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# Gravitational Two-Body Problem with Acceleration-Dependent Spin Terms

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We generalize our previous work, on the gravitational two-body post-Newtonian Lagrangian with spin and parametrized post-Newtonian parameters $\gamma$ and $\beta$, by adding *acceleration-dependent spin terms* corresponding to an *arbitrary spin supplementary condition*. For the purpose of constructing the corresponding Hamiltonian we make use of our recently developed *method of the double zero*. Using this method, we remove the acceleration-dependent spin terms from the Lagrangian and, in the process, create new non-acceleration-dependent terms. Use of this new Lagrangian enables us to construct the Hamiltonian corresponding to the arbitrary spin supplementary condition. Energy constants of the motion are also discussed.

## 1. INTRODUCTION

In a previous work [1] we discussed the gravitational two-body post-Newtonian Lagrangian (and Hamiltonian) with spin and parametrized post-Newtonian (PPN) parameters $\gamma$ and $\beta$. We gave the Lagrangian (and Hamiltonian) in terms of center-of-mass Einstein–Infeld–Hoffmann (EIH) coordinates,[3] $\mathbf{r}_E$ and in terms of more general coordinates,[3] $\mathbf{r}_*$, where

$$\mathbf{r}_E = \mathbf{r}_*(1 - \alpha GM/c^2 r_*) \tag{1}$$

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[3] We have changed our notation slightly. The coordinates $\mathbf{r}_E$, $\mathbf{r}_*$, and $\mathbf{r}$ of our present paper correspond to $\mathbf{r}_{EIH;\gamma\beta}$, $\mathbf{r}$, and $\mathbf{r}_{(\lambda1,\lambda2)}$, respectively, of ref. [1]. See equations (5) and (43) of ref. [1], which correspond to equations (1) and (2), respectively, of our present paper.

and $\alpha$ is an arbitrary dimensionless parameter. We also considered the coordinate transformation [1]

$$\mathbf{r} = \mathbf{r}_* + \lambda_1 \frac{m_2 \mathbf{v}_* \times \mathbf{S}^{(1)}}{m_1 M c^2} + \lambda_2 \frac{m_1 \mathbf{v}_* \times \mathbf{S}^{(2)}}{m_2 M c^2} \tag{2}$$

due to a shift in the centers of mass of the two bodies corresponding to an arbitrary spin supplementary condition [1] determined by the arbitrary dimensionless parameters $\lambda_1$ and $\lambda_2$. In our original work on this subject [2], we also emphasized the fact that apparently different equations of motion appearing in the literature were in fact consistent with each other if one took into account the fact that they were expressed in different coordinates based on different *implicit* choices for the centers of mass of the spinning bodies, resulting from different choices for the spin supplementary condition. Using parameters $\lambda_1$ and $\lambda_2$, we were able to generalize [1] our original one-body considerations [2] to the two-body case and then to write down very general equations of motion that reduce to the various special cases found in the literature for special values of $\lambda_1$ and $\lambda_2$. By the same token, the corresponding Lagrangian and Hamiltonian that we derive in this paper will also be very general.

Explicitly, we used equation (2) to obtain the equations of motion[4] [1, 3] in terms of $\mathbf{r}$

$$\mathbf{a} + G\bar{M}\mathbf{r}/r^3 = \mathbf{B}^{(E)}(\alpha) + \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \mathbf{B}^{(\lambda 1)} + \mathbf{B}^{(\lambda 2)} + \mathbf{B}^{(1,2)} + \mathbf{B}^{(Q1)} + \mathbf{B}^{(Q2)} \tag{3}$$

from the equations of motion in terms of $\mathbf{r}_*$

$$\mathbf{a}_* + G\bar{M}\mathbf{r}_*/r_*^3 = \mathbf{B}^{(E)}(\alpha) + \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \mathbf{B}^{(1,2)} + \mathbf{B}^{(Q1)} + \mathbf{B}^{(Q2)} \tag{4}$$

We did *not*, however, write the Lagrangian in terms of $\mathbf{r}$.

In Section 2 of this paper we given the Lagrangian in terms of $\mathbf{r}$. This Lagrangian has two acceleration-dependent terms $V_{\lambda 1}$ and $V_{\lambda 2}$ proportional to $\lambda_1$ and $\lambda_2$, respectively. Some authors [4] have mistakenly thought it to be correct procedure to simplify a Lagrangian by using the lowest-order equations of motion in the highest-order terms of the

[4] We define $\mathbf{B}^{(\lambda 1)} \equiv \mathbf{B}^{(1)}(\lambda_1) - \mathbf{B}^{(1)}$ and $\mathbf{B}^{(\lambda 2)} \equiv \mathbf{B}^{(2)}(\lambda_2) - \mathbf{B}^{(2)}$; see equations (46) and (47) of ref. [1]. For completeness we have added the quadrupole terms $\mathbf{B}^{(Q1)}$ and $\mathbf{B}^{(Q2)}$ to the right-hand-side of equations (3) and (4); see equation (55) and (56) of Ref. [3]. The eight terms on the right-hand-side of Equation (3) correspond to the last eight terms on the right-hand-side of Equation (7), respectively. We have also added a bar over the $M$ to indicate a correction due to the Nordtvedt effect; see equation (14) and (25).

Lagrangian. If one uses the lowest-order equations of motion, $\mathbf{a}+GM\mathbf{r}/r^3=0$ in $V_{\lambda 1}$ and $V_{\lambda 2}$ they will vanish, and thus the equations of motion will be

$$\mathbf{a}+G\bar{M}\mathbf{r}/r^3=\mathbf{B}^{(E)}(\alpha)+\mathbf{B}^{(1)}+\mathbf{B}^{(2)} + \mathbf{B}^{(1,2)}+\mathbf{B}^{(Q1)}+\mathbf{B}^{(Q2)} \tag{5}$$

which are *clearly inconsistent* with equation (3).

The above inconsistency has been completely clarified by the recent work of Schäfer [5] who showed that using the lowest-order equations of motion in the highest-order terms of a Lagrangian is equivalent to carrying out a coordinate transformation (i.e., during this procedure and *implicit* coordinate transformation has been made). Clearly the *implicit* coordinate transformation made in obtaining equation (5) was to go from $\mathbf{r}$ coordinates back to $\mathbf{r}_*$ coordinates. We would like to eliminate acceleration-dependent terms from the Lagrangian and do it in such a way that there are no implicit coordinate [5, 6] or time [6] transformations taking place. We have already developed such a procedure (prior to the work of Schäfer [5]) and have called it the *method of the double zero* [6–9]. This method was subsequently used to eliminate the acceleration-dependent terms in the two-body post-post-Newtonian electromagnetic Lagrangian [9], and in Section 3 of this paper we use it to eliminate the acceleration-dependent terms $V_{\lambda 1}$ and $V_{\lambda 2}$ from the Lagrangian. In the process of eliminating the $V_{\lambda 1}$ and $V_{\lambda 2}$ terms, new non-acceleration-dependent terms $V'_{\lambda 1}$ and $V'_{\lambda 2}$ are created. Having removed the acceleration-dependent terms from the Lagrangian we then go on to obtain the corresponding Hamiltonian in Section 4.

