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QUANTUM DISTRIBUTION FUNCTIONS IN NON-EQUILIBRIUM STATISTICAL MECHANICS

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I. INTRODUCTION

Quantum distribution functions provide a means of expressing quantum mechanical averages in a form which is very similar to that for classical averages. Also, the Bloch equation for the density matrix for a canonical ensemble is replaced by a classical equation and, turning to dynamics, the von Neumann equation describing the time development of the density matrix is replaced by a classical equation which is similar in form to the Liouville equation but contains exactly the same information as the quantum von Neumann equation.

These distribution functions have been applied in essentially all areas of non-equilibrium statistical mechanics, from the more traditional branches to more recent branches such as quantum optics and synergetics. We have recently given an overview of such applications[1,2] and, in addition, participated in the presentation of a detailed review of the fundamentals underlying the theory of distribution functions.[3] Thus, to avoid repetition, we will concentrate here on a subject not treated in detail in our review viz. the inter-relationships between some of the more commonly used quantum distribution functions.

The original quantum distribution function was that introduced by Wigner[4] and which we will designate as $P_w(q,p)$, where q and p refer to position and momentum coordinates, respectively. Many authors have considered other distribution functions. In q-p phase space language the more commonly discussed functions include the Kirkwood function, the Husimi function, as well as the standard and anti-standard functions.

With applications to quantum optics in mind, Glauber[5] and Sudarshan[6] introduced a function, commonly designated as $P(\alpha)$, where α is a complex variable. This function has enjoyed wide usage. In complex variable language, other functions often considered include the $Q(\alpha)$ distribution function, the normal and anti-normal functions and the Cahill-Glauber[7] class of generalized functions. In addition, the New Zealand group has made extensive use of a function of two complex variables,[8] but this will not concern us here.

We have recently concerned ourselves with the general question of what criteria one should use in the selection of a quantum distribution function when one is faced with carrying out a specific calculation.[2,9] In a similar vein, one might have intermediate results pertaining to some calculation and might want to consider switching from the use of one distribution function to another. For this and other reasons it is of interest to obtain the relationships between the more commonly used functions. This will form the focus of our present considerations.

As we have emphasized in our previous investigations of this question[2,9] the optimum approach to the problem is via the use of characteristic functions, which are nothing more than Fourier transforms of distribution distributions. The next step is to write $P_w(q,p)$ in terms of the complex variable α. Then we obtain relationships between the various distribution functions, which are readily converted into relationships between the distribution functions, either in q-p or α language.

For simplicity, we treat a one-dimensional system in a pure state since the case of a mixture in multi-dimensions presents no essential complications. Since the touchstone of our considerations is the Wigner distribution, we will write down its properties in Section II, repeating of necessity some of the material in Ref. 9 since it is necessary for the purpose of establishing our notation. In Section III, we summarize the properties of the generalized distribution functions of the Cahill-Glauber[7] class and their relation to the Wigner distribution. In addition, we discuss the Husimi (smoothed) distribution function and point out its exact equivalence with the anti-normal distribution. In Section IV, we discuss the Glauber-Sudarshan $P(\alpha)$ function as well as the $Q(\alpha)$ function and also the relation between them and P_w. Then we show that the $Q(\alpha)$, Husimi, and anti-normal distributions are identical to each other and, similarly, the Glauber-Sudarshan and normal distributions are identical. Finally, in Section V, we introduce a new class of generalized distribution functions, with emphasis on two particular cases of same viz. the standard of anti-standard distributions, and we also demonstrate the equivalence of the latter to the Kirkwood distribution. In Section VI we discuss results.

II. THE WIGNER DISTRIBUTION FUNCTION

This is the original distribution function and is given by[4]

$$P_w(q,p) = (\pi\hbar)^{-1} \int \psi(q+y)^* \psi(q-y) e^{2ipy/\hbar} dy. \tag{1}$$

If we now introduce the characteristic function

$$C_w(\sigma,\tau) = \langle\psi|\exp[\frac{i}{\hbar}(\sigma\hat{q} + \tau\hat{p})]|\psi\rangle, \tag{2}$$

where the hats denote operators, then it follows[3] that P_w is the Fourier transform of C_w i.e.

