

## RELATIVISTIC KEPLER'S THIRD LAW

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### ABSTRACT

We give a post-Newtonian generalization of Kepler's third law that includes spin and quadrupole moment effects, parameterized post-Newtonian parameters  $\gamma$  and  $\beta$ , and the Nordtvedt effect. The time from periastron to apastron is also found, and it turns out, in general, *not* to be equal to half the period. Our results are given in a very general coordinate system which is specified by three arbitrary dimensionless parameters.

*Subject headings:* pulsars — relativity — stars: binaries — stars: stellar dynamics

### I. INTRODUCTION

In a series of papers (Barker and O'Connell 1975, 1976, 1981, 1985, hereafter papers I–III and V, respectively; Barker, Byrd, and O'Connell 1982, hereafter Paper IV) on the gravitational two-body problem with spin, we have discussed the following topics: (1) the Lagrangian-Hamiltonian formalism and coordinate transformations (Papers I, II, V); (2) nonrelativistic (Papers I, IV, V) and relativistic (Papers II, III) quadrupole moment effects; (3) parameterized post-Newtonian (PPN) parameters  $\gamma$  and  $\beta$  and the Nordtvedt effect (Papers II, V); (4) equations of motion (Papers I, II, V) and constants of the motion (Paper V); (5) precession of the orbit (Papers I–III), precession of the spin (Papers I–V), and spin nutation (Paper IV). However, we have *not* yet discussed the topic of the *mean anomaly*. In § II of this paper we will consider the mean anomaly and obtain a relativistic generalization of Kepler's third law. In § III we will find the time from periastron to apastron, which, in general, turns out *not* to be equal to half the period. All our results are given to the post-Newtonian approximation and include spin and quadrupole moment effects, parameterized post-Newtonian (PPN) parameters  $\gamma$  and  $\beta$ , and the Nordtvedt effect. In § IV we give our conclusions.

We will be working in very general coordinates,<sup>1</sup>  $r$ , which are related to the center-of-mass Einstein-Infeld-Hoffmann (EIH) coordinates,  $r_E$ , by the coordinate transformation

$$r_E = r \left( 1 - \alpha \frac{GM}{c^2 r} \right) - \sum_{N=1}^2 \lambda_N \frac{\mu v \times S^{(N)}}{m_N^2 c^2}, \quad (1.1)$$

where  $\alpha$ ,  $\lambda_1$ , and  $\lambda_2$  are arbitrary dimensionless parameters,  $c$  is the speed of light,  $G$  is Newton's constant of gravitation,  $r = r_1 - r_2$ ,  $v = v_1 - v_2$ ,  $\mu = m_1 m_2 / M$ ,  $M = m_1 + m_2$ , and  $r_N$ ,  $v_N$ ,  $m_N$ , and  $S^{(N)}$  are the position, velocity, mass, and spin angular momentum, respectively, of body  $N$ . The equations of motion in the  $r$  coordinate system are (Papers I, II, V)

$$a + G\bar{M}r/r^3 = B, \quad (1.2)$$

where  $a = a_1 - a_2$ ,  $\bar{M} = m_1^G m_2^G / \mu$ , and  $a_N$  is the acceleration of body  $N$ . The gravitational mass (Will 1981) of body  $N$  is given by  $m_N^G = m_N + \eta U_N / c^2$ , where  $\eta = 4\beta - \gamma - 3$  is the Nordtvedt parameter and  $U_N$  is the gravitational binding energy of body  $N$ . The mass  $m_N$  that is used in calculating  $\bar{M}$  is given by (Paper V)  $m_N = m_{0N} + \frac{1}{2} I^{(N)} \omega^{(N)2} / c^2$ , where  $m_{0N}$  is the mass of body  $N$  in its rest frame when it is not spinning and  $I^{(N)}$  and  $\omega^{(N)}$  are the moment of inertia and angular velocity, respectively, of body  $N$ . Because we are working only to the post-Newtonian approximation (order  $c^{-2}$ ), the masses  $m_1$  and  $m_2$  used in equation (1.1) can be approximated by  $m_{01}$  and  $m_{02}$ , respectively. The  $B$  term on the right-hand side of equation (1.2) can be expressed as (Papers I, II, V)

$$B = B^{(E)}(\alpha) + B^{(1)} + B^{(2)} + B^{(\lambda_1)} + B^{(\lambda_2)} + B^{(1,2)} + B^{(Q1)} + B^{(Q2)}, \quad (1.3)$$

and sometimes it is convenient to set (Papers II, V)

$$B^{(N)}(\lambda_N) \equiv B^{(N)} + B^{(\lambda_N)}, \quad (1.4)$$

where throughout this paper  $N$  always is 1 or 2. The  $B^{(E)}(\alpha)$  term is the Einstein term and is independent of the spins and quadrupole moments of the two bodies. The  $B^{(N)}(\lambda_N)$  term and  $B^{(QN)}$  term depend on the spin and quadrupole moment, respectively, of body  $N$ . The  $B^{(1,2)}$  term depends on the spins of both bodies. In Appendix A we give expressions for the  $B$  terms along with the  $(r \cdot B)$  and  $(v \cdot B)$  terms.

<sup>1</sup> The coordinates  $r$  and  $r_E$  of our present paper and Paper V correspond to  $r_{(\lambda_1, \lambda_2)}$  and  $r_{\text{EIH}, \gamma\beta}$ , respectively, of Paper II.

If the  $\mathbf{B}$  term in equation (1.2) is set equal to zero, we have the following constants of the motion:

$$E/\mu \equiv \frac{1}{2}v^2 - G\bar{M}/r, \quad (1.5)$$

$$\mathbf{L}/\mu \equiv \mathbf{r} \times \mathbf{v}, \quad (1.6)$$

$$\mathbf{A}/\mu \equiv \mathbf{v} \times (\mathbf{r} \times \mathbf{v}) - G\bar{M}\mathbf{r}/r, \quad (1.7)$$

where  $E$ ,  $\mathbf{L}$ , and  $\mathbf{A}$  are the energy, orbital angular momentum, and Runge-Lenz vector, respectively. The semimajor axis,  $a$ , and the eccentricity,  $e$ , of the osculating ellipse can be obtained from

$$E/\mu = -G\bar{M}/2a, \quad (1.8)$$

$$L/\mu = (G\bar{M})^{1/2}a^{1/2}(1 - e^2)^{1/2}, \quad (1.9)$$

$$A/\mu = G\bar{M}e. \quad (1.10)$$

The mean motion,  $\bar{\omega}$ , is defined as

$$\bar{\omega} \equiv \left(\frac{G\bar{M}}{a^3}\right)^{1/2} = \frac{L/\mu}{a^2(1 - e^2)^{1/2}}. \quad (1.11)$$

The mean anomaly,  $\theta$ , is defined in terms of the eccentric anomaly,  $\xi$ , by the relation

$$\theta \equiv \xi - e \sin \xi, \quad (1.12)$$

where

$$\cos \xi \equiv \frac{\mathbf{r} \cdot \mathbf{n}^{(I)}}{a} + e, \quad \sin \xi \equiv \frac{\mathbf{r} \cdot \mathbf{n}^{(II)}}{a(1 - e^2)^{1/2}}, \quad (1.13)$$

and  $\mathbf{n}^{(I)}$ ,  $\mathbf{n}^{(II)}$ ,  $\mathbf{n}^{(III)}$  are an orthogonal set of unit vectors defined as

$$\mathbf{n}^{(I)} \equiv \mathbf{n}^{(A)} \equiv \frac{\mathbf{A}}{A}, \quad \mathbf{n}^{(II)} \equiv \frac{\mathbf{L} \times \mathbf{A}}{LA}, \quad \mathbf{n}^{(III)} \equiv \mathbf{n} \equiv \frac{\mathbf{L}}{L}. \quad (1.14)$$

For the case when  $\mathbf{B} = 0$  we have  $\theta = (t - T)\bar{\omega}$ , where  $T$  is the time when the bodies arrive at periastron. We shall set  $T = 0$ , henceforth. We are, thus, led to Kepler's third law in the form  $2\pi = \bar{\omega}P$ , where  $P$  is the period (i.e., time from periastron to periastron).

