On the completion of the post-Newtonian gravitational two-body problem with spin

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Previous work by the authors [B. M. Barker and R. F. O'Connell, Phys. Rev. D **12**, 329 (1975); **14**, 861 (1976); B. M. Barker, G. G. Byrd, and R. F. O'Connell, Astrophys. J. **305**, 623 (1986); B. M. Barker and R. F. O'Connell, Gen. Relativ. Gravit. **18**, 1055 (1986)] on the post-Newtonian (order c^{-2}) gravitational two-body problem with spin and parametrized post-Newtonian parameters γ and β was concerned with the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Here this work is completed by finding the individual positions $\mathbf{r}_1$ and $\mathbf{r}_2$, which is necessary for the interpretation of certain binary-system observations. First the center of inertia $\mathbf{r}_{CI}$ is found. This makes it possible to obtain the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ and the center of mass $\mathbf{r}_{CM}$ as a function of the relative position $\mathbf{r}$, relative velocity $\mathbf{v}$, and spin angular momenta $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ of the two bodies. Thus, if a solution $\mathbf{r} = \mathbf{r}(t)$ can be obtained, then solutions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$ can also be obtained. The final results are given in a very general coordinate system specified by four arbitrary dimensionless parameters. In particular, the spin-orbit potential energy terms V_{S1} and V_{S2} are given *without* going to a frame of reference where the total momentum is zero.

I. INTRODUCTION

In our previous work[1–4] involving equations of motion arising from the post-Newtonian gravitational two-body problem with spin, we were interested only in the relative position $\mathbf{r}$. However, for some binary systems—such as the binary pulsar[5,6] PSR 1913 + 16—it is necessary to have equations of motion for the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ in order to connect theory and observation.[7–9] In Sec. II we define three coordinate systems. Our most general coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ are related to the Einstein–Infeld–Hoffman (EIH) coordinates $\mathbf{r}_{E1}$ and $\mathbf{r}_{E2}$ by four arbitrary dimensionless parameters. In Sec. III we give the spin-orbit potential energy terms V_{S1} and V_{S2} in a frame of reference where the total momentum is *not* zero and include parametrized post-Newtonian (PPN) parameters γ and β. In the Appendix, we give a more elaborate treatment of V_{S1} and V_{S2} for general relativity. In Sec. IV, we find the center of inertia $\mathbf{r}_{CI}$ for our most general coordinate system and display the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ and the center of mass $\mathbf{r}_{CM}$ as a function of the relative position $\mathbf{r}$, relative velocity $\mathbf{v}$, and spin angular momenta $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ of the two bodies. In Sec. V we present our conclusions.

II. COORDINATE SYSTEMS

In this paper, we use coordinates $\mathbf{r}_{EN}$, $\mathbf{r}_{*N}$, and $\mathbf{r}_N$, where N for body N always equals 1 or 2. The relative coordinates $\mathbf{r}_E$, $\mathbf{r}_*$, and $\mathbf{r}$ are given by

$$\mathbf{r}_E = \mathbf{r}_{E1} - \mathbf{r}_{E2}, \quad \mathbf{r}_* = \mathbf{r}_{*1} - \mathbf{r}_{*2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{2.1}$$

The $\mathbf{r}_{*N}$ coordinates are related to the EIH coordinates[1,2] $\mathbf{r}_{EN}$ by the coordinate transformation[10]

$$\mathbf{r}_{EN} = \mathbf{r}_{*N} + (-1)^N \alpha G\left[(1-a_0)m_N + a_0\frac{m_1 m_2}{m_N}\right]\frac{\mathbf{r}_*}{c^2 r_*}, \tag{2.2}$$

where α and a_0 are arbitrary dimensionless parameters, m_N is the mass of body N, G is Newton's constant of gravitation, and c is the speed of light. From Eqs. (2.1) and (2.2), we obtain[1,2,10]

$$\mathbf{r}_E = \mathbf{r}_*(1 - \alpha GM/c^2 r_*), \tag{2.3}$$

where $M \equiv m_1 + m_2$.

The $\mathbf{r}_N$ coordinates are related to the $\mathbf{r}_{*N}$ coordinates by the coordinate transformation[2]

$$\mathbf{r}_{*N} = \mathbf{r}_N - \lambda_N \mathbf{v}_N \times \mathbf{S}^{(N)}/m_N c^2, \tag{2.4}$$

where λ_1 and λ_2 are arbitrary dimensionless parameters, $\mathbf{v}_N$ is the velocity of body N, and $\mathbf{S}^{(N)}$ is the spin angular momentum of body N. From Eqs. (2.1) and (2.4), we obtain[2]

$$\mathbf{r}_* = \mathbf{r} + \sum_{N=1}^{2}(-1)^N \lambda_N \frac{\mathbf{v}_N \times \mathbf{S}^{(N)}}{m_N c^2}. \tag{2.5}$$

If we are in a frame of reference where the total momentum is equal to zero (i.e., center-of-mass system) then to first order $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$. We then obtain $\mathbf{v}_N = -(-1)^N \mu\mathbf{v}/m_N$, where $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ and $\mu \equiv m_1 m_2/M$. Using the above in Eqs. (2.4) and (2.5) we obtain,[2] respectively (correct to the post-Newtonian approximation),

$$\mathbf{r}_{*N} = \mathbf{r}_N + (-1)^N \lambda_N \frac{\mu\mathbf{v}\times\mathbf{S}^{(N)}}{m_N^2 c^2}, \tag{2.6}$$

$$\mathbf{r}_* = \mathbf{r} - \sum_{N=1}^{2}\lambda_N \frac{\mu\mathbf{v}\times\mathbf{S}^{(N)}}{m_N^2 c^2}. \tag{2.7}$$

To the same approximation, we also obtain

$$\mathbf{r}_{EN} = \mathbf{r}_N + (-1)^N \alpha G\left[(1-a_0)m_N + a_0\frac{m_1 m_2}{m_N}\right]\frac{\mathbf{r}}{c^2 r} + (-1)^N \lambda_N \frac{\mu\mathbf{v}\times\mathbf{S}^{(N)}}{m_N^2 c^2}, \tag{2.8}$$

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$$\mathbf{r}_E = \mathbf{r}\left(1 - \alpha \frac{GM}{c^2 r}\right) - \sum_{N=1}^{2} \lambda_N \frac{\mu \mathbf{v} \times \mathbf{S}^{(N)}}{m_N^2 c^2}, \qquad (2.9)$$

where Eqs. (2.8) and (2.9) hold only for center-of-mass system. Our most general coordinates $\mathbf{r}_N$ are, thus, related to the EIH coordinates $\mathbf{r}_{EN}$ by the four arbitrary dimensionless parameters α, a_0, λ_1, and λ_2 as given by Eq. (2.8).