$$P_w(q,p) = (2\pi\hbar)^{-2} \iint d\sigma d\tau \exp[-\frac{i}{\hbar}(\sigma q + \tau p)]\, C_w(\sigma,\tau). \tag{3}$$

Next we introduce the language of creation and annihilation operators by defining, as usual.

$$\hat{a} = \tfrac{1}{2} \left[\frac{\hat{q}}{q_o} + i \frac{\hat{p}}{p_o} \right] \tag{4a}$$

$$\hat{a}^{+} = \tfrac{1}{2} \left[\frac{\hat{q}}{q_o} - i \frac{\hat{p}}{p_o} \right], \tag{4b}$$

where

$$q_o = (\hbar/2m\omega)^{\frac{1}{2}} \tag{5a}$$

and

$$p_o = (m\hbar\omega/2)^{\frac{1}{2}} = m\omega q_o = (\hbar/2q_o). \tag{5b}$$

In addition, we define

$$\alpha = \tfrac{1}{2} \left[\frac{q}{q_o} + i \frac{p}{p_o} \right] \tag{6a}$$

$$\alpha^{*} = \tfrac{1}{2} \left[\frac{q}{q_o} - i \frac{p}{p_o} \right], \tag{6b}$$

and

$$\eta = (\sigma q_o - i\,\tau\, p_o)/\hbar \tag{7a}$$

$$\eta^{*} = (\sigma q_o + i\,\tau\, p_o)/\hbar\ , \tag{7b}$$

from which it follows that

$$\hbar^{-1}(\sigma\hat{q} + \tau\hat{p}) = \eta\hat{a} + \eta^{*}\hat{a}^{+} \tag{8a}$$

and

$$\hbar^{-1}(\sigma q + \tau p) = \eta\alpha + \eta^{*}\alpha^{*}\ . \tag{8b}$$

Hence

$$C_W(\sigma,\tau) = \langle\psi|\ \exp[\ i\,(\eta\hat{a} + \eta^{*}\hat{a}^{+}\ )|\psi\rangle$$

$$\equiv C_W(\eta,\eta^{*})\ . \tag{9}$$

Substituting this result in Eq. (3), and using Eq. (8b) and the fact (see Eq. (7)) that $d\sigma\ d\tau = \hbar d\eta d\eta^{*}$, it follows that

$$P_W(q,p) = \hbar(2\pi\hbar)^{-2} \iint d\eta d\eta^* \exp[-i\ (\eta\alpha + \eta^*\alpha^*)] C_W(\eta,\eta^*)$$

$$\equiv \hbar^{-1} P_W(\alpha,\alpha^*) \equiv (2\hbar)^{-1}\ P_W(\alpha). \qquad (10)$$

The normalization factors of $\hbar^{-1}$ and $(2\hbar)^{-1}$ in this equation are introduced to ensure that

$$\iint P_W(q,p) dq\ dp = 1 \qquad (11a)$$

$$\iint P_W(\alpha,\alpha^*) d\alpha\ d\alpha^* = 1 \qquad (11b)$$

$$\int P_W(\alpha) d^2\alpha = 1 \quad , \qquad (11c)$$

where we have used the fact that (see Eqs. 5 and 6)

$$dq\ dp = \hbar\ d\alpha\ d\alpha^*$$
$$= 2\hbar\ d^2\alpha \quad , \qquad (12)$$

where

$$\int d^2\alpha \equiv \iint d(\mathrm{Re}\alpha)\ d(\mathrm{Im}\alpha). \qquad (13)$$

Similar relations to Eqs. (10) and (11) exist for the other distribution functions to be considered below.

III. THE CAHILL-GLAUBER CLASS OF GENERALIZED DISTRIBUTION FUNCTIONS

A generalized distribution function $P_g(q,p,s)$, where s is a parameter, is defined by replacing $C_W(\eta,\eta^*)$ in Eq. (9) by $C_g(\eta,\eta^*)$, defined as follows[7]

$$C_g(\eta,\eta^*) \equiv \langle \psi \mid \exp\ [\ \frac{s|\eta|^2}{2} + i\ (\eta\hat{a} + \eta^*\ \hat{a}^+)] | \psi\rangle. \qquad (14)$$