When the  $\mathbf{B}$  term in equation (1.2) is *not* set equal to zero, equations (1.5)–(1.14) still hold. Differentiating equations (1.5), (1.6), (1.7), and (1.12), we obtain

$$\dot{E}/\mu = \mathbf{v} \cdot \mathbf{B}, \quad (1.15)$$

$$\dot{\mathbf{L}}/\mu = \mathbf{r} \times \mathbf{B}, \quad (1.16)$$

$$\dot{\mathbf{A}}/\mu = \mathbf{v} \times (\mathbf{r} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{r} \times \mathbf{v}), \quad (1.17)$$

$$\dot{\theta} = \bar{\omega} + \dot{\sigma}_1, \quad (1.18)$$

where a dot denotes differentiation with respect to time and  $\dot{\sigma}_1$  is given by

$$\dot{\sigma}_1 = \bar{\omega} \left\{ \frac{[a(1 - e^2) - r]}{(G\bar{M})e^2} (\mathbf{r} \cdot \mathbf{B}) - \frac{a[a(1 - e^2) + r]}{(G\bar{M})^2e^2} (\mathbf{v} \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{B}) \right\}. \quad (1.19)$$

The  $\dot{\sigma}_1$  term is discussed further in Appendix B along with other useful celestial-mechanics results. Appendices C and D contain needed time-averaged results. Throughout this paper we shall assume  $e^2 \gg GM/c^2a$  and  $(1 - e)^2 \gg GM/c^2a$ , i.e., the orbit is not extremely close to being circular or extremely close to being parabolic. This avoids very small divisors in our equations that could vary substantially with time.

## II. INTEGRATION OF $\dot{\theta}$

Using equation (A17) of Appendix A in equation (B13) of Appendix B yields

$$\dot{\omega} = -\frac{3}{\bar{\omega}a^2} \dot{g}. \quad (2.1)$$

To carry out the integration of the above equation, let us define  $\theta_0 = \theta(t_0)$ ,  $\bar{\omega}_0 = \bar{\omega}(t_0)$ ,  $a_0 = a(t_0)$ ,  $E_0 = E(t_0)$ , and  $g_0 = g(t_0)$ . We then have

$$\begin{aligned} \bar{\omega} &= \bar{\omega}_0 + \int_{t_0}^t \dot{\omega} dt \\ &= \bar{\omega}_0 + \frac{3}{\bar{\omega}a^2} (g_0 - g). \end{aligned} \quad (2.2)$$

Integrating equation (1.18) from  $t_0$  to  $t$  gives us

$$\begin{aligned}\theta - \theta_0 &= (t - t_0)\bar{\omega}_0 + \int_{t_0}^t \left( \int_{t_0}^t \dot{\bar{\omega}} dt \right) dt + \int_{t_0}^t \dot{\sigma}_1 dt \\ &= (t - t_0)\bar{\omega}_0 + \frac{3}{\bar{\omega}a^2} \int_{t_0}^t (g_0 - g) dt + \int_{t_0}^t \dot{\sigma}_1 dt.\end{aligned}\quad (2.3)$$

It should be noted that  $g$  is a higher order term because it is a function of  $\mathbf{B}$ . Thus, when equation (2.1) was integrated, the  $-3/\bar{\omega}a^2$  factor could be regarded as a constant because we are working only to the post-Newtonian approximation.

Let us now set  $t = t_0 + P$ , where  $P$  is the period. Because  $g$  is a higher order term, we can set  $g(t_0 + P) = g_0$ . We then obtain, from equation (2.2),  $\bar{\omega}(t_0 + P) = \bar{\omega}_0$ , which implies that  $a(t_0 + P) = a_0$  and  $E(t_0 + P) = E_0$ . We conclude that to the post-Newtonian approximation there is no secular change in  $\bar{\omega}$ ,  $a$ , and  $E$  and, thus,  $P$  is a constant. From equation (2.3) we obtain (after setting  $t = t_0 + P$  and dividing through by  $P$ )

$$\frac{2\pi}{P} = \bar{\omega}_0 + \frac{3}{\bar{\omega}a^2} (g_0 - g_{av}) + \dot{\sigma}_{1av}, \quad (2.4)$$

where

$$g_{av} = \frac{1}{P} \int_{t_0}^{t_0+P} g dt, \quad \dot{\sigma}_{1av} = \frac{1}{P} \int_{t_0}^{t_0+P} \dot{\sigma}_1 dt, \quad (2.5)$$

and the above averages are independent of the value of  $t_0$ .

Sometimes equation (1.18) is replaced by

$$\dot{\theta} = \bar{\omega} + (t - t_0)\dot{\bar{\omega}} + \dot{\sigma}, \quad (2.6)$$

where  $\dot{\sigma} = \dot{\sigma}_1 - (t - t_0)\dot{\bar{\omega}}$ . Integrating equation (2.6) from  $t_0$  to  $t$  results in

$$\theta - \theta_0 = (t - t_0)\bar{\omega} + \int_{t_0}^t \dot{\sigma} dt. \quad (2.7)$$

From equation (2.7) we obtain (after setting  $t = t_0 + P$  and dividing through by  $P$ )

$$\frac{2\pi}{P} = \bar{\omega}_0 + \dot{\sigma}_{av(0)} \quad (2.8)$$

where

$$\begin{aligned}\dot{\sigma}_{av(0)} &= \frac{1}{P} \int_{t_0}^{t_0+P} \dot{\sigma} dt \\ &= \frac{3}{\bar{\omega}a^2} (g_0 - g_{av}) + \dot{\sigma}_{1av}.\end{aligned}\quad (2.9)$$

Note that  $\dot{\sigma}_{av(0)}$  does depend on the value of  $t_0$ . Equation (2.4) or the equivalent equation (2.8) is a relativistic generalization of Kepler's third law.

Periastron passage occurs at  $t = t_p = 0$ , and apastron passage occurs at  $t = t_a = \frac{1}{2}P + \delta t$ , where  $\delta t$  is a higher order term (see § III). Let us put  $t_0 = t_p$  and  $t = t_a$  in equation (2.2) and obtain

$$\bar{\omega}_a = \bar{\omega}_p + \frac{3}{\bar{\omega}a^2} (g_p - g_a), \quad (2.10)$$

which is equivalent to

$$a_a = a_p - \frac{2}{\bar{\omega}^2 a} (g_p - g_a). \quad (2.11)$$

Subscripts  $p$  and  $a$  always stand for periastron and apastron, respectively. Setting  $t_0$  equal to  $t_p$  and then  $t_a$  in equations (2.8) and 2.9 gives us, respectively (after using eq. [C17] of Appendix C),

$$\frac{2\pi}{P} = \bar{\omega}_p + \dot{\sigma}_{av(p)}, \quad (2.12)$$

$$\dot{\sigma}_{av(p)} = \frac{3g_p}{\bar{\omega}a^2} + \frac{GM\bar{\omega}}{c^2 a} \left( -2 - \gamma + \frac{7\mu}{2M} \right), \quad (2.13)$$

and

$$\frac{2\pi}{P} = \bar{\omega}_a + \dot{\sigma}_{av(a)}, \quad (2.14)$$

$$\dot{\sigma}_{\text{av}}(a) = \frac{3g_a}{\bar{\omega}a^2} + \frac{GM\bar{\omega}}{c^2a} \left( -2 - \gamma + \frac{7\mu}{2M} \right). \quad (2.15)$$