The notation of Refs. 3 and 4 is consistent with this paper. The coordinates $\mathbf{r}_{EN}$, $\mathbf{r}_{*N}$, and $\mathbf{r}_N$ of this paper correspond to $\mathbf{r}_{\mathrm{EIH},\alpha\beta,N}$, $\mathbf{r}_N$, and $\mathbf{r}_{N(\lambda N)}$ of Ref. 2, respectively. The coordinates $\mathbf{r}_{EN}$ and $\mathbf{r}_{*N}$ of this paper correspond to $\mathbf{r}_{NB}$ and $\mathbf{r}_N$ of Ref. 10 (Sec. I and II), respectively, if the electromagnetic part of Ref. 10 is omitted. Reference 1 is all in EIH coordinates.

We also will be using coordinates for the center of mass $\mathbf{r}_{\mathrm{ECM}}$, $\mathbf{r}_{*\mathrm{CM}}$, and $\mathbf{r}_{\mathrm{CM}}$, where

$$\mathbf{r}_{\mathrm{ECM}} = \sum_{N=1}^{2} \nu_N \mathbf{r}_{EN}, \quad \mathbf{r}_{*\mathrm{CM}} = \sum_{N=1}^{2} \nu_N \mathbf{r}_{*N}, \quad \mathbf{r}_{\mathrm{CM}} = \sum_{N=1}^{2} \nu_N \mathbf{r}_N, \qquad (2.10)$$

and where in general ν_1 and ν_2 can take any values such that $\nu_1 + \nu_2 = 1$. However, in this paper we will always set $\nu_N = m_N/M$. Using Eqs. (2.2) and (2.10), we obtain

$$\mathbf{r}_{\mathrm{ECM}} = \mathbf{r}_{*\mathrm{CM}} - \alpha G(1 - a_0)\delta m \mathbf{r}_*/c^2 r_*, \qquad (2.11)$$

where $\delta m \equiv m_1 - m_2$. Using Eqs. (2.6), (2.8), and (2.10), we obtain

$$\mathbf{r}_{*\mathrm{CM}} = \mathbf{r}_{\mathrm{CM}} + \sum_{N=1}^{2} (-1)^N \lambda_N \frac{\mu \mathbf{v} \times \mathbf{S}^{(N)}}{m_N M c^2}, \qquad (2.12)$$

$$\mathbf{r}_{\mathrm{ECM}} = \mathbf{r}_{\mathrm{CM}} - \alpha G(1 - a_0)\delta m \frac{\mathbf{r}}{c^2 r} + \sum_{N=1}^{2} (-1)^N \lambda_N \frac{\mu \mathbf{v} \times \mathbf{S}^{(N)}}{m_N M c^2}, \qquad (2.13)$$

where Eqs. (2.12) and (2.13) hold only for a center-of-mass system.

III. SPIN-ORBIT TERMS

Cho and Dass[11] have given the potential energy terms V_{S1} and V_{S2} for general relativity in EIH coordinates and in a frame of reference where the total momentum is *not* zero. Their results—derived from Schwinger's source theory[12]—are

$$V_{S1} = (Gm_2/c^2 r_E^3) \times [\tfrac{3}{2}\mathbf{S}^{(1)}\cdot(\mathbf{r}_E \times \mathbf{v}_{E1}) - 2\mathbf{S}^{(1)}\cdot(\mathbf{r}_E \times \mathbf{v}_{E2})], \qquad (3.1)$$

$$V_{S2} = (Gm_1/c^2 r_E^3) \times [-\tfrac{3}{2}\mathbf{S}^{(2)}\cdot(\mathbf{r}_E \times \mathbf{v}_{E2}) + 2\mathbf{S}^{(2)}\cdot(\mathbf{r}_E \times \mathbf{v}_{E1})]. \qquad (3.2)$$

Using the time derivative Eq. (2.10), we can put Eqs. (3.1) and (3.2) in the form

$$V_{SN} = \frac{G\mu}{c^2 r_E^3}\left(\frac{3}{2}\frac{m_1 m_2}{m_N^2} + 2\right)\mathbf{S}^{(N)}\cdot(\mathbf{r}_E \times \mathbf{v}_E) + (-1)^N \frac{G\mu}{c^2 r_E^3}\left(\frac{M}{2m_N}\right)\mathbf{S}^{(N)}\cdot(\mathbf{r}_E \times \mathbf{v}_{ECM}), \qquad (3.3)$$

which is in agreement with the earlier general relativity results of Tulczyjew[13] [see his Eq. (3.10) and his Errata] who derived them using an "improved" EIH formalism.

The generalization of Eq. (3.3) to include PPN parameters γ and β is

$$V_{SN} = \frac{G\mu}{c^2 r_E^3}\left[\left(\gamma + \frac{1}{2}\right)\frac{m_1 m_2}{m_N^2} + \gamma + 1\right]\mathbf{S}^{(N)}\cdot(\mathbf{r}_E \times \mathbf{v}_E) + (-1)^N \frac{G\mu}{c^2 r_E^3}\left(\frac{M}{2m_N}\right)\mathbf{S}^{(N)}\cdot(\mathbf{r}_E \times \mathbf{v}_{ECM}). \qquad (3.4)$$

The γ and β dependence of the first term in Eq. (3.4)—it turns out to be independent of β—has been given by us[2] previously and is consistent with the results of Börner, Ehlers, and Rudolph.[14] The V_{SN} term will contribute a term $-\partial V_{SN}/\partial \mathbf{v}_{\mathrm{ECM}}$ to the total momentum $\mathbf{P}_{\mathrm{ECM}}$, where

$$-\frac{\partial V_{SN}}{\partial \mathbf{v}_{\mathrm{ECM}}} = (-1)^N \frac{G\mu}{c^2 r_E^3}\left(\frac{M}{2m_N}\right)\mathbf{r}_E \times \mathbf{S}^{(N)}. \qquad (3.5)$$

The second term in Eq. (3.4) must be γ (and β) independent so that its contribution to the total momentum will be γ (and β) independent. We shall now explain why the total momentum $\mathbf{P}_{\mathrm{ECM}}$ must be independent of γ (and β). Consider the n-body post-Newtonian Lagrangian with PPN parameters γ and β for (uncharged) point bodies (see Sec. IV C of Ref. 10). For this case[10] $\mathbf{P}_{\mathrm{ECM}}$ is independent of γ (and β). Because our two spinning bodies can be considered to be made up of n-point bodies, the total momentum $\mathbf{P}_{\mathrm{ECM}}$ for the two spinning bodies must also be independent of γ (and β).