In Section 5 we discuss the equations of motion and the energy constants derived from them, while in Section 6 the equations of motion of spin and the energy constants derived from them are presented. In Section 7 we discuss the total conserved energy and show that it is a sum of quantities that are independently conserved. We present our conclusions in Section 8.

## 2. ACCELERATION-DEPENDENT LAGRANGIAN

If one starts with the total two-body post-Newtonian Lagrangian [1] with spin and PPN parameters $\gamma$ and $\beta$ in center-of-mass EIH coordinates $\mathbf{r}_E$ and then makes the coordinate transformation

$$\mathbf{r}_E=\mathbf{r}\left(1-\alpha\frac{GM}{c^2 r}\right)-\sum_{N=1}^{2}\lambda_N\frac{\mu\mathbf{v}\times\mathbf{S}^{(N)}}{m_N^2c^2} \tag{6}$$

to the more general coordinates $\mathbf{r}$, one obtains the total Lagrangian (with acceleration-dependent spin terms)

$$\begin{aligned}\mathscr{L}_t(\alpha, \lambda_1, \lambda_2) = \sum_{N=1}^{2} &\left[-m_{0N}c^2 + \tfrac{1}{2}I^{(N)}\omega^{(N)2} + \tfrac{1}{8}J^{(N)}\omega^{(N)4}/c^2\right] \\ &+ \mathscr{L}(\alpha) - V_{S1} - V_{S2} - V_{\lambda 1} - V_{\lambda 2} \\ &- V_{S1,S2} - V_{Q1} - V_{Q2} \end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathscr{L}(\alpha) = &\frac{1}{2}\mu v^2 + \frac{1}{8}(1 - 3\mu/M)\mu v^4/c^2 + Gm_1^G m_2^G/r \\ &+ \left(\frac{1}{2} + \gamma - \alpha + \frac{\mu}{2M}\right)\frac{GM\mu v^2}{c^2 r} \\ &+ \left(\alpha + \frac{\mu}{2M}\right)\frac{GM\mu(\mathbf{v}\cdot\mathbf{r})^2}{c^2 r^3} + (1 - 2\beta + 2\alpha)\frac{G^2M^2\mu}{2c^2r^2}\end{aligned} \tag{8}$$

$$V_{SN} = \frac{G}{c^2r^3}\left[\gamma + 1 + \left(\gamma + \frac{1}{2}\right)\frac{m_1 m_2}{m_N^2}\right]\mathbf{S}^{(N)}\cdot(\mathbf{r}\times\mu\mathbf{v}) \tag{9}$$

$$V_{\lambda N} = -C_N\mathbf{S}^{(N)}\cdot\left[\left(\mathbf{a} + \frac{GM}{r^3}\mathbf{r}\right)\times\mu\mathbf{v}\right] \tag{10}$$

$$V_{S1,S2} = \frac{(\gamma+1)G}{2c^2r^3}\left[\frac{3(\mathbf{S}^{(1)}\cdot\mathbf{r})(\mathbf{S}^{(2)}\cdot\mathbf{r})}{r^2} - \mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}\right] \tag{11}$$

$$V_{QN} = \frac{Gm_1m_2\Delta I^{(N)}}{2m_N r^3}\left[\frac{3(\mathbf{n}^{(N)}\cdot\mathbf{r})^2}{r^2} - 1\right] \tag{12}$$

$$C_N \equiv \frac{\lambda_N\mu}{m_N^2c^2} \qquad \mathbf{S}^{(N)} \equiv I^{(N)}\boldsymbol{\omega}^{(N)} \qquad \mathbf{n}^{(N)} \equiv \frac{\mathbf{S}^{(N)}}{S^{(N)}} \tag{13}$$

$$\mu \equiv m_1m_2/M \qquad M \equiv m_1 + m_2 \qquad \bar{M} \equiv m_1^G m_2^G/\mu \tag{14}$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \qquad \mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2 \tag{15}$$

$$I^{(N)} = \sum_i m_{0Ni}r_{Ni}''^2 \qquad J^{(N)} = \sum_i m_{0Ni}r_{Ni}''^4 \tag{16}$$

and $N$ always is 1 or 2. We also note that $m_{0N}$ is the mass of body $N$ in its rest frame when it is not spinning, $\boldsymbol{\omega}^{(N)}$ is the angular velocity of body $N$ as measured in the system where the bodies are moving, $m_{0Ni}$ is the rest mass of the $i$th particle in body $N$, $r''_{Ni}$ is the distance of the $i$th particle in body $N$ form its axis of rotation, $\mathbf{r}_N$, $\mathbf{v}_N$, and $\mathbf{a}_N$ are the position, velocity, and acceleration, respectively, of body $N$, $G$ is Newton's contant of gravitation, $c$ is the speed of light, and $\Delta I^{(N)}$ in the quadrupole term $V_{QN}$ is [3, 10, 11], the moment of inertia of body $N$ about its polar axis minus the moment of intertia of body $N$ about its equatorial axis.

The angular velocity $\boldsymbol{\omega}^{(N)}$ can be expressed in terms of the Euler angles $\phi^{(N)}$, $\theta^{(N)}$, and $\psi^{(N)}$ and their time derivatives as [12]

$$\omega_x^{(N)} = \dot{\theta}^{(N)} \cos \phi^{(N)} + \dot{\psi}^{(N)} \sin \theta^{(N)} \sin \phi^{(N)} \tag{17a}$$

$$\omega_y^{(N)} = \dot{\theta}^{(N)} \sin \phi^{(N)} - \dot{\psi}^{(N)} \sin \theta^{(N)} \cos \phi^{(N)} \tag{17b}$$

$$\omega_z^{(N)} = \dot{\psi}^{(N)} \cos \theta^{(N)} + \dot{\phi}^{(N)} \tag{17c}$$

and the unit vector $\mathbf{n}^{(N)}$ can be expressed as [10]

$$\mathbf{n}^{(N)} = (\sin \theta^{(N)} \sin \phi^{(N)}, -\sin \theta^{(N)} \cos \phi^{(N)}, \cos \theta^{(N)}) \tag{18}$$

We find it usefull to define

$$q_1^{(N)} \equiv \phi^{(N)} \qquad q_2^{(N)} \equiv \theta^{(N)} \qquad q_3^{(N)} \equiv \psi^{(N)} \tag{19}$$