We can express  $g_p$  and  $g_a$  as (see eqs. [A13]–[A16])

$$g_p = g_p^{(E)}(\alpha) + g_p^{(1,2)} + \sum_{N=1}^2 (g_p^{(\lambda N)} + g_p^{(Q_N)}), \quad (2.16)$$

$$g_a = g_a^{(E)}(\alpha) + g_a^{(1,2)} + \sum_{N=1}^2 (g_a^{(\lambda N)} + g_a^{(Q_N)}). \quad (2.17)$$

We then find, using  $r_p = a(1 - e)$  and  $r_a = a(1 + e)$ , that

$$\frac{3g_p^{(E)}(\alpha)}{\bar{\omega}a^2} = \frac{3GM\bar{\omega}}{c^2a(1 - e)^2} \left[ \left( -\beta - \gamma + 2\alpha - \frac{\mu}{2M} \right) + \left( -2 - \gamma + \alpha + \frac{4\mu}{M} \right) e \right], \quad (2.18)$$

$$\frac{3g_p^{(\lambda N)}}{\bar{\omega}a^2} = \frac{6G\lambda_N m_1 m_2 (1 + e)^{1/2}}{c^2 a^3 m_N^2 (1 - e)^{5/2}} \mathbf{S}^{(N)} \cdot \mathbf{n}, \quad (2.19)$$

$$\frac{3g_p^{(1,2)}}{\bar{\omega}a^2} = \frac{3(\gamma + 1)G}{2c^2 a^5 \bar{\omega} \mu (1 - e)^3} [\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} - 3(\mathbf{S}^{(1)} \cdot \mathbf{n}^{(A)})(\mathbf{S}^{(2)} \cdot \mathbf{n}^{(A)})], \quad (2.20)$$

$$\frac{3g_p^{(Q_N)}}{\bar{\omega}a^2} = \frac{3GM\Delta I^{(N)}}{2a^5 \bar{\omega} m_N (1 - e)^3} [1 - 3(\mathbf{n}^{(N)} \cdot \mathbf{n}^{(A)})^2], \quad (2.21)$$

and  $g_a$  can be obtained from  $g_p$  by replacing  $e$  in  $g_p$  by  $-e$ .

Differentiating equation (1.1) with respect to time yields

$$\mathbf{v}_E = \mathbf{v} - \alpha \frac{GM}{c^2 r} \left( \mathbf{v} - \frac{(\mathbf{v} \cdot \mathbf{r})\mathbf{r}}{r^2} \right) - \sum_{N=1}^2 \lambda_N \frac{\mu \mathbf{a} \times \mathbf{S}^{(N)}}{m_N^2 c^2}, \quad (2.22)$$

and after setting  $\mathbf{a} = -GM\mathbf{r}/r^3$ , we obtain

$$\mathbf{v}_E \cdot \mathbf{r}_E = \mathbf{v} \cdot \mathbf{r} \left( 1 - \alpha \frac{GM}{c^2 r} \right). \quad (2.23)$$

If  $\mathbf{v}_E \cdot \mathbf{r}_E = 0$  (periastron and apastron in the EIH coordinate system), we must have  $\mathbf{v} \cdot \mathbf{r} = 0$  (periastron and apastron in our general coordinate system). We conclude that if the bodies are at periastron (apastron) in one coordinate system they are at periastron (apastron) in all coordinate systems defined by equation (1.1). Thus, while  $\theta = 0, \pi$  corresponds to  $\theta_E = 0, \pi$  we will *not* in general have  $\theta_E = \theta$ . A subscript E always indicates that results are in the EIH coordinate system.

From equations (2.12)–(2.15) and the same with  $\alpha = \lambda_1 = \lambda_2 = 0$ , we obtain

$$\bar{\omega}_{p,E} = \bar{\omega}_p + 3\alpha \frac{GM\bar{\omega}(2 + e)}{c^2 a(1 - e)^2} + \sum_{N=1}^2 \frac{3g_p^{(\lambda N)}}{\bar{\omega}a^2}, \quad (2.24)$$

$$\bar{\omega}_{a,E} = \bar{\omega}_a + 3\alpha \frac{GM\bar{\omega}(2 - e)}{c^2 a(1 + e)^2} + \sum_{N=1}^2 \frac{3g_a^{(\lambda N)}}{\bar{\omega}a^2}, \quad (2.25)$$

and from the above two equations we obtain

$$a_{p,E} = a_p - 2\alpha \left( \frac{GM}{c^2} \right) \frac{2 + e}{(1 - e)^2} - \sum_{N=1}^2 \frac{2g_p^{(\lambda N)}}{\bar{\omega}^2 a}, \quad (2.26)$$

$$a_{a,E} = a_a - 2\alpha \left( \frac{GM}{c^2} \right) \frac{2 - e}{(1 + e)^2} - \sum_{N=1}^2 \frac{2g_a^{(\lambda N)}}{\bar{\omega}^2 a}. \quad (2.27)$$

Thus, the semimajor axis depends on the coordinate system and on the mean anomaly, which in general also depends on the coordinate system.

### III. EVALUATION OF $\delta t$

Setting  $t_0 = t_p = 0$  in equation (2.3), we obtain

$$\theta = t\bar{\omega}_p + \frac{3}{\bar{\omega}a^2} \int_0^t (g_p - g) dt + \int_0^t \dot{\sigma}_1 dt. \quad (3.1)$$

Setting  $t = t_a = \frac{1}{2}P + \delta t$  in equation (3.1) and then multiplying the resulting equation by 2 gives us (to order required)

$$2\pi = P\bar{\omega}_p + 2(\delta t)\bar{\omega} + \frac{6}{\bar{\omega}a^2} \int_0^{P/2} (g_p - g) dt + 2 \int_0^{P/2} \dot{\sigma}_1 dt, \quad (3.2)$$

while setting  $t = P$  in equation (3.1) yields

$$2\pi = P\bar{\omega}_p + \frac{3}{\bar{\omega}a^2} \int_0^P (g_p - g)dt + \int_0^P \dot{\sigma}_1 dt. \quad (3.3)$$

Subtracting equation (3.3) from equation (3.2) and dividing by  $2\bar{\omega}P$  results in

$$\frac{\delta t}{P} = \frac{3}{2\bar{\omega}^2 a^2} \left( \frac{1}{P/2} \int_0^{P/2} g dt - \frac{1}{P} \int_0^P g dt \right) - \frac{1}{2\bar{\omega}} \left( \frac{1}{P/2} \int_0^{P/2} \dot{\sigma}_1 dt - \frac{1}{P} \int_0^P \dot{\sigma}_1 dt \right). \quad (3.4)$$

Let us put

$$\delta t = \delta t^{(E)}(\alpha) + \delta t^{(1)} + \delta t^{(2)} + \delta t^{(\lambda_1)} + \delta t^{(\lambda_2)} + \delta t^{(1,2)} + \delta t^{(Q1)} + \delta t^{(Q2)}, \quad (3.5)$$

where the  $\delta t$ 's on the right-hand side of equation (3.5) correspond, respectively, to the  $B$ 's on the right-hand side of equation (1.3). We have corresponding expansions for  $g$  (see eq. [A16] and note that  $g^{(N)} \equiv 0$ ) and for  $\dot{\sigma}_1$  (see eq. [C1]). Using equations (C22)–(C26), we find

$$\delta t^{(E)}(\alpha) = \delta t^{(N)} = \delta t^{(\lambda_N)} = 0. \quad (3.6)$$