IV. CENTER OF INERTIA

Let us start in the $\mathbf{r}_{*N}$ coordinate system where $\mathbf{r}_{*\mathrm{CI}}$, $\mathbf{v}_{*\mathrm{CI}}$, and $\mathbf{a}_{*\mathrm{CI}}$ are the position, velocity, and acceleration, respectively, of the center of inertia, and where $\mathscr{E}_*$, $\mathbf{P}_{*\mathrm{CM}}$, and $\mathbf{P}_{*N}$ are the total conserved energy, total conserved (canonical) momentum, and (canonical) momentum of body N, respectively. The center of inertia must satisfy the equation[10,15,16]

$$\frac{d}{dt}((\mathscr{E}_*/c^2)\mathbf{r}_{*\mathrm{CI}}) = \mathbf{P}_{*\mathrm{CM}} = \mathbf{P}_{*1} + \mathbf{P}_{*2}, \qquad (4.1)$$

from which it follows that $(\mathscr{E}_*/c^2)\mathbf{v}_{*\mathrm{CI}} = \mathbf{P}_{*\mathrm{CM}}$ and $\mathbf{a}_{*\mathrm{CI}} = 0$.

In order to satisfy Eq. (4.1), we set

$$\mathscr{E}_* \mathbf{r}_{*\mathrm{CI}} = \sum_{N=1}^{2}\left[\mathscr{E}_{*N}\mathbf{r}_{*N} + (-1)^N \frac{\mu}{2m_N}\mathbf{S}^{(N)} \times \mathbf{v}_*\right], \qquad (4.2)$$

where

$$\mathscr{E}_{*N} = m_N c^2 + \tfrac{1}{2} m_N v_{*N}^2 - \left[\frac{1}{2} - (-1)^N \alpha(1 - a_0)\frac{\delta m}{\mu}\right]\frac{Gm_1 m_2}{r_*}. \qquad (4.3)$$

We must also have

$$\mathscr{E}_* = \mathscr{E}_{*1} + \mathscr{E}_{*2} = Mc^2 + \sum_{N=1}^{N} \frac{1}{2} m_N v_{*N}^2 - \frac{Gm_1 m_2}{r_*}, \qquad (4.4)$$

which is in agreement with Eq. (4.3). The terms $m_N c^2$ in Eq. (4.3) and Mc^2 in Eq. (4.4) must include rotational kinetic energy in order that these equations be accurate to the Newtonian approximation (i.e., order c^0). Thus we have

$$m_N c^2 = \left(m_{0N} + \tfrac{1}{2}\frac{I^{(N)}\omega^{(N)2}}{c^2}\right)c^2, \tag{4.5}$$

$$Mc^2 = (m_1 + m_2)c^2, \tag{4.6}$$

where m_{0N}, $I^{(N)}$, and $\boldsymbol{\omega}^{(N)}$ are the nonrotating rest mass, moment of inertia, and angular velocity, respectively, of body N. We deduced Eqs. (4.2) and (4.3) by using the results for nonrotating bodies given in Secs. IV B and IV C of Ref. 10 and then adding spin terms to Eq. (4.2) that are consistent with Eq. (3.5). The spin terms in Eq. (4.2) are consistent with Eq. (3.5) because

$$\frac{d}{dt}\left[(-1)^N \frac{\mu}{2m_N c^2}\mathbf{S}^{(N)}\times\mathbf{v}_*\right] = (-1)^N \frac{G\mu M}{2r_*^3 m_N c^2}\mathbf{r}_*\times\mathbf{S}^{(N)}. \tag{4.7}$$

In evaluating the left-hand side of Eq. (4.7) which is a post-Newtonian term (i.e., of order c^{-2}), we have used $d\mathbf{S}^{(N)}/dt = 0$ and $\mathbf{a}_* = -GM\mathbf{r}_*/r_*^3$, which are correct to first order. The spin-orbit terms and Eq. (3.5) in Sec. III are of order c^{-2} and, thus, to this order we can replace the $\mathbf{r}_{EN}$ coordinates with $\mathbf{r}_{*N}$ coordinates. It is an interesting fact[10] that $\mathbf{P}_{*\mathrm{CM}}$ expressed in $\mathbf{r}_{*N}$ coordinates is explicitly independent of the parameters γ and β (and α if $a_0 = 1$). It should be noted that the post-Newtonian potential energy terms for the spin–spin interaction[1–4] $V_{S1,S2}$ and the Nordtvedt effect[2–4,9] as well as the quadrupole moment interactions[1,3,4] (small Newtonian terms and thus treated as if they were post-Newtonian terms from an order of magnitude point of view) V_{Q1} and V_{Q2} are velocity independent and hence will not contribute to $\mathbf{P}_{*\mathrm{CM}}$. We conclude that Eqs. (4.1)–(4.3) are still valid when these terms are included in the Lagrangian.

In the center-of-mass coordinate system $\mathbf{P}_{*\mathrm{CM}} = 0$ and, thus, $\mathbf{v}_{*\mathrm{CI}} = 0$ and $\mathbf{r}_{*\mathrm{CI}}$ is a constant. We shall now set $\mathbf{r}_{*\mathrm{CI}} = 0$ and obtain from Eq. (4.2)

$$\sum_{N=1}^{2}\left[\mathscr{E}_{*N}\mathbf{r}_{*N} + (-1)^N\frac{\mu}{2m_N}\mathbf{S}^{(N)}\times\mathbf{v}_*\right] = 0. \tag{4.8}$$

From Eq. (4.8), it follows (to post-Newtonian order) that

$$\mathbf{r}_{*1} = -\frac{\mathscr{E}_{*2}}{\mathscr{E}_{*1}}\mathbf{r}_{*2} - \frac{1}{m_1c^2}\sum_{N=1}^{2}(-1)^N\frac{\mu}{2m_N}S^{(N)}\times\mathbf{v}_*, \tag{4.9}$$

$$\mathbf{r}_{*2} = -\frac{\mathscr{E}_{*1}}{\mathscr{E}_{*2}}\mathbf{r}_{*1} - \frac{1}{m_2c^2}\sum_{N=1}^{2}(-1)^N\frac{\mu}{2m_N}\mathbf{S}^{(N)}\times\mathbf{v}_*, \tag{4.10}$$

and thus

$$\mathbf{r}_* = -(-1)^N\left[\frac{\mathscr{E}_*\mathscr{E}_{*N}}{\mathscr{E}_{*1}\mathscr{E}_{*2}}\mathbf{r}_{*N} + \frac{m_N}{m_1m_2c^2}\times\sum_{N=1}^{2}(-1)^N\frac{\mu}{2m_N}\mathbf{S}^{(N)}\times\mathbf{v}_*\right], \tag{4.11}$$

which can be inverted to give us

$$\mathbf{r}_{*N} = -(-1)^N\frac{\mathscr{E}_{*1}\mathscr{E}_{*2}}{\mathscr{E}_*\mathscr{E}_{*N}}\mathbf{r} - \frac{1}{Mc^2}\times\sum_{N=1}^{2}(-1)^N\frac{\mu}{2m_N}\mathbf{S}^{(N)}\times\mathbf{v}_*. \tag{4.12}$$