The gravitational mass of body $N$ is given by [1, 13]

$$m_N^G = m_N + \eta\, U_N/c^2 \tag{20}$$

where $U_N$, the gravitational binding energy of body $N$ and $\eta$, the Nordtvedt parameter, are given by

$$U_N = -\frac{G}{2} \int \frac{\rho_N(\mathbf{R})\, \rho_N(\mathbf{R}')\, dV\, dV'}{|\mathbf{R} - \mathbf{R}'|} \qquad \eta = 4\beta - \gamma - 3 \tag{21}$$

and $\rho_N$ is the mass density of body $N$. We thus have

$$\bar{M} = M \left(1 + \frac{\eta U_1}{m_1 c^2} + \frac{\eta U_2}{m_2 c^2} + \frac{\eta^2 U_1 U_2}{m_1 m_2 c^4}\right) \tag{22}$$

The terms $\frac{1}{2}\mu v^2$ and $G m_1^G m_2^G / r$ on the right-hand-side of equation (8) as well as $\bar{M}$ have *hidden rotational kinetic energy terms*[5] in the factor $m_N$, the relativistic mass of body $N$ in its rest frame. We have (from Appendix A)

$$m_N = m_{0N} + \frac{1}{2} I^{(N)} \omega^{(N)2} / c^2 \tag{23}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \mu_0 + \sum_{N=1}^{2} \frac{1}{2} I^{(N)} \omega^{(N)2} \frac{\mu_0^2}{m_{0N}^2 c^2} \tag{24}$$

where $\mu_0 = m_{01} m_{02} / M_0$ and $M_0 = m_{01} + m_{02}$.

[5] This was not mentioned in Ref. [1]; see equation (22) or Ref. [3], equation (38). Compare these two equations with equation (7) of our present paper.

We also have the post-Newtonian approximation

$$\bar{M} = M + M_0\left(\frac{\eta U_1}{m_{01}c^2} + \frac{\eta U_2}{m_{02}c^2}\right) \tag{25}$$

$$\frac{Gm_1^G m_2^G}{r} \equiv \frac{G\bar{M}\mu}{r} = \frac{G(m_{01}m_2^G + m_1^G m_{02} - m_{01}m_{02})}{r} \tag{26}$$

In the post-Newtonian terms on the right-hand-side of equation (7) we can set $m_N = m_{0N}$.

In the $\mathbf{r}_*$ coordinate system we have

$$\mathscr{L}_I(\alpha) = \mathscr{L}_{\text{free}}(\mathbf{r}_*, \mathbf{v}_*, q_j^{(1)}, \dot{q}_j^{(1)}, q_k^{(2)}, \dot{q}_l^{(2)}) - V(\alpha) \tag{27}$$

$$\mathscr{H}_I(\alpha) = \mathscr{H}_{\text{free}}(\mathbf{r}_*, \mathbf{P}_*, q_j^{(1)}, p_j^{(1)}, q_k^{(2)}, p_l^{(2)}) + V(\alpha) \tag{28}$$

where $\mathbf{P}_* = \partial\mathscr{L}_I(\alpha)/\partial\mathbf{v}_*$ and $p_i^{(N)} = \partial\mathscr{L}_I(\alpha)/\partial\dot{q}_i^{(N)}$. The potential energy term $V(\alpha)$ expressed as a function of coordinates and momentum, can be derived [1, 3, 12, 14] from quantum field theory. One then adds $\mathscr{H}_{\text{free}}$ (see Appendix A) to $V(\alpha)$ to obtain $\mathscr{H}_I(\alpha)$. From $\mathscr{H}_I(\alpha)$ we can then obtain $\mathscr{L}_I(\alpha)$. If we then make the coordinate transformation of equation (2) we can obtain from equation (27)

$$\mathscr{L}_I(\alpha, \lambda_1, \lambda_2) = \mathscr{L}_{\text{free}}(\mathbf{r}, \mathbf{v}, q_j^{(1)}, \dot{q}_j^{(1)}, q_k^{(2)}, \dot{q}_l^{(2)}) - V(\alpha, \lambda_1, \lambda_2) \tag{29}$$

which is just another way of writing equation (7). Because $V(\alpha, \lambda_1, \lambda_2)$ is *acceleration-dependent* we cannot obtain the Hamiltonian corresponding to equation (29) in the usual way. Because $V(\alpha, \lambda_1, \lambda_2)$ has only *linear* acceleration terms it is *not* even possible to obtain an Ostrogradsky Hamiltonian.[6]

## 3. REMOVAL OF ACCELERATION TERMS

We remove [6–9] the acceleration-dependent terms from $\mathscr{L}_I(\alpha, \lambda_1, \lambda_2)$ by adding *double-zero terms* and *total time-derivative terms* to $-V_{\lambda N}$. First let us consider the double-zero term

$$(ZZ)_N \equiv C_N[\mathbf{S}^{(0N)} - \mathbf{S}^{(N)}] \cdot \left[\left(\mathbf{a} + \frac{GM}{r^3}\mathbf{r}\right) \times \mu\mathbf{v}\right] \tag{30}$$

[6] See paragraph before conclusion in first item of Ref. [9].

where the lowest-order equations of motions are $\mathbf{a}+GM\mathbf{r}/r^3=0$ and $\dot{\mathbf{S}}^{(N)}=0$. Thus, $\mathbf{S}^{(N)}=\mathbf{S}^{(0N)}$ where $\mathbf{S}^{(0N)}$ is to be regarded as a constant. We then have

$$-V_{\lambda N}+(ZZ)_N=C_N\mathbf{S}^{(0N)}\cdot\left[\left(\mathbf{a}+\frac{GM}{r^3}\mathbf{r}\right)\times\mu\mathbf{v}\right] \quad (31)$$

It is not difficult to show that

$$\mathbf{a}\times\mu\mathbf{v}\equiv-\frac{d}{dt}\left(\frac{(\mathbf{v}\cdot\mathbf{r})}{r^2}\mathbf{L}\right)+\left(\frac{v^2}{r^2}-\frac{2(\mathbf{v}\cdot\mathbf{r})^2}{r^4}\right)\mathbf{L}$$
$$+\frac{2}{r^2}(\mathbf{a}\cdot\mathbf{r})\,\mathbf{L}-\frac{1}{r^2}(\mathbf{a}\cdot\mathbf{L})\,\mathbf{r} \quad (32)$$

where $\mathbf{L}\equiv\mathbf{r}\times\mu\mathbf{v}$.To first order $\dot{\mathbf{L}}=0$ and, thus, $\mathbf{L}=\mathbf{L}_0$ where $\mathbf{L}_0$ is to be regarded as a constant. Let us consider the double-zero term

$$(\mathbf{ZZ})\equiv\frac{2}{r^2}\left[\left(\mathbf{a}+\frac{GM}{r^3}\mathbf{r}\right)\cdot\mathbf{r}\right](\mathbf{L}_0-\mathbf{L})$$
$$-\frac{1}{r^2}\left[\left(\mathbf{a}+\frac{GM}{r^3}\mathbf{r}\right)\cdot(\mathbf{L}_0-\mathbf{L})\right]\mathbf{r} \quad (33)$$

and the total time-derivative term

$$(\mathbf{TTD})\equiv\frac{d}{dt}\left(\frac{(\mathbf{v}\cdot\mathbf{r})}{r^2}\mathbf{L}-\frac{2(\mathbf{v}\cdot\mathbf{r})}{r^2}\mathbf{L}_0+\frac{(\mathbf{v}\cdot\mathbf{L}_0)}{r^2}\mathbf{r}\right) \quad (34)$$