After using equation (A15) in equation (C18), we obtain from equation (3.4)

$$\frac{\delta t^{(1,2)}}{P} = \frac{1}{2\bar{\omega}^2 a^2 e^2} \left[ \frac{1}{P/2} \int_0^{P/2} \left( 1 - e^2 - \frac{r}{a} \right) g^{(1,2)} dt - \frac{1}{P} \int_0^P \left( 1 - e^2 - \frac{r}{a} \right) g^{(1,2)} dt \right], \quad (3.7)$$

$$\frac{\delta t^{(QN)}}{P} = \frac{1}{2\bar{\omega}^2 a^2 e^2} \left[ \frac{1}{P/2} \int_0^{P/2} \left( 1 - e^2 - \frac{r}{a} \right) g^{(QN)} dt - \frac{1}{P} \int_0^P \left( 1 - e^2 - \frac{r}{a} \right) g^{(QN)} dt \right]. \quad (3.8)$$

Next, using equations (C27)–(C30) in equations (3.7) and (3.8), we obtain

$$\frac{\delta t^{(1,2)}}{P} = \frac{-(\gamma + 1)/2}{\pi e(1 - e^2)^{1/2}} \left( \frac{S^{(1)}S^{(2)}}{m_1 m_2 c^2 a^2} \right) [(\mathbf{n}^{(1)} \cdot \mathbf{n}^{(II)})(\mathbf{n}^{(2)} \cdot \mathbf{n}^{(III)}) + (\mathbf{n}^{(1)} \cdot \mathbf{n}^{(III)})(\mathbf{n}^{(2)} \cdot \mathbf{n}^{(II)})], \quad (3.9)$$

$$\frac{\delta t^{(QN)}}{P} = \frac{-1}{\pi e(1 - e^2)^{1/2}} \left( \frac{\Delta I^{(N)}}{m_N a^2} \right) [(\mathbf{n}^{(N)} \cdot \mathbf{n}^{(I)})(\mathbf{n}^{(N)} \cdot \mathbf{n}^{(II)})]. \quad (3.10)$$

Thus,  $\delta t$  is generally not equal to zero. The final result for  $\delta t$  does not depend on  $\alpha$ ,  $\lambda_1$ ,  $\lambda_2$  (i.e.,  $\delta t$  is independent of the coordinate system used). Clearly the time  $t_a - t_p = \frac{1}{2}P + \delta t$  had to be coordinate independent because the times of periastron and apastron passage are independent of the coordinate system.

#### IV. CONCLUSION

We have given a relativistic generalization of Kepler's third law (see eqs. [2.12]–[2.15]). Because we are using the *framework of the osculating ellipse* in traditional celestial mechanics  $E$ ,  $L$ , and  $A$  and, thus,  $a$  (and  $\bar{\omega}$ ) and  $e$  (see eqs. [1.5]–[1.19]) are *not* constants of the motion but will vary slightly as the mean anomaly goes from 0 to  $2\pi$ . From equation (2.11) we see that

$$a_a = a_p + O(GM/c^2) \quad (4.1)$$

and similarly<sup>2</sup>

$$e_a = e_p + O(GM/c^2 a). \quad (4.2)$$

An *alternative framework* has been given by Wagoner and Will (1976). They give post-Newtonian equations of motion (see their eq. [61]), for the case of general relativity without spin or quadrupole moment effects, that contain two constants of integration,  $p$  and  $e$ , which they designate "semi-latus rectum"<sup>3</sup> and "eccentricity," respectively. These constants are equivalent to the geometric  $p$  and  $e$  of an ellipse in the Newtonian limit. This framework leads to an  $E$  and  $L$  which are constant to the post-Newtonian approximation and can be expressed in terms of the constants  $p$  and  $e$  (see their eqs. [66] and [67]). Epstein (1977), in a paper considering post-Newtonian timing effects in the binary pulsar PSR 1913+16 (see Hulse and Taylor 1975; Taylor and Weisburg 1982), continues with the *alternative framework* and derives a relativistic Kepler's third law (see his eq. [A12]) where the periastron-to-periastron period is given as a function of the constants  $a$  and  $e$ . The advantages of the *osculating ellipse framework* are (1) that  $a$  and  $e$  have the standard meanings and are related to an ellipse, and (2) that results that include spin and quadrupole moment effects can readily be included because no modification to the classical celestial-mechanics formalism is necessary (i.e., spin and quadrupole moment effects can cause the orbit plane to change, but this is within the framework of classical celestial mechanics). One could also include spin and quadrupole moment effects in the *alternative framework*, but this would require an extension of the *alternative framework* formalism from solving differential equations in two dimensions to solving differential equations in three dimensions. The advantages of the *alternative framework* are (1) that  $a$  and  $e$  do not change as the bodies move along their trajectory, and (2) that it is the basis for software actually used to interpret observations of the binary pulsar. Thus, effects calculated within it can be most

<sup>2</sup> For the results needed in this paper we did not need to find the eccentricity version of eq. (2.11), which could be derived by integrating eq. (B14).

<sup>3</sup> One can then define a "semimajor axis,"  $a$ , as  $a \equiv p/(1 - e^2)$ , as is done in Epstein (1977).

readily compared with observations. Of course, one could also make a complete timing model based on the *osculating ellipse framework* of orbital analysis.

We have also found that  $\delta t \equiv (t_a - t_p) - P/2$  can have only nonzero contributions from the  $\delta t^{(Q1)}$ ,  $\delta t^{(Q2)}$ , and  $\delta t^{(1,2)}$  terms. Even these terms can be zero for various alignments of the unit vectors in equations (3.9) and (3.10). For example,  $\delta t^{(1,2)} = \delta t^{(Q1)} = 0$  if  $\mathbf{n}^{(1)}$  is perpendicular to the plane of the orbit. For the case of the binary pulsar PSR 1913+16, it appears that the pulsar's spin axis is perpendicular to the plane of the orbit (see Appendix B of Taylor and Weisberg 1982).

## APPENDIX A

The terms on the right-hand side of equation (1.3) are (Papers I, II, V)

$$\mathbf{B}^{(E)}(\alpha) = \frac{GM}{c^2 r^3} \left[ \left( 2\beta + 2\gamma - 2\alpha + \frac{2\mu}{M} \right) \frac{GM\mathbf{r}}{r} - \left( \gamma + \alpha + \frac{3\mu}{M} \right) v^2 \mathbf{r} + \left( 2 + 2\gamma - 2\alpha - \frac{2\mu}{M} \right) (\mathbf{v} \cdot \mathbf{r}) \mathbf{v} + \left( 3\alpha + \frac{3\mu}{2M} \right) \frac{(\mathbf{v} \cdot \mathbf{r})^2 \mathbf{r}}{r^2} \right], \quad (\text{A1})$$

$$\mathbf{B}^{(N)} = \frac{G}{c^2 r^5} \left[ 2\gamma + 2 + (2\gamma + 1) \frac{m_1 m_2}{m_N^2} \right] \left\{ \frac{3}{2} [\mathbf{S}^{(N)} \cdot (\mathbf{r} \times \mathbf{v})] \mathbf{r} + r^2 \mathbf{S}^{(N)} \times \mathbf{v} - \frac{3}{2} (\mathbf{v} \cdot \mathbf{r}) \mathbf{S}^{(N)} \times \mathbf{r} \right\}, \quad (\text{A2})$$

$$\mathbf{B}^{(\lambda N)} = \frac{3G\lambda_N}{c^2 r^5} \left( \frac{m_1 m_2}{m_N^2} \right) \{ (\mathbf{v} \cdot \mathbf{r}) \mathbf{r} \times \mathbf{S}^{(N)} - [\mathbf{S}^{(N)} \cdot (\mathbf{r} \times \mathbf{v})] \mathbf{r} \}, \quad (\text{A3})$$

