *the convolution of a quantum state with a classical state is still a quantum state*.

As we mentioned earlier, Gracia-Bondía and Várilly give an essentially linear method for generating non-negative, non-gaussian WDFs. Here we will provide an alternative, *non-linear* way of doing the same thing. Let  $P_0$  be any non-negative WDF. From (i) and (ii), we have that  $\{0, \hbar\} \subseteq W(\tilde{P}_0)$ . Moreover by (v),  $\{0, \hbar\} \subseteq W(\tilde{P}_0^k)$  for  $k=1, 2, \dots$ . If  $\alpha_1, \alpha_2, \dots$  is a sequence of non-negative numbers for which  $\sum \alpha_k = 1$ , then it is a straightforward consequence of the definition of  $\eta$ -positive type that

$$\tilde{P}(a) \equiv \sum_{k=1}^{\infty} \alpha_k [\tilde{P}_0(a)]^k \quad (11)$$

has both 0 and  $\hbar$  in its Wigner spectrum. Hence,  $P(z)$  – the inverse symplectic Fourier transform of  $\tilde{P}$  – will be a non-negative WDF. For example, if we set

$$\tilde{P}_0(a) = \exp(-\frac{1}{4}\hbar|a|^2), \quad \alpha_k = \frac{1}{e(k-1)!}, \quad (12)$$

then from (11) we find that

$$\tilde{P}(a) = \exp[\exp(-\frac{1}{4}\hbar|a|^2) - \frac{1}{4}\hbar|a|^2 - 1] \quad (13)$$

is the symplectic Fourier transform of a non-negative WDF that is manifestly non-gaussian.

We close by pointing out that Smith [14] has re-

cently used conditions equivalent [6,14] to the KLM conditions to obtain results concerning the class of distribution functions introduced by Cohen [15]. We will discuss this topic in a later work.

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