We then find that

$$\left(\mathbf{a}+\frac{GM}{r^3}\mathbf{r}\right)\times\mu\mathbf{v}+(\mathbf{ZZ})+(\mathbf{TTD})$$
$$\equiv\left[\frac{v^2}{r^2}-\frac{GM}{r^3}-\frac{2(\mathbf{v}\cdot\mathbf{r})^2}{r^4}\right](\mathbf{L}-2\mathbf{L}_0)$$
$$-\left[\frac{GM}{r^5}(\mathbf{r}\cdot\mathbf{L}_0)+\frac{2(\mathbf{v}\cdot\mathbf{r})}{r^4}(\mathbf{v}\cdot\mathbf{L}_0)\right]\mathbf{r}+\frac{1}{r^2}(\mathbf{v}\cdot\mathbf{L}_0)\,\mathbf{v} \quad (35)$$

We next define

$$-V'_{\lambda N}\equiv-V_{\lambda N}+(ZZ)_N+C_N\mathbf{S}^{(0N)}\cdot[(\mathbf{ZZ})+(\mathbf{TTD})]$$
$$=C_N\left(\frac{v^2}{r^2}-\frac{GM}{r^3}-\frac{2(\mathbf{v}\cdot\mathbf{r})^2}{r^4}\right)\mathbf{S}^{(0N)}\cdot(\mathbf{L}-2\mathbf{L}_0)$$
$$-C_N\left[\frac{GM}{r^5}(\mathbf{r}\cdot\mathbf{L}_0)+\frac{2(\mathbf{v}\cdot\mathbf{r})}{r^4}(\mathbf{v}\cdot\mathbf{L}_0)\right]\mathbf{S}^{(0N)}\cdot\mathbf{r}$$
$$+\frac{C_N}{r^2}(\mathbf{v}\cdot\mathbf{L}_0)(\mathbf{S}^{(0N)}\cdot\mathbf{v}) \quad (36)$$

and note that $C_N \mathbf{S}^{(0N)} \cdot \textbf{(TTD)}$ is a total time derivative because $C_N \mathbf{S}^{(0N)}$ is a constant. Our total Lagrangian, thus, becomes

$$\mathscr{L}'_t(\alpha, \lambda_1, \lambda_2) = \sum_{N=1}^{2} \left[ -m_{0N}c^2 + \frac{1}{2} I^{(N)}\omega^{(N)2} + \frac{1}{8} I^{(N)}\omega^{(N)4}/c^2 \right] + \mathscr{L}(\alpha) - V_{S1} - V_{S2} - V'_{\lambda 1} - V'_{\lambda 2} - V_{S1,S2} - V_{Q1} - V_{Q2} \tag{37}$$

In summary, we have eliminated the acceleration-dependent terms $V_{\lambda 1}$ and $V_{\lambda 2}$ from the Lagrangian and, in the process, new non-acceleration-dependent terms $V'_{\lambda 1}$ and $V'_{\lambda 2}$ were created. The terms $V'_{\lambda 1}$ and $V'_{\lambda 2}$ contain the constants $\mathbf{S}^{(01)}$, $\mathbf{S}^{(02)}$, and $\mathbf{L}_0$ and, thus, the Lagrangian $\mathscr{L}'_t(\alpha, \lambda_1, \lambda_2)$ now depends on something other than the usual variables of the system. These constants are determined by the initial conditions for the problem. After the equations of motion have been obtained from $\mathscr{L}'_t(\alpha, \lambda_1, \lambda_2)$ one can then replace $\mathbf{S}^{(01)}$, $\mathbf{S}^{(02)}$, and $\mathbf{L}_0$ in the equations of motion by $\mathbf{S}^{(1)}$, $\mathbf{S}^{(2)}$, and $\mathbf{L}$, respectively.

Tulczyjew [15] has given a Lagrangian that contains acceleration-dependent spin terms. To avoid these terms he used a spin supplementary condition which in our notation is determined by setting $\lambda_1 = \lambda_2 = 0$. For this special case $V_{\lambda N}$ and $V'_{\lambda N}$ are identically zero. For a further discussion of Lagrangians with acceleration-dependent spin terms see Damour [16].

## 4. HAMILTONIAN

The Hamiltonian corresponding to the Lagrangian of equation (37) can be written as

$$\mathscr{H}'_t(\alpha, \lambda_1, \lambda_2) = \sum_{N=1}^{2} \left\{ m_{0N}c^2 + \frac{1}{2} a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} - \frac{J^{(N)}}{8 I^{(N)2} c^2} \left[ a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} \right]^2 \right\} + \mathscr{H}(\alpha) + V_{S1} + V_{S2} + V'_{\lambda 1} + V'_{\lambda 2} + V_{S1,S2} + V_{Q1} + V_{Q2} \tag{38}$$

where

$$\mathscr{H}(\alpha) = \frac{P^2}{2\mu} - \frac{1}{8}\left(1 - \frac{3\mu}{M}\right)\frac{P^4}{\mu^3 c^2} - \frac{G m_1^G m_2^G}{r} - \left(\frac{1}{2} + \gamma - \alpha + \frac{\mu}{2M}\right)\frac{GMP^2}{\mu c^2 r} - \left(\alpha + \frac{\mu}{2M}\right)\frac{GM(\mathbf{P}\cdot\mathbf{r})^2}{\mu c^2 r^3} - (1 - 2\beta + 2\alpha)\frac{G^2 M^2 \mu}{2c^2 r^2} \tag{39}$$

and (see equations (A3) and (A6) of Appendix A)

$$a_{ij}^{(N)-1}p_i^{(N)}p_j^{(N)}=\frac{1}{I^{(N)}}\left[p_1^{(N)2}\csc^2 q_2^{(N)}+p_2^{(N)2}+p_3^{(N)2}\csc^2 q_2^{(N)}\right.$$
$$\left.-2p_1^{(N)}p_3^{(N)}\cos q_2^{(N)}\csc^2 q_2^{(N)}\right] \tag{40}$$