$$\mathbf{B}^{(1,2)} = -\frac{3(\gamma + 1)G}{2c^2 r^5 \mu} [(\mathbf{S}^{(2)} \cdot \mathbf{r}) \mathbf{S}^{(1)} + (\mathbf{S}^{(1)} \cdot \mathbf{r}) \mathbf{S}^{(2)} - 5(\mathbf{S}^{(1)} \cdot \mathbf{r})(\mathbf{S}^{(2)} \cdot \mathbf{r}) \mathbf{r} / r^2 + (\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}) \mathbf{r}], \quad (\text{A4})$$

$$\mathbf{B}^{(QN)} = -\frac{3GM\Delta I^{(N)}}{2r^5 m_N} \left\{ \left[ 1 - \frac{5(\mathbf{n}^{(N)} \cdot \mathbf{r})^2}{r^2} \right] \mathbf{r} + 2(\mathbf{n}^{(N)} \cdot \mathbf{r}) \mathbf{n}^{(N)} \right\}, \quad (\text{A5})$$

where  $\mathbf{n}^{(N)} \equiv \mathbf{S}^{(N)}/S^{(N)}$  and  $\Delta I^{(N)}$  is (Papers I, III) the moment of inertia of body  $N$  about its polar axis minus the moment of inertia of body  $N$  about its equatorial axis. In the above  $\mathbf{B}$  terms we can set  $m_N \approx m_{0N}$  and  $\mathbf{S}^{(N)} \approx I^{(N)} \boldsymbol{\omega}^{(N)}$  because we are working only to the post-Newtonian approximation.

We will also need the  $\mathbf{r} \cdot \mathbf{B}$  terms

$$\begin{aligned} \mathbf{r} \cdot \mathbf{B}^{(E)}(\alpha) &= \frac{GM}{c^2 r} \left[ \left( 2\beta + 2\gamma - 2\alpha + \frac{2\mu}{M} \right) \frac{GM}{r} - \left( \gamma + \alpha + \frac{3\mu}{M} \right) v^2 + \left( 2 + 2\gamma + \alpha - \frac{\mu}{2M} \right) \frac{(\mathbf{v} \cdot \mathbf{r})^2}{r^2} \right] \\ &= \frac{GM}{c^2 r} \left[ \left( 4 + 2\beta + 4\gamma - 2\alpha - \frac{5\mu}{M} \right) \frac{GM}{r} - \left( 2 + \gamma - \frac{7\mu}{2M} \right) \frac{GM}{a} - \left( 2 + 2\gamma + \alpha - \frac{\mu}{2M} \right) \frac{L^2/\mu^2}{r^2} \right], \end{aligned} \quad (\text{A6})$$

$$\mathbf{r} \cdot \mathbf{B}^{(N)}(\lambda_N) = \frac{G}{c^2 r^3} \left[ \gamma + 1 + \left( \gamma + \frac{1}{2} - 3\lambda_N \right) \frac{m_1 m_2}{m_N^2} \right] \frac{\mathbf{S}^{(N)} \cdot \mathbf{L}}{\mu}, \quad (\text{A7})$$

$$\mathbf{r} \cdot \mathbf{B}^{(1,2)} = \frac{3(\gamma + 1)G}{2c^2 r^3 \mu} \left[ \frac{3(\mathbf{S}^{(1)} \cdot \mathbf{r})(\mathbf{S}^{(2)} \cdot \mathbf{r})}{r^2} - \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} \right], \quad (\text{A8})$$

$$\mathbf{r} \cdot \mathbf{B}^{(QN)} = -\frac{3GM\Delta I^{(N)}}{2r^3 m_N} \left[ 1 - \frac{3(\mathbf{n}^{(N)} \cdot \mathbf{r})^2}{r^2} \right], \quad (\text{A9})$$

and the  $\mathbf{v} \cdot \mathbf{B}$  terms (Paper V)

$$\mathbf{v} \cdot \mathbf{B}^{(E)}(\alpha) = \dot{g}^{(E)}(\alpha), \quad (\text{A10})$$

$$\dot{g}^{(N)} = \mathbf{v} \cdot \mathbf{B}^{(N)} \equiv 0, \quad \mathbf{v} \cdot \mathbf{B}^{(\lambda N)} = \dot{g}^{(\lambda N)}, \quad (\text{A11})$$

$$\mathbf{v} \cdot \mathbf{B}^{(1,2)} = \dot{g}^{(1,2)}, \quad \mathbf{v} \cdot \mathbf{B}^{(QN)} = \dot{g}^{(QN)}, \quad (\text{A12})$$

where

$$\begin{aligned} g^{(E)}(\alpha) &= \frac{GM}{c^2 r} \left[ \left( 2 - \beta + \alpha - \frac{9\mu}{2M} \right) \frac{GM}{r} + \left( -2 - \gamma + \alpha + \frac{4\mu}{M} \right) v^2 + \left( -\alpha - \frac{\mu}{2M} \right) \frac{(\mathbf{v} \cdot \mathbf{r})^2}{r^2} \right] \\ &= \frac{GM}{c^2 r} \left[ \left( -2 - \beta - 2\gamma + \alpha + \frac{5\mu}{2M} \right) \frac{GM}{r} + \left( 2 + \gamma - \frac{7\mu}{2M} \right) \frac{GM}{a} + \left( \alpha + \frac{\mu}{2M} \right) \frac{L^2/\mu^2}{r^2} \right], \end{aligned} \quad (\text{A13})$$

$$g^{(\lambda N)} = \frac{2G\lambda_N}{c^2 r^3} \left( \frac{m_1 m_2}{m_N^2} \right) \frac{\mathbf{S}^{(N)} \cdot \mathbf{L}}{\mu}, \quad (\text{A14})$$

$$g^{(1,2)} = -\frac{1}{3} \mathbf{r} \cdot \mathbf{B}^{(1,2)}, \quad g^{(QN)} = -\frac{1}{3} \mathbf{r} \cdot \mathbf{B}^{(QN)}, \quad (\text{A15})$$

and  $g^{(N)} \equiv 0$ . Let us define

$$g \equiv g^{(E)}(\alpha) + g^{(\lambda 1)} + g^{(\lambda 2)} + g^{(1,2)} + g^{(Q1)} + g^{(Q2)}, \quad (\text{A16})$$

so that

$$\mathbf{v} \cdot \mathbf{B} = \dot{g}. \quad (\text{A17})$$

## APPENDIX B

The time derivatives of  $\mathbf{L}$  and  $\mathbf{A}$  can be put in the form

$$\dot{\mathbf{L}} = \boldsymbol{\Omega}^* \times \mathbf{L} + \dot{\mathbf{L}}\mathbf{n}^{(III)}, \quad (\text{B1})$$

$$\dot{\mathbf{A}} = \boldsymbol{\Omega}^* \times \mathbf{A} + \dot{\mathbf{A}}\mathbf{n}^{(I)}, \quad (\text{B2})$$

where

$$\boldsymbol{\Omega}^* = \Omega_I^* \mathbf{n}^{(I)} + \Omega_{II}^* \mathbf{n}^{(II)} + \Omega_{III}^* \mathbf{n}^{(III)}, \quad (\text{B3})$$

$$\Omega_I^* = -(\mathbf{n}^{(III)} \cdot \dot{\mathbf{L}}/L), \quad (\text{B4})$$

$$\Omega_{II}^* = (\mathbf{n}^{(I)} \cdot \dot{\mathbf{L}}/L) = -(\mathbf{n}^{(III)} \cdot \dot{\mathbf{A}}/A), \quad (\text{B5})$$

$$\Omega_{III}^* = (\mathbf{n}^{(III)} \cdot \dot{\mathbf{A}}/A), \quad (\text{B6})$$

$$\dot{\mathbf{L}} = \mathbf{n}^{(III)} \cdot \dot{\mathbf{L}}, \quad \dot{\mathbf{A}} = \mathbf{n}^{(I)} \cdot \dot{\mathbf{A}}. \quad (\text{B7})$$