Because we are in center-of-mass system, we can use $\mathbf{v}_{*N} = -(-1)^N\mu\mathbf{v}_*/m_N$ in Eqs. (4.4) and (4.3) to obtain, respectively,

$$\mathscr{E}_* = Mc^2 + \tfrac{1}{2}\mu v_*^2 - GM\mu/r_*, \tag{4.13}$$

$$\mathscr{E}_{*N} = m_Nc^2 + \tfrac{1}{2}\mu^2v_*^2/m_N - [\tfrac{1}{2} - (-1)^N\alpha(1-a_0)\delta m/\mu]Gm_1m_2/r_*. \tag{4.14}$$

Inverting Eq. (2.10) we obtain

$$\mathbf{r}_{*N} = -(-1)^N(m_1m_2/Mm_N)\mathbf{r}_* + \mathbf{r}_{*\mathrm{CM}}. \tag{4.15}$$

Using Eqs. (4.13) and (4.14) in (4.12) and comparing the result with Eq. (4.15), we obtain (to the post-Newtonian approximation)

$$\mathbf{r}_{*\mathrm{CM}} = \frac{\mu\delta m}{2M^2c^2}\left[v_*^2 - \frac{GM}{r_*} + \frac{2\alpha(1-a_0)GM^2}{\mu r_*}\right]\mathbf{r}_* - \frac{1}{Mc^2}\sum_{N=1}^{2}(-1)^N\frac{\mu}{2m_N}\mathbf{S}^{(N)}\times\mathbf{v}_*. \tag{4.16}$$

The masses M and m_N in the first terms (terms of order c^2) in Eqs. (4.13) and (4.14), respectively, are given by Eqs. (4.5) and (4.6) and the same must be true for the masses in the first term of Eq. (4.15) (i.e., for the masses in $\nu_N = m_N/M$) if Eq. (4.16) is to be correct.

The EIH coordinate versions of Eqs. (4.15) and (4.16) are given by setting $\alpha = 0$. We obtain

$$\mathbf{r}_{EN} = -(-1)^N(m_1m_2/Mm_N)\mathbf{r}_E + \mathbf{r}_{E\mathrm{CM}}, \tag{4.17}$$

$$\mathbf{r}_{E\mathrm{CM}} = \frac{\mu\delta m}{2M^2c^2}\left[v_E^2 - \frac{GM}{r_E}\right]\mathbf{r}_E - \frac{1}{Mc^2}\sum_{N=1}^{2}(-1)^N\frac{\mu}{2m_N}\mathbf{S}^{(N)}\times\mathbf{v}_E. \tag{4.18}$$

The above with $\mathbf{S}^{(N)} = 0$ (the result for nonrotating bodies) has been given by Wagoner and Will.[7] Using Eqs. (2.11) and (4.18), we can regain Eq. (4.16) correct to the post-Newtonian approximation.

Let us next consider our most general coordinate system, the $\mathbf{r}_N$ coordinate system. Inverting Eq. (2.10) and combining Eqs. (2.12) and (4.16), we obtain (to the post-Newtonian approximation)

$$\mathbf{r}_N = -(-1)^N\frac{m_1m_2}{Mm_N}\mathbf{r} + \mathbf{r}_{\mathrm{CM}}, \tag{4.19}$$

$$\mathbf{r}_{\mathrm{CM}} = \frac{\mu\delta m}{2M^2c^2}\left[v^2 - \frac{GM}{r} + \frac{2\alpha(1-a_0)GM^2}{\mu r}\right]\mathbf{r} - \frac{1}{Mc^2}\sum_{N=1}^{N}(-1)^N\left(\frac{\mu}{2m_N}\right)(1-2\lambda_N)\mathbf{S}^{(N)}\times\mathbf{v}, \tag{4.20}$$

where the above are valid in the center-of-mass coordinate system where the total momentum is zero. Tulczyjew[13] [see his Eq. (3.14) and his Errata] has given an EIH coordinate system version (i.e., α and λ_N are zero) of our Eq. (4.20). If we start with a Lagrangian in the $\mathbf{r}_{*N}$ coordinate system and make the coordinate transformation of Eq. (2.4) to the $\mathbf{r}_N$ coordinate system, we will obtain an acceleration-dependent Lagrangian.[4] To obtain this acceleration-dependent Lagrangian start with the Lagrangian in the $\mathbf{r}_{*N}$ coordinate system, replace $\mathbf{r}_{*N}$ by $\mathbf{r}_N$ and $\mathbf{v}_{*N}$ by $\mathbf{v}_N$, and then add the terms $-V_{\lambda 1}$ and $-V_{\lambda 2}$, where

$$V_{\lambda N} = -\frac{\lambda_N}{m_N c^2}\mathbf{S}^{(N)}\cdot\left\{\left[\mathbf{a}_N - (-1)^N\frac{Gm_1m_2}{r^3 m_N}\mathbf{r}\right]\times m_N\mathbf{v}_N\right\}. \tag{4.21}$$

The center of inertia must then satisfy the equation[16]

$$\frac{d}{dt}\left(\frac{\mathscr{E}}{c^2}\mathbf{r}_{\rm CI}\right) = \mathbf{\Pi}_1 + \mathbf{\Pi}_2, \tag{4.22}$$

where

$$\mathbf{\Pi}_N = \frac{\partial\mathscr{L}}{\partial\mathbf{v}_N} - \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{a}_N}\right), \tag{4.23}$$

and it follows from Eq. (4.22) that $(\mathscr{E}/c^2)\mathbf{v}_{\rm CI} = \mathbf{\Pi}_1 + \mathbf{\Pi}_2$ and $\mathbf{v}_{\rm CI}$ is a constant. In order to satisfy Eq. (4.22), we set

$$\mathscr{E}\mathbf{r}_{\rm CI} = \sum_{N=1}^{2}\left[\mathscr{E}_N\mathbf{r}_N + (-1)^N \times\left(\frac{\mu}{2m_N}\right)(1-2\lambda_N)\mathbf{S}^{(N)}\times\mathbf{v}\right], \tag{4.24}$$

where $\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2$ and

$$\mathscr{E}_N = m_N c^2 + \tfrac{1}{2}m_N v_N^2 - \left[\tfrac{1}{2} - (-1)^N\alpha(1-a_0)\delta m/\mu\right]Gm_1m_2/r. \tag{4.25}$$

It should be noted that we could also add the constant terms $K_1\mathbf{S}^{(1)}\times\mathbf{v}_{\rm CM}$ and $K_2\mathbf{S}^{(2)}\times\mathbf{v}_{\rm CM}$, where K_1 and K_2 are arbitrary dimensionless constants, to the right-hand side of Eq. (4.24) and also to the right-hand side of Eq. (4.2). This would not alter our other results because we use Eqs. (4.2) and (4.24) in a frame of reference where $\mathbf{v}_{\rm CM}$ is zero. We can also obtain Eqs. (4.24) and (4.25) by noting that these results are exactly what is needed to obtain Eq. (4.20). This can easily be seen by comparing Eqs. (4.2), (4.3), and (4.16) with Eqs. (4.24), (4.25), and (4.20), respectively.