The term $V_{ON}$ is exactly as given by equation (12), and the term $V'_{\lambda N}$ is given by equation (36) with $\mathbf{v}$ replaced by $\mathbf{P}/\mu$. The terms $V_{SN}$ and $V_{S1,S2}$ are given by equations (9) and (11) with $\mathbf{v}$ replaced by $\mathbf{P}/\mu$ and $\mathbf{S}^{(N)}\equiv I^{(N)}\boldsymbol{\omega}^{(N)}$ now replaced by

$$S_x^{(N)}=p_2^{(N)}\cos q_1^{(N)}+(p_3^{(N)}-p_1^{(N)}\cos q_2^{(N)})\csc q_2^{(N)}\sin q_1^{(N)} \tag{41a}$$

$$S_y^{(N)}=p_2^{(N)}\sin q_1^{(N)}-(p_3^{(N)}-p_1^{(N)}\cos q_2^{(N)})\csc q_2^{(N)}\cos q_1^{(N)} \tag{41b}$$

$$S_z^{(N)}=p_1^{(N)} \tag{41c}$$

We also note that

$$\mathbf{P}=\frac{\partial\mathscr{L}'_i(\alpha,\lambda_1,\lambda_2)}{\partial\mathbf{v}} \qquad p_i^{(N)}=\frac{\partial\mathscr{L}'_i(\alpha,\lambda_1,\lambda_2)}{\partial\dot q_i^{(N)}} \tag{42}$$

$$\mathbf{v}=\frac{\partial\mathscr{H}'_i(\alpha,\lambda_1,\lambda_2)}{\partial\mathbf{P}} \qquad \dot q_i^{(N)}=\frac{\partial\mathscr{H}'_i(\alpha,\lambda_1,\lambda_2)}{\partial p_i^{(N)}} \tag{43}$$

$$\dot{\mathbf{P}}=-\frac{\partial\mathscr{H}'_i(\alpha,\lambda_1,\lambda_2)}{\partial\mathbf{r}} \qquad \dot p_i^{(N)}=-\frac{\partial\mathscr{H}'_i(\alpha,\lambda_1,\lambda_2)}{\partial q_i^{(N)}} \tag{44}$$

Equations (41a, b, c) are correct to lowest order, which is all that is required. These equations are most easily derived by using $\dot q_i^{(N)}=a_{ij}^{(N)-1}p_j^{(N)}$, correct to lowest order, in equations (17a, b, c).

Again we must consider the *hidden rotational kinetic energy terms* in the factor $m_N$ contained in the terms $P^2/2\mu$ and $Gm_1^G m_2^G/r$ on the right-hand-side of equation (39). We have (see Appendix A)

$$m_N=m_{0N}+\tfrac{1}{2}a_{ij}^{(N)-1}p_i^{(N)}p_j^{(N)}/c^2 \tag{45}$$

$$\frac{1}{\mu}=\frac{1}{\mu_0}-\sum_{N=1}^{2}\frac{1}{2}a_{ij}^{(N)-1}p_i^{(N)}p_j^{(N)}\frac{1}{m_{0N}^2c^2} \tag{46}$$

## 5. ENERGY CONSTANTS FROM EQUATIONS OF MOTION

The equations of motion from a Lagrangian $\mathscr{L}=\mathscr{L}(\mathbf{r},\mathbf{v},\mathbf{a})$ can be written as

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{v}}\right)-\frac{\partial\mathscr{L}}{\partial\mathbf{r}}=\frac{d^2}{dt^2}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{a}}\right) \tag{47}$$

If one uses either $\mathscr{L}_i(\alpha, \lambda_1, \lambda_2)$ or $\mathscr{L}'_i(\alpha, \lambda_1, \lambda_2)$ in the above equation, one will obtain equation (3) *after* (i) dividing through by $\mu$, (ii) moving everything except $\mathbf{a} + G\bar{M}\mathbf{r}/r^3$ to the right-hand-side, (iii) using $\mathbf{a} = -GM\mathbf{r}/r^3$ and $\dot{\mathbf{S}}^{(N)} = 0$ in the right-hand-side (note that $\dot{\mu} = 0$ and $\dot{M} = 0$ to the post-Newtonian approximation), and (iv) replacing $\mathbf{L}_0$ and $\mathbf{S}^{(0N)}$ by $\mathbf{L}$ and $\mathbf{S}^{(N)}$, respectively. It should be noted that $\bar{M} = M$ plus a post-Newtonian correction [see equation (25)]. One could move the post-Newtonian part of $G\bar{M}\mathbf{r}/r^3$ to the right-hand-side, but it is more convenient not to do so.

Taking the dot product of equation (3) with $\mathbf{v}$ we obtain

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - G\bar{M}/r\right) = (\mathbf{a} + G\bar{M}\mathbf{r}/r^3)\cdot\mathbf{v} = \frac{d}{dt}\left(\bar{g}^{(E)}(\alpha) + g^{(\lambda 1)} + g^{(\lambda 2)} + g^{(1,2)} + g^{(Q1)} + g^{(Q2)}\right) \quad (48)$$

where

$$\mathbf{v}\cdot\mathbf{B}^{(E)}(\alpha) = \dot{\bar{g}}^{(E)}(\alpha) = \dot{g}^{(E)}(\alpha) \quad (49)$$

$$\mathbf{v}\cdot\mathbf{B}^{(N)} \equiv 0, \qquad \mathbf{v}\cdot\mathbf{B}^{(\lambda N)} = \dot{g}^{(\lambda N)} \quad (50)$$

$$\mathbf{v}\cdot\mathbf{B}^{(1,2)} = \dot{g}^{(1,2)} \qquad \mathbf{v}\cdot\mathbf{B}^{(QN)} = \dot{g}^{(QN)} \quad (51)$$

and

$$\bar{g}^{(E)}(\alpha) \equiv \bar{K}_0\frac{v^4}{c^4} + \bar{K}_1\frac{G^2M^2}{c^2r^2} + \bar{K}_2\frac{GMv^2}{c^2r} + \bar{K}_3\frac{GM(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^3} \quad (52)$$

$$g^{(E)}(\alpha) \equiv K_0\frac{v^4}{c^4} + K_1\frac{G^2M^2}{c^2r^2} + K_2\frac{GMv^2}{c^2r} + K_3\frac{GM(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^3} \quad (53)$$

$$\bar{K}_0 \equiv -\frac{3}{8} + \frac{9\mu}{8M} \qquad K_0 \equiv 0 \quad (54)$$

$$\bar{K}_1 \equiv K_1 - \frac{3}{2} + \frac{9\mu}{2M} \qquad K_1 \equiv 2 - \beta + \alpha - \frac{9\mu}{2M} \quad (55)$$

$$\bar{K}_2 \equiv K_2 + \frac{3}{2} - \frac{9\mu}{2M} \qquad K_2 \equiv -2 - \gamma + \alpha + \frac{4\mu}{M} \quad (56)$$

$$\bar{K}_3 = K_3 \qquad K_3 \equiv -\alpha - \frac{\mu}{2M} \quad (57)$$

$$g^{(\lambda N)} \equiv 2C_N\left(\frac{GM}{r^3}\right)\mathbf{S}^{(N)}\cdot(\mathbf{r}\times\mathbf{v}) \quad (58)$$

$$g^{(1,2)} \equiv -V_{S1,S2}/\mu \qquad g^{(QN)} \equiv -V_{QN}/\mu \quad (59)$$