Using the results of equations (1.16) and (1.17), we obtain

$$\Omega_I^* = (\mu/L)(\mathbf{r} \cdot \mathbf{n}^{(I)})(\mathbf{B} \cdot \mathbf{n}^{(III)}), \quad (\text{B8})$$

$$\Omega_{II}^* = (\mu/L)(\mathbf{r} \cdot \mathbf{n}^{(III)})(\mathbf{B} \cdot \mathbf{n}^{(III)}), \quad (\text{B9})$$

$$\Omega_{III}^* = \bar{\omega} \left\{ \frac{a[a(1-e^2)+r]}{(GM)^2 e^2 (1-e^2)^{1/2}} (\mathbf{v} \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{B}) - \frac{[a(1-e^2)-r+2ae^2]}{(GM)e^2(1-e^2)^{1/2}} (\mathbf{r} \cdot \mathbf{B}) \right\}, \quad (\text{B10})$$

$$\dot{\mathbf{L}} = (\mu^2/L)[r^2(\mathbf{v} \cdot \mathbf{B}) - (\mathbf{v} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{B})], \quad (\text{B11})$$

$$\dot{\mathbf{A}} = (\mu/ae)\{[a^2(1-e^2)-r^2](\mathbf{v} \cdot \mathbf{B}) + (\mathbf{v} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{B})\}. \quad (\text{B12})$$

From equations (1.8), (1.11), and (1.15) we obtain

$$\dot{a} = \frac{2}{\bar{\omega}^2 a} (\mathbf{v} \cdot \mathbf{B}), \quad \dot{\bar{\omega}} = -\frac{3}{\bar{\omega} a^2} (\mathbf{v} \cdot \mathbf{B}), \quad (\text{B13})$$

and from equations (1.10), (1.11), and (B12) we obtain

$$\dot{e} = \frac{1}{\bar{\omega}^2 a^4 e} \{[a^2(1-e^2)-r^2](\mathbf{v} \cdot \mathbf{B}) + (\mathbf{v} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{B})\}. \quad (\text{B14})$$

We can also write  $\boldsymbol{\Omega}^*$  in the form

$$\boldsymbol{\Omega}^* = \dot{\Omega} \mathbf{n}_0 + \dot{\omega} \mathbf{n}^{(III)} + \dot{I}(\mathbf{n}_0 \times \mathbf{n}^{(III)})/|\mathbf{n}_0 \times \mathbf{n}^{(III)}|, \quad (\text{B15})$$

where  $\Omega$ ,  $\omega$ , and  $I$  denote the longitude of the ascending node, the argument of the periastron, and the inclination of the orbit, respectively. The unit vector  $\mathbf{n}_0$  is perpendicular to the plane of the sky and directed from the center of inertia of the binary system away from the Earth. We also define the unit vector  $\mathbf{n}_N$  in the plane of the sky directed from the center of inertia of the binary system toward the north. The angle between  $\mathbf{n}_N$  and  $\mathbf{n}_0 \times \mathbf{n}^{(III)}$  is  $\Omega$ , the angle between  $\mathbf{n}_0 \times \mathbf{n}^{(III)}$  and  $\mathbf{n}^{(I)}$  is  $\omega$ , and the angle between  $\mathbf{n}_0$  and  $\mathbf{n}^{(III)}$  is  $I$ . Let us now set

$$\mathbf{n}^{(I)} = \alpha_I \mathbf{n}_0 + \beta_I \mathbf{n}^{(III)} + \gamma_I (\mathbf{n}_0 \times \mathbf{n}^{(III)})/\sin I, \quad (\text{B16})$$

$$\mathbf{n}^{(III)} = \alpha_{III} \mathbf{n}_0 + \beta_{III} \mathbf{n}^{(III)} + \gamma_{III} (\mathbf{n}_0 \times \mathbf{n}^{(III)})/\sin I, \quad (\text{B17})$$

where

$$\alpha_I = \sin \omega / \sin I, \quad \alpha_{III} = \cos \omega / \sin I, \quad (\text{B18})$$

$$\beta_I = -\alpha_I \cos I, \quad \beta_{III} = -\alpha_{III} \cos I, \quad (\text{B19})$$

$$\gamma_I = \cos \omega, \quad \gamma_{III} = -\sin \omega. \quad (\text{B20})$$



Using equations (B16) and (B17) in equation (B3) and then comparing the result with equation (B15), one obtains

$$\dot{\Omega} = \alpha_I \Omega_I^* + \alpha_{II} \Omega_{II}^* , \quad (\text{B21})$$

$$\dot{\omega} = -\dot{\Omega} \cos I + \Omega_{III}^* , \quad (\text{B22})$$

$$\dot{I} = \gamma_I \Omega_I^* + \gamma_{II} \Omega_{II}^* . \quad (\text{B23})$$

The longitude of the periastron,  $\varpi$ , is given by

$$\varpi = \omega + \Omega , \quad (\text{B24})$$

and, thus,

$$\dot{\varpi} - 2\dot{\Omega} \sin^2 (I/2) = \dot{\omega} + \dot{\Omega} \cos I = \Omega_{III}^* . \quad (\text{B25})$$

It should be noted that  $\Omega_{III}^*$  is the precession of the periastron *in the plane of the orbit* (see Barker and O'Connell 1974). From equations (1.19) and (B10) one finds that

$$\dot{\sigma}_1 = -(1 - e^2)^{1/2} \Omega_{III}^* - \frac{2\bar{\omega}a(\mathbf{r} \cdot \mathbf{B})}{GM} . \quad (\text{B26})$$

The mean longitude,  $l$ , is given by

$$l = \theta + \varpi , \quad (\text{B27})$$

and thus, using equation (1.18), we have

$$\dot{l} = \dot{\omega} + \dot{\epsilon}_1 , \quad (\text{B28})$$

where

$$\dot{\epsilon}_1 = \dot{\varpi} + \dot{\sigma}_1 . \quad (\text{B29})$$

Using equations (B25) and (B26) in the above, one obtains

$$\dot{\epsilon}_1 = [1 - (1 - e^2)^{1/2}] \dot{\varpi} + 2(1 - e^2)^{1/2} \dot{\Omega} \sin^2 (I/2) - \frac{2\bar{\omega}a(\mathbf{r} \cdot \mathbf{B})}{GM} . \quad (\text{B30})$$

### APPENDIX C

First, in this appendix we wish to find  $\dot{\sigma}_{1av}$  and  $(3/\bar{\omega}a^2)g_{av}$ , where  $g$  is given in Appendix A and

$$\dot{\sigma}_1 = \dot{\sigma}_1^{(E)}(\alpha) + \dot{\sigma}_1^{(1)} + \dot{\sigma}_1^{(2)} + \dot{\sigma}_1^{(\lambda 1)} + \dot{\sigma}_1^{(\lambda 2)} + \dot{\sigma}_1^{(1,2)} + \dot{\sigma}_1^{(Q1)} + \dot{\sigma}_1^{(Q2)} . \quad (\text{C1})$$