The spin supplementary condition[2,17] of Price[18] and of Newton and Wigner[19] corresponds to setting λ_N equal to zero and has the advantage that the Langrangian will have no acceleration-dependent spin terms.[4] The spin supplementary condition[2,17] of Corinaldesi–Papapetrou[20] corresponds to setting $\lambda_N = \frac{1}{2}$ and has the advantage that Eqs. (4.20) and (4.24) will have no spin-dependent terms. The spin supplementary condition[2,17] of Pirani[21] corresponds to setting $\lambda_N = -\frac{1}{2}$ and has the advantage that the equations of motion take on a particularly simple form. The equations of motion can be written as

$$\begin{aligned}\mathbf{a}_N - (-1)^N Gm_P\mathbf{r}/r^3 &= \text{SPIN-INDEPENDENT TERMS}\\ &\quad + \text{SPIN TERMS} + \text{SPIN–SPIN TERMS}\\ &\quad + \text{QUADRUPOLE MOMENT TERMS},\end{aligned} \tag{4.26}$$

where the SPIN TERMS, which depend on λ_1 and λ_2, are given by

$$\begin{aligned}\text{SPIN TERMS} = \frac{Gm_P}{m_N c^2 r^3}\bigg[&(2\gamma+1)\mathbf{S}^{(N)}\times\mathbf{v}_N - \left(2\gamma+\frac{3}{2}+\lambda_N\right)\mathbf{S}^{(N)}\times\mathbf{v}_P + \left(3\gamma+\frac{3}{2}-3\lambda_N\right)\frac{\mathbf{S}^{(N)}\cdot(\mathbf{r}\times\mathbf{v}_N)}{r^2}\mathbf{r}\\ &- (3\gamma+3)\frac{\mathbf{S}^{(N)}\cdot(\mathbf{r}\times\mathbf{v}_P)}{r^2}\mathbf{r} + (-1)^N\left(3\gamma+\frac{3}{2}+3\lambda_N\right)\frac{\mathbf{v}\cdot\mathbf{r}}{r^2}\mathbf{S}^{(N)}\times\mathbf{r}\bigg]\\ + \frac{G}{c^2 r^3}\bigg[&(2\gamma+2)\mathbf{S}^{(P)}\times\mathbf{v}_N + \left(\lambda_P-2\gamma-\frac{3}{2}\right)\mathbf{S}^{(P)}\times\mathbf{v}_P + \left(3\lambda_P-3\gamma-\frac{3}{2}\right)\frac{\mathbf{S}^{(P)}\cdot(\mathbf{r}\times\mathbf{v}_P)}{r^2}\mathbf{r}\\ &+ (3\gamma+3)\frac{\mathbf{S}^{(P)}\cdot(\mathbf{r}\times\mathbf{v}_N)}{r^2}\mathbf{r} + (-1)^N(3\gamma+3)\frac{\mathbf{v}\cdot\mathbf{r}}{r^2}\mathbf{S}^{(P)}\times\mathbf{r}\bigg],\end{aligned} \tag{4.27}$$

and where $P\equiv 3-N$. If one now sets $\lambda_1 = \lambda_2 = -\frac{1}{2}$, the above can be put in the form

$$\begin{aligned}\text{SPIN TERMS} &= \frac{Gm_P}{m_N c^2 r^3}(-1)^P\bigg[(2\gamma+1)\mathbf{S}^{(N)}\times\mathbf{v} + (3\gamma+3)\\ &\quad\times\frac{\mathbf{S}^{(N)}\cdot(\mathbf{r}\times\mathbf{v})}{r^2}\mathbf{r} - 3\gamma\frac{\mathbf{v}\cdot\mathbf{r}}{r^2}\mathbf{S}^{(N)}\times\mathbf{r}\bigg]\\ &\quad + \frac{G}{c^2r^3}(-1)^P\bigg[(2\gamma+2)\mathbf{S}^{(P)}\times\mathbf{v} + (3\gamma+3)\\ &\quad\times\frac{\mathbf{S}^{(P)}\cdot(\mathbf{r}\times\mathbf{v})}{r^2}\mathbf{r} - (3\gamma+3)\frac{\mathbf{v}\cdot\mathbf{r}}{r^2}\mathbf{S}^{(P)}\times\mathbf{r}\bigg].\end{aligned} \tag{4.28}$$

It should be noted that Eqs. (4.26)–(4.28) are correct for a non-center-of-mass coordinate system and that Eq. (4.28) takes on the identical form in a center-of-mass coordinate system where $\mathbf{v}_{\rm CM}$ is zero (to first order). If one sets $\gamma = 1$ and uses the vector identity

$$(\mathbf{r}\times\mathbf{v})\mathbf{S}^{(N)}\cdot\mathbf{r} \equiv [\mathbf{S}^{(N)}\cdot(\mathbf{r}\times\mathbf{v})]\mathbf{r} + (\mathbf{S}^{(N)}\times\mathbf{v})r^2 - (\mathbf{v}\cdot\mathbf{r})(\mathbf{S}^{(N)}\times\mathbf{r}), \tag{4.29}$$

and the center-of-mass system relation $\mathbf{v}_N = -(-1)^N \times\mu\mathbf{v}/m_N$ in Eq. (4.28) one can put this equation into the form of the center-of-mass general relativity result of D'Eath[22] [see his Eq. (6.7)].

V. CONCLUSIONS

We completed the solution to the post-Newtonian gravitational two-body problem with spin and PPN parameters γ and β by giving [see Eqs. (4.19) and (4.20)] the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ and the center of mass $\mathbf{r}_{\rm CM}$ of the two bodies as a function of the relative position $\mathbf{r}$, relative velocity $\mathbf{v}$, and spin angular momenta $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ of the two bodies. Thus, if we have a solution $\mathbf{r} = \mathbf{r}(t)$, correct to the post-Newtonian approximation, we also have solutions $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, correct to the post-Newtonian approximation. For coordinate systems [see Eq. (2.8)] where $\alpha \neq 0$ and $a_0 \neq 1$, the position $\mathbf{r}_{\rm CM}$ has a nonzero term $2\alpha(1-a_0) \times GM^2/\mu r$ inside the square bracket of Eq. (4.20) that is due to the potential energy term $-Gm_1m_2/r$ not being split[10] equally between $\mathscr{E}_1$ and $\mathscr{E}_2$ of Eq. (4.25). Clearly Eqs. (4.19), (4.20), (4.24), and (4.25) remain valid if the potential energy terms for the spin–spin interaction[1–4] $V_{S1,S2}$ and the Nordtvedt effect[2–4,9] as well as the quadrupole moment interactions[1,3,4] V_{Q1} and V_{Q2} (which are all velocity and acceleration independent and hence will not contribute to the total momentum $\mathbf{\Pi}_1 + \mathbf{\Pi}_2$) are included in the Lagrangian.