From equations (48) and (49), we can construct the following two constants of the motion

$$\mathscr{E}(\alpha, \lambda_1, \lambda_2) \equiv \tfrac{1}{2}\mu v^2 - (G\bar{M}\mu/r) - \mu[\bar{g}^{(E)}(\alpha) + g^{(\lambda 1)} + g^{(\lambda 2)} + g^{(1,2)} + g^{(Q1)} + g^{(Q2)}] \tag{60}$$

$$\mathscr{K}(\alpha, \lambda_1, \lambda_2) \equiv \tfrac{1}{2}\mu v^2 - (G\bar{M}\mu/r) - \mu[g^{(E)}(\alpha) + g^{(\lambda 1)} + g^{(\lambda 2)} + g^{(1,2)} + g^{(Q1)} + g^{(Q2)}] \tag{61}$$

If we next define $\mathscr{E}(\alpha)$ as

$$\mathscr{E}(\alpha) \equiv \frac{\partial \mathscr{L}(\alpha)}{\partial \mathbf{v}} \cdot \mathbf{v} - \mathscr{L}(\alpha) \tag{62}$$

We find that

$$\mathscr{E}(\alpha) = \tfrac{1}{2}\mu v^2 - (G\bar{M}\mu/r) - \mu\bar{g}^{(E)}(\alpha) \tag{63}$$

We are thus led to define $\mathscr{K}(\alpha)$ as

$$\mathscr{K}(\alpha) \equiv \tfrac{1}{2}\mu v^2 - (G\bar{M}\mu/r) - \mu g^{(E)}(\alpha) \tag{64}$$

Clearly from equation (49), $\mathscr{E}(\alpha)$ and $\mathscr{K}(\alpha)$ can only differ by a constant, and it can easily be shown that

$$\mathscr{K}(\alpha) = \mathscr{E}(\alpha) - \frac{3}{2}\left(1 - \frac{3\mu}{M}\right)\left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right)^2 \Big/ \mu c^2 \tag{65}$$

Using equations (63) and (64) in (60) and (61) we obtain

$$\mathscr{E}(\alpha, \lambda_1, \lambda_2) = \mathscr{E}(\alpha) + V_{S1,S2} + V_{Q1} + V_{Q2} - \mu(g^{(\lambda 1)} + g^{(\lambda 2)}) \tag{66}$$

$$\mathscr{K}(\alpha, \lambda_1, \lambda_2) = \mathscr{K}(\alpha) + V_{S1,S2} + V_{Q1} + V_{Q2} - \mu(g^{(\lambda 1)} + g^{(\lambda 2)}) \tag{67}$$

Clearly $\mathscr{E}(\alpha, \lambda_1, \lambda_2)$ and $\mathscr{K}(\alpha, \lambda_1, \lambda_2)$ are but two out of an infinite number of possible constants of the motion that could be constructed in a similar way. It should also be noted that the constant that $\mathscr{E}(\alpha, \lambda_1, \lambda_2)$ and $\mathscr{K}(\alpha, \lambda_1, \lambda_2)$ differ by is not a *trivial* constant (like a rest energy) but is a function of $r$ and $v^2$. The constant of motion $\mathscr{E}(\alpha, \lambda_1, \lambda_2)$ is unique among the infinite possibilities due to the definition of equation (62). However, $g^{(E)}(\alpha)$, which leads to $\mathscr{K}(\alpha, \lambda_1, \lambda_2)$, has a simpler form than $\bar{g}^{(E)}(\alpha)$ because $K_0$ was chosen to be zero, and this makes $g^{(E)}(\alpha)$ the best form to use in certain relativistic celestial mechanics problems [17].

## 6. ENERGY CONSTANTS FROM EQUATIONS OF MOTION OF SPIN

Let[7] $\mathscr{L}_i(q_i^{(N)}, \dot{q}_j^{(N)})$ be the terms in $\mathscr{L}'_i(\alpha, \lambda_1, \lambda_2)$ that depend on $q_i^{(N)}$ and $\dot{q}_j^{(N)}$ with the same value of $N$. Note that $V'_{\lambda N}$ does not depend on either $q_i^{(N)}$ or $\dot{q}_j^{(N)}$, and hence ther will be no $\lambda_N$ dependence in $\mathscr{L}_i(q_i^{(N)}, \dot{q}_j^{(N)})$. We find that

$$\mathscr{L}_i(q_i^{(N)}, \dot{q}_j^{(N)}) = \frac{1}{2} I^{(N)} \omega^{(N)2} \left(1 + \frac{\mu_0^2 v^2}{2m_{0N}^2 c^2} + \frac{Gm_1 m_2}{c^2 r m_N}\right) + \frac{1}{8} J^{(N)} \omega^{(N)4}/c^2 - \mathbf{S}^{(N)} \cdot (\mathbf{\Omega}_{dS}^{(N)} + \mathbf{\Omega}_{LT}^{(N)}) - V_{QN} \tag{68}$$

where [1]

$$\mathbf{\Omega}_{dS}^{(N)} = \frac{G}{c^2 r^3}\left[\gamma + 1 + \left(\gamma + \frac{1}{2}\right)\frac{m_1 m_2}{m_N^2}\right]\mathbf{L} \tag{69}$$

$$\mathbf{\Omega}_{LT}^{(1)} = \frac{(\gamma+1)\,G}{2c^2 r^3}\left[\frac{3(\mathbf{S}^{(2)} \cdot \mathbf{r})\,\mathbf{r}}{r^2} - \mathbf{S}^{(2)}\right] \tag{70}$$

$$\mathbf{\Omega}_{LT}^{(2)} = \frac{(\gamma+1)\,G}{2c^2 r^3}\left[\frac{3(\mathbf{S}^{(1)} \cdot \mathbf{r})\,\mathbf{r}}{r^2} - \mathbf{S}^{(1)}\right] \tag{71}$$

We also note (to the lowest order) that $\mu_0 \mathbf{v} = \mathbf{P} = m_{01}\mathbf{v}_1 = -m_{02}\mathbf{v}_2$, and hence the factor $\mu_0^2 v^2/2m_{0N}^2 c^2$ in equation (68) can be replaced by $v_N^2/2c^2$.