The terms on the right-hand side of equation (C1) correspond, respectively, to the terms on the right-hand side of equation (1.3). From equation (B26) we obtain (note that  $\mathbf{n} \equiv \mathbf{n}^{(III)}$ )

$$\dot{\sigma}_{1av} = -(1 - e^2)^{1/2} (\mathbf{\Omega}_{av}^* \cdot \mathbf{n}) - \frac{2}{\bar{\omega}a^2} (\mathbf{r} \cdot \mathbf{B})_{av} , \quad (\text{C2})$$

where

$$\mathbf{\Omega}^* = \mathbf{\Omega}^{*(E)}(\alpha) + \mathbf{\Omega}^{*(1)} + \mathbf{\Omega}^{*(2)} + \mathbf{\Omega}^{*(\lambda 1)} + \mathbf{\Omega}^{*(\lambda 2)} + \mathbf{\Omega}^{*(1,2)} + \mathbf{\Omega}^{*(Q1)} + \mathbf{\Omega}^{*(Q2)} , \quad (\text{C3})$$

and our notation follows the form of equations (1.3) and (C1). We find<sup>4</sup> directly from Papers I and II that

$$(1 - e^2)^{1/2} \mathbf{\Omega}_{av}^{*(E)} \cdot \mathbf{n} = \frac{(2 + 2\gamma - \beta)GM\bar{\omega}}{c^2 a (1 - e^2)^{1/2}} , \quad (\text{C4})$$

$$(1 - e^2)^{1/2} \mathbf{\Omega}_{av}^{*(N)} \cdot \mathbf{n} = -\frac{2G}{c^2 a^3 (1 - e^2)} \left[ \gamma + 1 + \left( \gamma + \frac{1}{2} \right) \frac{m_1 m_2}{m_N^2} \right] \mathbf{S}^{(N)} \cdot \mathbf{n} , \quad (\text{C5})$$

$$(1 - e^2)^{1/2} \mathbf{\Omega}_{av}^{*(1,2)} \cdot \mathbf{n} = \frac{3(\gamma + 1)G}{4c^2 a^5 \bar{\omega} \mu (1 - e^2)^{3/2}} [3(\mathbf{S}^{(1)} \cdot \mathbf{n})(\mathbf{S}^{(2)} \cdot \mathbf{n}) - \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}] , \quad (\text{C6})$$

$$(1 - e^2)^{1/2} \mathbf{\Omega}_{av}^{*(QN)} \cdot \mathbf{n} = \frac{3GM\Delta I^{(N)}}{4a^5 \bar{\omega} m_N (1 - e^2)^{3/2}} [3(\mathbf{n}^{(N)} \cdot \mathbf{n})^2 - 1] , \quad (\text{C7})$$

where  $\mathbf{\Omega}_{av}^{*(E)} \equiv \mathbf{\Omega}_{av}^{*(E)}(\alpha)$  and  $\mathbf{\Omega}_{av}^{*(\lambda N)} = 0$ .

<sup>4</sup> The  $\mathbf{\Omega}_{av}^*$ 's of this paper correspond to the  $\mathbf{\Omega}^*$ 's of Papers I and II. We did not give unaveraged  $\mathbf{\Omega}^*$ 's in Papers I and II.



There is no  $\alpha$  or  $\lambda_N$  dependence in  $\Omega_{av}^*$ . Using the various time averages in Appendix D and equations (A6)–(A9) and (A13)–(A15), we obtain

$$\frac{2}{\bar{\omega}a^2} [\mathbf{r} \cdot \mathbf{B}^{(E)}(\alpha)]_{av} = \frac{GM\bar{\omega}}{c^2a(1-e^2)^{1/2}} \left[ \left( 4 + 4\beta + 4\gamma - 6\alpha - \frac{9\mu}{M} \right) - \left( 4 + 2\gamma - \frac{7\mu}{M} \right) (1-e^2)^{1/2} \right], \quad (C8)$$

$$\frac{2}{\bar{\omega}a^2} (\mathbf{r} \cdot \mathbf{B}^{(N)})_{av} = -(1-e^2)^{1/2} \Omega_{av}^{*(N)} \cdot \mathbf{n}, \quad (C9)$$

$$\frac{2}{\bar{\omega}a^2} (\mathbf{r} \cdot \mathbf{B}^{(\lambda N)})_{av} = -\frac{6G\lambda_N m_1 m_2}{c^2 a^3 (1-e^2) m_N^2} \mathbf{S}^{(N)} \cdot \mathbf{n}, \quad (C10)$$

$$\frac{2}{\bar{\omega}a^2} (\mathbf{r} \cdot \mathbf{B}^{(1,2)})_{av} = -2(1-e^2)^{1/2} \Omega_{av}^{*(1,2)} \cdot \mathbf{n}, \quad (C11)$$

$$\frac{2}{\bar{\omega}a^2} (\mathbf{r} \cdot \mathbf{B}^{(QN)})_{av} = -2(1-e^2)^{1/2} \Omega_{av}^{*(QN)} \cdot \mathbf{n}, \quad (C12)$$

and

$$\frac{3}{\bar{\omega}a^2} g_{av}^{(E)}(\alpha) = \frac{GM\bar{\omega}}{c^2a(1-e^2)^{1/2}} \left[ \left( -6 - 3\beta - 6\gamma + 6\alpha + \frac{9\mu}{M} \right) + \left( 6 + 3\gamma - \frac{21\mu}{2M} \right) (1-e^2)^{1/2} \right], \quad (C13)$$

$$\frac{3}{\bar{\omega}a^2} g_{av}^{(\lambda N)} = -\frac{2}{\bar{\omega}a^2} (\mathbf{r} \cdot \mathbf{B}^{(\lambda N)})_{av}, \quad g_{av}^{(N)} = g^{(N)} \equiv 0, \quad (C14)$$

$$\frac{3}{\bar{\omega}a^2} g_{av}^{(1,2)} = (1-e^2)^{1/2} \Omega_{av}^{*(1,2)} \cdot \mathbf{n}, \quad (C15)$$

$$\frac{3}{\bar{\omega}a^2} g_{av}^{(QN)} = (1-e^2)^{1/2} \Omega_{av}^{*(QN)} \cdot \mathbf{n}. \quad (C16)$$

From equations (C1)–(C16) we find, after a lot of cancellation,

$$\dot{\sigma}_{1av} - \frac{3}{\bar{\omega}a^2} g_{av} = \frac{GM\bar{\omega}}{c^2a} \left( -2 - \gamma + \frac{7\mu}{2M} \right). \quad (C17)$$

It can easily be shown that  $\dot{\sigma}_1$ , given by equation (1.19), can be put in the form (correct to the post-Newtonian approximation)

$$\dot{\sigma}_1 = \frac{1}{\bar{\omega}a^2 e^2} \left\{ \left[ (1-e^2) - \frac{r}{a} \right] (\mathbf{r} \cdot \mathbf{B}) + \left[ (2+e^2) - \frac{2r}{a} \right] g \right\} + \bar{\omega} \dot{Q}, \quad (C18)$$

where

$$Q \equiv -\frac{a[a(1-e^2) + r]}{(GM)^2 e^2} (\mathbf{v} \cdot \mathbf{r})g, \quad (C19)$$

and

$$\frac{1}{P} \int_0^P \dot{Q} dt = \frac{1}{P/2} \int_0^{P/2} \dot{Q} dt = 0 \quad (C20)$$

because  $\mathbf{v} \cdot \mathbf{r}$  is zero both at periastron and at apastron. Using the various time averages in Appendix D, we have verified (as a cross check) that  $\dot{\sigma}_{1av}$  as obtained from equation (C18) is in agreement with  $\dot{\sigma}_{1av}$  as obtained from equation (C2). We shall set (notation follows the form of eqs. [1.3] and [A16])

$$Q = Q^{(E)}(\alpha) + Q^{(1,2)} + \sum_{N=1}^2 (Q^{(\lambda N)} + Q^{(QN)}), \quad (C21)$$

and  $Q^{(N)} \equiv 0$  because  $g^{(N)} \equiv 0$ .