In the Appendix, we gave a quantum field theory derivation of the spin-orbit potential energy terms V_{S1} and V_{S2} for general relativity in a frame of reference where the total momentum was *not* zero and our results were in agreement with those derived by Cho and Dass[11] from Schwinger's source theory[12] and those derived by Tulczyjew[13] using an "improved" EIH formalism. In Sec. III, we included PPN parameters γ and β in the spin-orbit potential energy terms and concluded that the second term in Eq. (3.4) had to be independent of γ and β.

APPENDIX: FIELD THEORY DERIVATION OF V_{S1} AND V_{S2}

In this Appendix, we shall give a quantum field theory derivation of the spin-orbit potential energy terms V_{S1} and V_{S2} of Eqs. (3.1) and (3.2), respectively, for general relativity in a frame of reference where the total momentum is not zero. Let us consider the one-graviton exchange interaction between a spin-$\frac{1}{2}$ particle of mass m_1 and a spin-0 particle of mass m_2. Let the initial and final propagation four-vectors for the spin-$\frac{1}{2}$ particle be p_μ and p'_μ, respectively, and those for the spin-0 particle be q_μ and q'_μ, respectively. We also have

$$\mathbf{P}_1 = \hbar\mathbf{p}, \quad E_1 = c\hbar p_0, \quad \lambda_1 = m_1c/\hbar, \quad p^2 = \mathbf{p}^2 - p_0^2 = -\lambda_1^2, \tag{A1}$$

$$\mathbf{P}_2 = \hbar\mathbf{q}, \quad E_2 = c\hbar q_0, \quad \lambda_2 = m_2c/\hbar, \quad q^2 = \mathbf{q}^2 - q_0^2 = -\lambda_2^2, \tag{A2}$$

where $\mathbf{P}_N$ and E_N are the momentum and energy, respectively, of particle N and $\hbar$ is Planck's constant divided by 2π. Note that the λ_1 and λ_2 of this Appendix are *not* the same quantities as the λ_1 and λ_2 used in the rest of this paper.

The graviton coupling constant κ is related to Newton's constant of gravitation G and the speed of light c by the relation

$$\kappa^2 = 16\pi G/c^4. \tag{A3}$$

Let ψ be a spin-$\frac{1}{2}$ field with mass m_1 and U_0 be a spin-0 field with mass m_2. The interaction terms (using ordered products[23]) with the graviton field $h_{\mu\nu}$ are[24,25] (to order κ)

$$:L_{\rm int}: = -\frac{1}{8}\kappa c\hbar :\left[\bar\psi\gamma_\mu \frac{\partial\psi}{\partial x_\nu} + \bar\psi\gamma_\nu \frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar\psi}{\partial x_\nu}\gamma_\mu\psi - \frac{\partial\bar\psi}{\partial x_\mu}\gamma_\nu\psi\right] h_{\mu\nu}: - \frac{1}{2}\kappa :\left[\frac{\partial U_0}{\partial x_\mu}\frac{\partial U_0}{\partial x_\nu} - \frac{\delta_{\mu\nu}}{2}\frac{\partial U_0}{\partial x_\rho}\frac{\partial U_0}{\partial x_\rho} - \frac{\delta_{\mu\nu}}{2}\lambda_2^2 U_0^2\right] h_{\mu\nu}:, \tag{A4}$$

and the contractions are[24,25]

$$h^{\cdot}_{\mu\nu}(x)h^{\cdot}_{\lambda\rho}(x') = -ic\hbar(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\rho})D_F(x-x'), \tag{A5}$$

where

$$D_F(x-x') = \lim_{\epsilon\to+0}\frac{1}{(2\pi)^4}\int dk\, e^{ik(x-x')}\frac{1}{k^2 - i\epsilon}. \tag{A6}$$

We also have[23]

$$S_2 = \frac{-1}{2c^2\hbar^2}\int dx \int dx'\, T[:H_{\rm int}(x)::H_{\rm int}(x'):], \tag{A7}$$

where $:H_{\rm int}:$ may be replaced[23,26] by $-:L_{\rm int}:$.

Using Eqs. (A4) in (A7), we obtain

$$S_2 = \frac{-\kappa^2}{16c\hbar}\int dx\int dx' :\left[\bar\psi(x)\gamma_\mu\frac{\partial\psi(x)}{\partial x_\nu} + \bar\psi(x)\gamma_\nu\frac{\partial\psi(x)}{\partial x_\mu} - \frac{\partial\bar\psi(x)}{\partial x_\nu}\gamma_\mu\psi(x) - \frac{\partial\bar\psi(x)}{\partial x_\mu}\gamma_\nu\psi(x)\right]h^{\cdot}_{\mu\nu}(x) \times\left[\frac{\partial U_0(x')}{\partial x'_\alpha}\frac{\partial U_0(x')}{\partial x'_\beta} - \frac{\delta_{\alpha\beta}}{2}\frac{\partial U_0(x')}{\partial x'_\lambda}\frac{\partial U_0(x')}{\partial x'_\lambda} - \frac{\delta_{\alpha\beta}}{2}\lambda_2^2 U_0(x')U_0(x')\right]h^{\cdot}_{\alpha\beta}(x'):. \tag{A8}$$

Using Eqs. (A5) and (A6) along with

$$\psi(x) = (1/V)^{1/2}\psi^+(\mathbf{p})e^{ipx}, \tag{A9}$$

$$\bar\psi(x) = (1/V)^{1/2}\bar\psi^-(\mathbf{p}')e^{-ip'x}, \tag{A10}$$

$$U_0(x') = (c\hbar/2q_0V)^{1/2}a(\mathbf{q})e^{iqx'} + (c\hbar/2q'_0V)^{1/2}a^*(\mathbf{q}')e^{-iq'x'}, \tag{A11}$$

in Eq. (A8), we find

$$S_2=\frac{ic\hbar\kappa^2(2\pi)^4}{4V^2(q_0'q_0)^{1/2}}\delta(p-p'+q-q')$$
$$\times\frac{1}{(p'-p)^2}:[-\lambda_1\lambda_2^2\bar{\psi}^-(\mathbf{p}')\psi^+(\mathbf{p})$$
$$+\tfrac{1}{4}(p+p')(q+q')\bar{\psi}^-(\mathbf{p}')$$
$$\times i(q+q')\gamma\psi^+(\mathbf{p})]a^*(\mathbf{q}')a(\mathbf{q}):, \qquad (A12)$$

where V is a volume factor, and only the appropriate terms in the Fourier expansion have been included on the right-hand side of Eqs. (A9), (A10), and (A11).