The equations of motion of spin are

$$\frac{d}{dt}\left[\frac{\partial \mathscr{L}_i(q_j^{(N)}, \dot{q}_k^{(N)})}{\partial \dot{q}_i^{(N)}}\right] = \frac{\partial \mathscr{L}_i(q_j^{(N)}, \dot{q}_k^{(N)})}{\partial q_i^{(N)}} \tag{72}$$

and using $i = 1$ gives the $z$ component [12] of the following vector equation[8]

$$\frac{d}{dt}\left[I^{(N)}\boldsymbol{\omega}_0^{(N)} + \frac{1}{2} J^{(N)} \omega^{(N)2} \boldsymbol{\omega}^{(N)}/c^2 - I^{(N)}(\mathbf{\Omega}_{dS}^{(N)} + \mathbf{\Omega}_{LT}^{(N)})\right] = (\mathbf{\Omega}_{dS}^{(N)} + \mathbf{\Omega}_{LT}^{(N)} + \mathbf{\Omega}_{QN}^{(N)}) \times \mathbf{S}^{(N)} \tag{73}$$

[7] Note that $\mathscr{L}_i(q_i^{(N)}, \dot{q}_j^{(N)})$ stands for $\mathscr{L}_i(q_1^{(1)}, q_2^{(1)}, q_3^{(1)}, \dot{q}_1^{(1)}, \dot{q}_2^{(1)}, \dot{q}_3^{(1)})$ or $\mathscr{L}_i(q_1^{(2)}, q_2^{(2)}, q_3^{(2)}, \dot{q}_1^{(2)}, \dot{q}_2^{(2)}, \dot{q}_3^{(2)})$. An index $N$ is never implicitly summed over.

[8] If we had started with $\mathscr{L}_i(\alpha, \lambda_1, \lambda_2)$, then the terms in the equations of motion of spin coming from the $V_{\lambda N}$ term would be proportional to $\mathbf{a} + GM\mathbf{r}/r^3$, which could then be set equal to zero, thus giving the same final result.

where [3, 12]

$$\mathbf{\Omega}_{QN}^{(N)} = \frac{Gm_1 m_2 \Delta I^{(N)}}{S^{(N)} m_N r^3}\left[\frac{3(\mathbf{n}^{(N)}\cdot\mathbf{r})\,\mathbf{r}}{r^2} - \mathbf{n}^{(N)}\right] \tag{74}$$

$$\boldsymbol{\omega}_0^{(N)} = \boldsymbol{\omega}^{(N)}\left(1 + \frac{v_N^2}{2c^2} + \frac{Gm_1 m_2}{c^2 r m_N}\right) \tag{75}$$

We next define

$$\mathbf{S}_0^{(N)} \equiv I^{(N)}\boldsymbol{\omega}_0^{(N)} + \tfrac{1}{2}J^{(N)}\omega^{(N)2}\boldsymbol{\omega}^{(N)}/c^2 - I^{(N)}(\mathbf{\Omega}_{dS}^{(N)} + \mathbf{\Omega}_{LT}^{(N)}) \tag{76}$$

and obtain (correct to the post-Newtonian approximation)

$$\dot{\mathbf{S}}_0^{(N)} = (\mathbf{\Omega}_{dS}^{(N)} + \mathbf{\Omega}_{LT}^{(N)} + \mathbf{\Omega}_{QN}^{(N)}) \times \mathbf{S}_0^{(N)} \tag{77}$$

and, hence, $|\mathbf{S}_0^{(N)}|$ will remain constant. It then follows that $S_0^{(N)2}/2I^{(N)}$ and $J^{(N)}\omega^{(N)4}/8c^2 \equiv J^{(N)}S^{(N)4}/8I^{(N)4}c^2$ will be constants of motion with *dimensions of energy* correct to the post-Newtonian approximation.

Finally we remark that the Nordvedt parameter $\eta$ does not enter into the equations of motion of spin, equation (77), to the post-Newtonian approximation.

## 7. TOTAL CONSERVED ENERGY

The total conserved energy from a Lagrangian $\mathscr{L}_t = \mathscr{L}_t(q_i^{(1)}, \dot{q}_i^{(1)}, q_k^{(2)}, \dot{q}_l^{(2)}, \mathbf{r}, \mathbf{v}, \mathbf{a})$ can be written as[9]

$$\mathscr{E}_t = \left[\frac{\partial \mathscr{L}_t}{\partial \mathbf{v}} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}_t}{\partial \mathbf{a}}\right)\right]\cdot\mathbf{v} + \frac{\partial \mathscr{L}_t}{\partial \mathbf{a}}\cdot\mathbf{a}$$
$$+ \frac{\partial \mathscr{L}_t}{\partial \dot{q}_i^{(1)}}\dot{q}_i^{(1)} + \frac{\partial \mathscr{L}_t}{\partial \dot{q}_i^{(2)}}\dot{q}_i^{(2)} - \mathscr{L}_t \tag{78}$$

If one uses either $\mathscr{L}_t(\alpha, \lambda_1, \lambda_2)$ or $\mathscr{L}'_t(\alpha, \lambda_1, \lambda_2)$ in the above equation one obtains (*after* using $\mathbf{a} = -GM\mathbf{r}/r_3$, $\dot{\mathbf{S}}^{(N)} = 0$ and replacing $\mathbf{L}_0$ and $\mathbf{S}^{(0N)}$ by $\mathbf{L}$ and $\mathbf{S}^{(N)}$, respectively)

$$\mathscr{E}_t = \sum_{N=1}^{2}\left[m_{0N}c^2 + \frac{S_0^{(N)2}}{2I^{(N)}} - \frac{J^{(N)}S^{(N)4}}{8I^{(N)4}c^2}\right] + \mathscr{E}(\alpha, \lambda_1, \lambda_2) \tag{79}$$

It should be noted that each of the seven terms of equation (79) is separately conserved to the post-Newtonian approximation.

[9] See Sec. III of Ref. [8].

## 8. CONCLUSIONS

We have given the gravitational two-body post-Newtonian Lagrangian with PPN parameters $\gamma$ and $\beta$ and acceleration-dependent spin terms corresponding to an arbitrary spin supplementary condition. After removing the acceleration terms by our method of the double zero we were then able to construct the Hamiltonian. We have shown that the total conserved energy consists of seven separately conserved quantities. Two are rest energies, two come from the equations of motion of spin, and one comes from the equations of motion.

## APPENDIX A

In this appendix we wish to obtain the two-body results for $\mathscr{H}_{\text{free}}$ and $\mathscr{L}_{\text{free}}$. This is accomplished by giving the two-body generalization of the one-body results of equations (6), (9), and (10) of Ref. [12], which are, respectively, equation (16) of this paper

$$\mathscr{H}_{\text{free}} = \sum_{N=1}^{2} \left[ m_{0N} c^2 + \frac{1}{2} I^{(N)} \omega^{(N)2} + \frac{3}{8} I^{(N)} \omega^{(N)4} / c^2 + \frac{1}{2} \left( m_{0N} + \frac{3}{2} I^{(N)} \omega^{(N)2} / c^2 \right) v_N^2 + \frac{3}{8} m_{0N} v_N^4 / c^2 \right] \tag{A1}$$

$$\mathscr{L}_{\text{free}} = \sum_{N=1}^{2} \left[ -m_{0N} c^2 + \frac{1}{2} I^{(N)} \omega^{(N)2} + \frac{1}{8} I^{(N)} \omega^{(N)4} / c^2 + \frac{1}{2} \left( m_{0N} + \frac{1}{2} I^{(N)} \omega^{(N)2} / c^2 \right) v_N^2 + \frac{1}{8} m_{0N} v_N^4 / c^2 \right] \tag{A2}$$