We note that  $g^{(E)}(\alpha)$  and  $g^{(\lambda N)}$  as well as  $\dot{\sigma}_1^{(E)}(\alpha) - \bar{\omega} \dot{Q}^{(E)}(\alpha)$ ,  $\dot{\sigma}_1^{(N)}$ , and  $\dot{\sigma}_1^{(\lambda N)} - \bar{\omega} \dot{Q}^{(\lambda N)}$  can be considered as functions of  $r$  (and constants). We thus obtain, using equation (D8) of Appendix D and equation (C20),

$$\frac{1}{P/2} \int_0^{P/2} g^{(E)}(\alpha) dt = \frac{1}{P} \int_0^P g^{(E)}(\alpha) dt, \quad (C22)$$

$$\frac{1}{P/2} \int_0^{P/2} g^{(\lambda N)} dt = \frac{1}{P} \int_0^P g^{(\lambda N)} dt, \quad (C23)$$

$$\frac{1}{P/2} \int_0^{P/2} \dot{\sigma}_1^{(E)}(\alpha) dt = \frac{1}{P} \int_0^P \dot{\sigma}_1^{(E)}(\alpha) dt, \quad (C24)$$

$$\frac{1}{P/2} \int_0^{P/2} \dot{\sigma}_1^{(N)} dt = \frac{1}{P} \int_0^P \dot{\sigma}_1^{(N)} dt, \quad (C25)$$

$$\frac{1}{P/2} \int_0^{P/2} \dot{\sigma}_1^{(\lambda N)} dt = \frac{1}{P} \int_0^P \dot{\sigma}_1^{(\lambda N)} dt. \quad (C26)$$

We also find, using equations (D7)–(D10), that

$$\frac{1}{P/2} \int_0^{P/2} \frac{r}{a} g^{(1,2)} dt = \frac{1}{P} \int_0^P \frac{r}{a} g^{(1,2)} dt, \quad (C27)$$

$$\frac{1}{P/2} \int_0^{P/2} \frac{r}{a} g^{(QN)} dt = \frac{1}{P} \int_0^P \frac{r}{a} g^{(QN)} dt, \quad (C28)$$

$$\frac{1}{P/2} \int_0^{P/2} g^{(1,2)} dt - \frac{1}{P} \int_0^P g^{(1,2)} dt = -\frac{e(\gamma+1)G}{\pi c^2 a^3 (1-e^2)^{3/2} \mu} [(S^{(1)} \cdot n^{(I)})(S^{(2)} \cdot n^{(II)}) + (S^{(1)} \cdot n^{(II)})(S^{(2)} \cdot n^{(I)})], \quad (C29)$$

$$\frac{1}{P/2} \int_0^{P/2} g^{(QN)} dt - \frac{1}{P} \int_0^P g^{(QN)} dt = -\frac{2eGM\Delta I^{(N)}}{\pi a^3 (1-e^2)^{3/2} m_N} [(n^{(N)} \cdot n^{(I)})(n^{(N)} \cdot n^{(II)})]. \quad (C30)$$

Finally, let us evaluate  $\dot{a}_{av}$  and  $\dot{e}_{av}$ , where  $[\ ]_{av} = (1/P) \int_0^P [\ ] dt$ . Noting that  $\mathbf{v} \cdot \mathbf{B} = \dot{g}$  (see eq. [A17]), and  $\dot{g}_{av} = 0$ , we obtain from equations (B13) and (1.15)

$$\dot{a}_{av} = \dot{\omega}_{av} = \dot{E}_{av} = 0. \quad (C31)$$

From equations (B11), (B12), and (B14) we obtain

$$L\dot{L}_{av}/\mu^2 = -ae\dot{A}_{av}/\mu = -\bar{\omega}^2 a^4 e \dot{e}_{av}, \quad (C32)$$

where

$$L\dot{L}_{av}/\mu^2 = [r^2 \dot{g} - (\mathbf{v} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{B})]_{av}. \quad (C33)$$

Noting that  $r^2 \dot{g} = d(r^2 g)/dt - 2(\mathbf{v} \cdot \mathbf{r})g$ , we obtain

$$L\dot{L}_{av}/\mu^2 = -\{(\mathbf{v} \cdot \mathbf{r})[(\mathbf{r} \cdot \mathbf{B}) + 2g]\}_{av}. \quad (C34)$$

Next, using equations (D6) and (D11), we obtain  $L\dot{L}_{av}/\mu^2 = 0$ , which implies that

$$\dot{e}_{av} = \dot{A}_{av} = \dot{L}_{av} = 0. \quad (C35)$$

## APPENDIX D

In calculating the time-averaged expressions in Appendix C, the time-averaged values of a number of quantities (given below) had to be determined. We shall introduce a special coordinate system so that the orbit is in the  $x$ - $y$  plane with the periastron in the  $x$ -direction and the orbital angular momentum in the  $z$ -direction. Because the expressions in Appendix C are higher order terms, we can consider the ellipse to be unperturbed. Using  $r = a(1 - e \cos \zeta)$ ,  $x = a(\cos \zeta - e)$ ,  $y = a(1 - e^2)^{1/2} \sin \zeta$ ,  $t = (P/2\pi)(\zeta - e \sin \zeta)$ , and  $dt = (P/2\pi)(r/a)d\zeta$ , we obtain

$$\frac{1}{P/2} \int_0^{P/2} \frac{dt}{r} = \frac{1}{P} \int_0^P \frac{dt}{r} = \frac{1}{a}, \quad (D1)$$

$$\frac{1}{P/2} \int_0^{P/2} \frac{dt}{r^2} = \frac{1}{P} \int_0^P \frac{dt}{r^2} = \frac{1}{a^2(1-e^2)^{1/2}}, \quad (D2)$$

$$\frac{1}{P/2} \int_0^{P/2} \frac{dt}{r^3} = \frac{1}{P} \int_0^P \frac{dt}{r^3} = \frac{1}{a^3(1-e^2)^{3/2}}, \quad (D3)$$

$$\frac{1}{P/2} \int_0^{P/2} \frac{x^2 dt}{r^5} = \frac{1}{P} \int_0^P \frac{x^2 dt}{r^5} = \frac{1}{2a^3(1-e^2)^{3/2}}, \quad (D4)$$

$$\frac{1}{P/2} \int_0^{P/2} \frac{y^2 dt}{r^5} = \frac{1}{P} \int_0^P \frac{y^2 dt}{r^5} = \frac{1}{2a^3(1-e^2)^{3/2}}. \quad (D5)$$

From symmetry it is clear (note that  $\mathbf{v} \cdot \mathbf{r} = r\dot{r}$ ) that

$$\frac{1}{P} \int_0^P (\mathbf{v} \cdot \mathbf{r}) f dt = 0, \quad (\text{D6})$$

$$\frac{1}{P} \int_0^P xy f dt = 0, \quad (\text{D7})$$

$$\frac{1}{P/2} \int_0^{P/2} f dt = \frac{1}{P} \int_0^P f dt, \quad (\text{D8})$$

for  $f = f(r, x^2, y^2, v^2, (\mathbf{v} \cdot \mathbf{r})^2, \dot{r}^2, \dot{x}^2, \dot{y}^2)$ . We also find that

$$\frac{1}{P/2} \int_0^{P/2} \frac{xy dt}{r^4} = 0, \quad (\text{D9})$$

$$\frac{1}{P/2} \int_0^{P/2} \frac{xy dt}{r^5} = \frac{2e}{3\pi a^3 (1 - e^2)^{3/2}}, \quad (\text{D10})$$

$$\frac{1}{P} \int_0^P \frac{(\mathbf{v} \cdot \mathbf{r}) xy dt}{r^5} = 0. \quad (\text{D11})$$

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