The quantity $V(\mathbf{k})$, correct to first order in G, is defined in terms of the second-order S matrix as[23]

$$S_2=(-i/c\hbar V^2)(2\pi)^4\delta(p+q-p'-q')$$
$$\times\psi_L^{*-}(\mathbf{p}')a^*(\mathbf{q}')V(\mathbf{k})a(\mathbf{q})\psi_L^+(\mathbf{p}), \qquad (A13)$$

where $\bar{\psi}^-(\mathbf{p}')=\psi^{*-}(\mathbf{p}')\gamma_4$ and [23]

$$\psi^{*-}(\mathbf{p}')=\left(\frac{\lambda_1+p_0'}{2p_0'}\right)^{1/2}(\psi_L^{*-}(\mathbf{p}')\quad \psi_S^{*-}(\mathbf{p}')), \qquad (A14)$$

$$\psi^+(\mathbf{p})=\left(\frac{\lambda_1+p_0}{2p_0}\right)^{1/2}\begin{pmatrix}\psi_L^+(\mathbf{p})\\ \psi_S^+(\mathbf{p})\end{pmatrix}, \qquad (A15)$$

$$\psi_S^{*-}(\mathbf{p}')=\psi_L^{*-}(\mathbf{p}')\frac{(\mathbf{p}'\cdot\boldsymbol{\sigma}^{(1)})}{(\lambda_1+p_0')}, \qquad (A16)$$

$$\psi_S^+(\mathbf{p})=\frac{(\mathbf{p}\cdot\boldsymbol{\sigma}^{(1)})}{(\lambda_1+p_0)}\psi_L^+(\mathbf{p}), \qquad (A17)$$

and where $k\equiv p'-p=q-q'$ so that

$$\mathbf{k}=\mathbf{p}'-\mathbf{p}=\mathbf{q}-\mathbf{q}',\quad k_0=p_0'-p_0=q_0-q_0'. \qquad (A18)$$

Let us now set

$$V(\mathbf{k})=V_1(\mathbf{k})+V_{S1}(\mathbf{k}), \qquad (A19)$$

where $V_1(\mathbf{k})$ is spin independent and $V_{S1}(\mathbf{k})$ is spin dependent (i.e., depends on $\boldsymbol{\sigma}^{(1)}$). Using Eqs. (A12)–(A19), we obtain

$$V_1(\mathbf{k})=\frac{-c^2\hbar^2\kappa^2}{4(q_0q_0')^{1/2}}\left(\frac{\lambda_1+p_0'}{2p_0'}\right)^{1/2}\left(\frac{\lambda_1+p_0}{2p_0}\right)^{1/2}\frac{1}{\mathbf{k}^2-k_0^2}\Bigg[-\lambda_1\lambda_2^2\left(1-\frac{\mathbf{p}'\cdot\mathbf{p}}{(\lambda_1+p_0')(\lambda_1+p_0)}\right)$$
$$-\frac{1}{4}(p+p')(q+q')(q_0+q_0')\left(1+\frac{\mathbf{p}'\cdot\mathbf{p}}{(\lambda_1+p_0')(\lambda_1+p_0)}\right)$$
$$+\frac{1}{4}(p+p')(q+q')\left(\frac{\mathbf{p}\cdot(\mathbf{q}+\mathbf{q}')}{\lambda_1+p_0}+\frac{\mathbf{p}'\cdot(\mathbf{q}+\mathbf{q}')}{\lambda_1+p_0'}\right)\Bigg], \qquad (A20)$$

$$V_{S1}(\mathbf{k})=\frac{-c^2\hbar^2\kappa^2}{4(q_0q_0')^{1/2}}\left(\frac{\lambda_1+p_0'}{2p_0'}\right)^{1/2}\left(\frac{\lambda_1+p_0}{2p_0}\right)^{1/2}\frac{1}{\mathbf{k}^2-k_0^2}\Bigg[\left(\lambda_1\lambda_2^2-\frac{1}{4}(p+p')(q+q')(q_0+q_0')\right)$$
$$\times\frac{i\boldsymbol{\sigma}^{(1)}\cdot[\mathbf{k}\times(\mathbf{p}+\mathbf{p}')]}{2(\lambda_1+p_0')(\lambda_1+p_0)}+\frac{1}{4}(p+p')(q+q')$$
$$\times\left(\frac{i\boldsymbol{\sigma}^{(1)}\cdot[\mathbf{k}\times(\mathbf{q}+\mathbf{q}')]}{2(\lambda_1+p_0)}+\frac{i\boldsymbol{\sigma}^{(1)}\cdot[\mathbf{k}\times(\mathbf{q}+\mathbf{q}')]}{2(\lambda_1+p_0')}-\frac{ik_0\boldsymbol{\sigma}^{(1)}\cdot[(\mathbf{p}+\mathbf{p}')\times(\mathbf{q}+\mathbf{q}')]}{2(\lambda_1+p_0)(\lambda_1+p_0')}\right)\Bigg]. \qquad (A21)$$

Using Eq. (A18), we can express k_0 as[10] $k_0=\mathbf{k}\cdot(\mathbf{p}'+\mathbf{p})/(p_0'+p_0)$ or as $k_0=\mathbf{k}\cdot(\mathbf{q}+\mathbf{q}')/(q_0+q_0')$. Similar to what was done in Ref. 10, we shall put the k_0^2 that is in Eq. (A20) and (A21) in the form

$$k_0^2=[1+2\alpha(a_{12}+a_{21})]\left(\frac{\mathbf{k}\cdot(\mathbf{p}'+\mathbf{p})}{p_0'+p_0}\right)\left(\frac{\mathbf{k}\cdot(\mathbf{q}+\mathbf{q}')}{q_0+q_0'}\right)$$
$$-2\alpha\left[a_{12}\left(\frac{\mathbf{k}\cdot(\mathbf{p}'+\mathbf{p})}{p_0'+p_0}\right)^2+a_{21}\left(\frac{\mathbf{k}\cdot(\mathbf{q}+\mathbf{q}')}{q_0+q_0'}\right)^2\right], \qquad (A22)$$

where

$$a_{12}=[(1-a_0)m_1+a_0m_2]/m_2, \qquad (A23)$$

$$a_{21}=[(1-a_0)m_2+a_0m_1]/m_1. \qquad (A24)$$

We also have a k_0 in Eq. (A21), a situation that did not arise in Ref. 10. Because there is no simple way to obtain a square root of the k_0^2 of Eq. (A22), we suggest that this k_0 be put in the form

$$k_0=\bar{a}_0\left(\frac{\mathbf{k}\cdot(\mathbf{p}'+\mathbf{p})}{p_0'+p_0}\right)+(1-\bar{a}_0)\left(\frac{\mathbf{k}\cdot(\mathbf{q}+\mathbf{q}')}{q_0+q_0'}\right), \qquad (A25)$$

where $\bar{a}_0$ is another arbitrary dimensionless constant. It should be noted that Eqs. (A20)–(A22) and (A25) have been put into a form such that they remain the same if $\mathbf{p}\rightleftarrows\mathbf{p}'$ and $p_0\rightleftarrows p_0'$ or if $\mathbf{q}\rightleftarrows\mathbf{q}'$ and $q_0\rightleftarrows q_0'$ while $\mathbf{k}$ is *not* altered (i.e., we do *not* change $\mathbf{k}$ into $-\mathbf{k}$). The quantity $V(\mathbf{k})$ can also be defined in terms of the potential energy $V(\mathbf{r})$, correct to first order in G, as

$$V(\mathbf{k})=\int d\mathbf{r}\,e^{-i(\mathbf{p}'\cdot\mathbf{r}_1+\mathbf{q}'\cdot\mathbf{r}_2)}V(\mathbf{r})e^{i(\mathbf{p}\cdot\mathbf{r}_1+\mathbf{q}\cdot\mathbf{r}_2)}. \qquad (A26)$$

The potential energy $V(\mathbf{r})$ is a Hermitian operator and is also momentum dependent [i.e., $V(\mathbf{r})\equiv V(\mathbf{r},\mathbf{p}_{op},\mathbf{q}_{op})$ where $\mathbf{p}_{op}$ and $\mathbf{q}_{op}$ are operators].