Let us define $a_{ij}^{(N)} = a_{ji}^{(N)}$ such that

$$\begin{aligned} \tfrac{1}{2} a_{ij}^{(N)} \dot{q}_i^{(N)} \dot{q}_j^{(N)} &\equiv \tfrac{1}{2} I^{(N)} \omega^{(N)2} \\ &= \tfrac{1}{2} I^{(N)} \left( \dot{\phi}^{(N)2} + \dot{\theta}^{(N)2} + \dot{\psi}^{(N)2} + 2 \dot{\phi}^{(N)} \dot{\psi}^{(N)} \cos \theta^{(N)} \right) \end{aligned} \tag{A3}$$

The momenta $\mathbf{P}_N$ and $p_i^{(N)}$ are given by

$$\mathbf{P}_N = \partial \mathscr{L}_{\text{free}} / \partial \mathbf{v}_N \qquad p_i^{(N)} = \partial \mathscr{L}_{\text{free}} / \partial \dot{q}_i^{(N)} \tag{A4}$$

and in terms of the momenta we can express $\mathscr{H}_{\text{free}}$ as

$$\mathscr{H}_{\text{free}} = \sum_{N=1}^{2} \left[ m_{0N} c^2 + \frac{1}{2} a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} - \frac{I^{(N)}}{8 I^{(N)2} c^2} \left( a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} \right)^2 + \frac{P_N^2}{2 m_{0N}} \left( 1 - \frac{1}{2} a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} / m_{0N} c^2 \right) - \frac{P_N^4}{8 m_{0N}^3 c^2} \right] \tag{A5}$$

where $[a^{(N)-1}]_{ij} \equiv a_{ij}^{(N)-1} = a_{ji}^{(N)-1}$ and

$$\frac{1}{2} a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} = \frac{1}{2I^{(N)}} (p_1^{(N)2} \csc^2\theta^{(N)} + p_2^{(N)2} + p_3^{(N)2} \csc^2\theta^{(N)} - 2p_1^{(N)} p_3^{(N)} \cos\theta^{(N)} \csc^2\theta^{(N)}) \tag{A6}$$

The relativistic mass $m_N$ of body $N$ in its rest frame is given by

$$m_N = m_{0N} + \Delta m_N \tag{A7}$$

where

$$\Delta m_N = \tfrac{1}{2} I^{(N)} \omega^{(N)2}/c^2 = \tfrac{1}{2} a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)}/c^2 \tag{A8}$$

which is correct to first order. We also have

$$\frac{1}{m_{0N}} (1 - \Delta m_N/m_{0N}) = \frac{1}{m_{0N}(1 + \Delta m_N/m_{0N})} = \frac{1}{m_N} \tag{A9}$$

Using equation (A7) in (A2) and equation (A9) in (A5) gives us

$$\mathscr{L}_{\text{free}} = \sum_{N=1}^{2} \left( -m_{0N} c^2 + \frac{1}{2} I^{(N)} \omega^{(N)2} + \frac{1}{8} J^{(N)} \omega^{(N)4}/c^2 + \frac{1}{2} m_N v_N^2 + \frac{1}{8} m_N v_N^4/c^2 \right) \tag{A10}$$

$$\mathscr{H}_{\text{free}} = \sum_{N=1}^{2} \left[ m_{0N} c^2 + \frac{1}{2} a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)} - \frac{J^{(N)}}{8I^{(N)2} c^2} \times (a_{ij}^{(N)-1} p_i^{(N)} p_j^{(N)})^2 + \frac{P_N^2}{2m_N} - \frac{P_N^4}{8m_N^3 c^2} \right] \tag{A11}$$

correct to the post-Newtonian approximation. Note that we *cannot* simplify equation (A1) in the same way that we simplified (A2) because of the $\frac{3}{2}$ factor in (A1). If we wish to find the equations of motion we can use equations (A10) or (A11) without expanding out $m_N$ because $m_N$ is not a function of $\mathbf{v}_N$ or $\mathbf{P}_N$. However, if we wish to find the equations of motion of spin we should use equations (A2) or (A5) where $m_N$ has been expanded.

Let us now consider the *center-of-mass* results where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ (and thus $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$), and $\mathbf{P} = \partial \mathscr{L}_{\text{free}}/\mathbf{v} = \mathbf{P}_1 = -\mathbf{P}_2$. Using $P_N^2 = P^2$ in equation (A5) and simplifying as before gives us equation (A11) with $P_N^2$ replaced by

$P^2$. The Lagrangian corresponding to equation (A5) with $P_N^2$ replaced by $P^2$ is

$$\mathscr{L}_{\text{free}} = \sum_{N=1}^{2} \left[ -m_{0N}c^2 + \frac{1}{2} I^{(N)}\omega^{(N)2} + \frac{1}{8} J^{(N)}\omega^{(N)4}/c^2 + \frac{\mu_0^2 v^2}{2m_{0N}} \left( 1 + \frac{1}{2} I^{(N)}\omega^{(N)2}/m_{0N}c^2 \right) + \frac{\mu_0^4 v^4}{8m_{0N}^3 c^2} \right] \tag{A12}$$

where $\mu_0 = m_{01}m_{02}/M_0$ and $M_0 = m_{01} + m_{02}$. We also have

$$\begin{aligned} \sum_{N=1}^{2} \frac{\mu_0^2}{m_{0N}} \left( 1 + \frac{\Delta m_N}{m_{0N}} \right) &= \sum_{N=1}^{2} \frac{\mu_0^2 m_N}{m_{0N}^2} = \frac{m_1 m_{02}^2 + m_2 m_{01}^2}{(m_{01} + m_{02})^2} \\ &= \frac{m_1 m_2 (m_{02} - \Delta m_2) + m_2 m_1 (m_{01} - \Delta m_1)}{(m_1 + m_2)(m_{01} + m_{02} - \Delta m_1 - \Delta m_2)} \\ &= \frac{m_1 m_2}{(m_1 + m_2)} = \mu \end{aligned} \tag{A13}$$

$$\sum_{N=1}^{2} \mu_0^4/m_{0N}^3 = (1 - 3\mu_0/M_0)\,\mu_0 \tag{A14}$$

and thus equation (A12) simplifies to

$$\mathscr{L}_{\text{free}} = \sum_{N=1}^{2} \left( -m_{0N}c^2 + \frac{1}{2} I^{(N)}\omega^{(N)2} + \frac{1}{8} J^{(N)}\omega^{(N)4}/c^2 \right) + \frac{1}{2}\mu v^2 + \frac{1}{8}(1 - 3\mu/M)\,\mu v^4/c^2 \tag{A15}$$

correct to the post-Newtonian approximation.

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