If we are only interested in the classical results, as we are in this paper, the ordering of the factors in $V(\mathbf{r},\mathbf{p}_{op},\mathbf{q}_{op})$

makes no difference (i.e., we can neglect delta function terms). To obtain the classical results corresponding to Eqs. (A20)–(A22) and (A25) set $\mathbf{p}' = \mathbf{p}, p_0' = p_0, \mathbf{q}' = \mathbf{q}$, and $q_0' = q_0$ while now considering $\mathbf{k}$ to be an independent variable (i.e., do *not* set $\mathbf{k}$ equal to zero). The classical results are

$$V_1(\mathbf{k}) = -\frac{c^2\hbar^2\kappa^2}{4p_0q_0}\frac{1}{\mathbf{k}^2 - k_0^2}[2(pq)^2 - \lambda_1^2\lambda_2^2], \quad (A27)$$

$$\begin{aligned} V_{S1}(\mathbf{k}) = &-\frac{c^2\hbar^2 q_0\kappa^2}{8(\lambda_1 + p_0)}\frac{1}{\mathbf{k}^2 - k_0^2} \\ &\times\Bigg\{\left[2 - \frac{2\mathbf{p}\cdot\mathbf{q}}{p_0q_0} + \lambda_1\left(\frac{1}{p_0} - \frac{\mathbf{q}^2}{p_0q_0^2}\right)\right] i\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{k}\times\mathbf{p}) \\ &+\left[-\frac{2p_0}{q_0} + \frac{2\mathbf{p}\cdot\mathbf{q}}{q_0^2} + \lambda_1\left(-\frac{2}{q_0} + \frac{2\mathbf{p}\cdot\mathbf{q}}{p_0q_0^2}\right)\right] i\boldsymbol{\sigma}^{(1)} \\ &\cdot(\mathbf{k}\times\mathbf{q}) + k_0\left(\frac{2}{q_0} - \frac{2\mathbf{p}\cdot\mathbf{q}}{p_0q_0^2}\right) i\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{p}\times\mathbf{q})\Bigg\}, \end{aligned} \quad (A28)$$

where

$$\begin{aligned} k_0^2 = &[1 + 2\alpha(a_{12} + a_{21})]\frac{(\mathbf{k}\cdot\mathbf{p})(\mathbf{k}\cdot\mathbf{q})}{p_0q_0} \\ &- 2\alpha\left[a_{12}\left(\frac{\mathbf{k}\cdot\mathbf{p}}{p_0}\right)^2 + a_{21}\left(\frac{\mathbf{k}\cdot\mathbf{q}}{q_0}\right)^2\right], \end{aligned} \quad (A29)$$

$$k_0 = \bar{a}_0\frac{\mathbf{k}\cdot\mathbf{p}}{p_0} + (1 - \bar{a}_0)\frac{\mathbf{k}\cdot\mathbf{q}}{q_0}. \quad (A30)$$

The inverse of Eq. (A26) for the classical result is

$$V(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} V(\mathbf{k}), \quad (A31)$$

and we shall put [as in Eq. (A19)]

$$V(\mathbf{r}) = V_1(\mathbf{r}) + V_{S1}(\mathbf{r}). \quad (A32)$$

In Ref. 10, we considered the one-gravitation exchange interaction between a spin-0 particle of mass m_1 and a spin-0 particle of mass m_2. The result [Eq. (5) of Ref. 10] corresponding to Eq. (A20) was not identical, but the classical result [Eq. (8) of Ref. 10] corresponding to Eq. (A27) was identical. The post-Newtonian result (i.e., result to order c^{-2}) for $V_1(\mathbf{r})$ is given by Eq. (33) of Ref. 10. Because of the choice of k_0^2 in the form of Eq. (A29), the post-Newtonian result for $V_1(\mathbf{r})$ is actually in the $\mathbf{r}_{*N}$ coordinate system as defined in Sec. II of this paper.

The post-Newtonian approximation to Eq. (A28) is

$$V_{S1}(\mathbf{k}) = -\frac{c^2\hbar^2\kappa^2}{4\mathbf{k}^2}\left[\frac{3\lambda_2}{4\lambda_1} i\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{k}\times\mathbf{p}) - i\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{k}\times\mathbf{q})\right]. \quad (A33)$$

Using Eq. (A1)–(A3) and letting $\frac{1}{2}\hbar\boldsymbol{\sigma}^{(1)} \to \mathbf{S}^{(1)}$ in Eq. (A33), we find, after using Eq. (A31) and

$$\frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\mathbf{k}}{\mathbf{k}^2} = \frac{i\mathbf{r}}{4\pi r^3}, \quad (A34)$$

that the post-Newtonian approximation to $V_{S1}(\mathbf{r})$ is

$$\begin{aligned} V_{S1}(\mathbf{r}) = &(G/c^2r^3)[(3m_2/2m_1)\mathbf{S}^{(1)}\cdot(\mathbf{r}\times\mathbf{P}_1) \\ &- 2\mathbf{S}^{(1)}\cdot(\mathbf{r}\times\mathbf{P}_2)]. \end{aligned} \quad (A35)$$

To obtain both $V_{S1}(\mathbf{r})$ and $V_{S2}(\mathbf{r})$, one could consider the one-graviton exchange interaction between a spin-$\frac{1}{2}$ particle of mass m_1 and a spin-$\frac{1}{2}$ particle of mass m_2. However, it is much easier to obtain $V_{S2}(\mathbf{r})$ from $V_{S1}(\mathbf{r})$ by letting $1\to 2$ and $2\to 1$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \to -\mathbf{r}$, which gives us

$$\begin{aligned} V_{S2}(\mathbf{r}) = &(G/c^2r^3)[-(3m_1/2m_2)\mathbf{S}^{(2)}\cdot(\mathbf{r}\times\mathbf{P}_2) \\ &+ 2\mathbf{S}^{(2)}\cdot(\mathbf{r}\times\mathbf{P}_1)]. \end{aligned} \quad (A36)$$

Finally, noting that $\mathbf{P}_N = m_N\mathbf{v}_N$ to first order, we see that Eqs. (A35) and (A36) are in agreement with Eqs. (3.1) and (3.2).

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