It follows that for s = 0 we get the Wigner (or as oft-times called, the symmetric) characteristic function C_W given in Eq. (9). Also, making use of the Baker-Hausdorff theorem:

$$\exp(\hat{A})\ \exp(\hat{B}) = \exp(\tfrac{1}{2}[\hat{A},\hat{B}])\ \exp(\hat{A}+\hat{B}) \qquad (15)$$

provided $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$, and the fact that $[\hat{a},\hat{a}^+] = 1$, it follows that

$$C_a(\eta,\eta^*) = \langle\psi|\exp[i\eta\hat{a}]\exp[i\eta^*\hat{a}^+]|\psi\rangle \tag{16}$$

and

$$C_n(\eta,\eta^*) = \langle\psi|\exp[i\eta^*\hat{a}^+]\exp[i\eta\hat{a}]|\psi\rangle\ , \tag{17}$$

where C_a and C_n are the so-called anti-normal and normal characteristic functions, corresponding to taking s = - 1 and s = 1, respectively.

Using these relations, we have recently shown[9] that the generalized distribution function may be written in terms of P_w, as follows:

$$P_g(q,p) = \exp\ [-\frac{s}{2}\, q_o^2 \frac{\partial^2}{\partial q^2} - \frac{s}{2}\, p_o^2 \frac{\partial^2}{\partial p^2}]\ P_w(q,p). \tag{18}$$

Thus, in particular, for s=1 and -1 (corresponding to normal and anti-normal respectively) we obtain

$$P_{n,a}(q,p) = \exp[\mp \tfrac{1}{2}(q_o^2 \frac{\partial^2}{\partial q^2} + p_o^2 \frac{\partial^2}{\partial p^2})], \tag{19}$$

a result which we have presented previously[2] without proof.

In addition, we have shown that

$$P_{-1}(q,p) \equiv P_a(q,p) = P_s(q,p), \tag{20}$$

where P_a is the anti-normal distribution and P_s is Husimi's distribution function.[10] The latter is a smoothed Wigner function, the smoothing being the Wigner function of the ground-state of a harmonic oscillator[11,12]. It is everywhere non-negative and is defined as follows:

$$P_s(q,p) = (\pi\hbar)^{-1} \iint_{-\infty}^{\infty} P_w(q',p')\ \exp[-(q'-q)^2/\lambda]\ \exp[-\lambda(p'-p)^2/\hbar^2]\,dq'\,dp'$$

where $\lambda = (\hbar/m\omega)$. (21)

It is also of interest to re-write the phase-space operator appearing in Eq. (18) in terms of α. From Eq. (6) it follows that

$$\frac{\partial}{\partial q} = \frac{1}{2q_o}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*}\right) \tag{22a}$$

$$\frac{\partial}{\partial p} = \frac{i}{2p_o}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\alpha^*}\right), \tag{22b}$$

so that

$$O(q,p,s) \equiv \exp[-\frac{s}{2}(q_o^2 \frac{\partial^2}{\partial q^2} + p_o^2 \frac{\partial^2}{\partial p^2})]$$

$$= \exp[-\frac{s}{2} \frac{\partial^2}{\partial\alpha\partial\alpha^*}]$$

$$\equiv O(\alpha,\alpha^*,s). \qquad (23)$$

Hence, using Eqs. (10) and (18), we obtain

$$P_g(\alpha,\alpha^*) = \exp[-\frac{s}{2} \frac{\partial^2}{\partial\alpha\partial\alpha^*}] P_w(\alpha,\alpha^*) \qquad (24a)$$

which for the specific cases of s=1 and s=-1 gives

$$P_{n,a}(\alpha,\alpha^*) = \exp[\mp 1/2 \frac{\partial^2}{\partial\alpha\partial\alpha^*}] P_w(\alpha,\alpha^*), \qquad (24b)$$

a result first obtained by Agarwal and Wolf [13] (noting that their superscripts A and N correspond to our subscripts "n" and "a", respectively).

Finally, we use our defining Eq. (21) for $P_s(q,p)$ in conjunction with Eqs. (6) and (10), to obtain

$$P_s(\alpha) = 2\hbar P_s(q,p)$$

$$= (2/\pi)\int d^2\beta\, P_w(\beta)\exp[-2|\beta-\alpha|^2], \qquad (25)$$

with

$$\int P_s(\alpha)d^2\alpha = 1. \qquad (26)$$

IV. THE P(α) AND Q(α) FUNCTIONS

As a preliminary to the introduction of these functions, we write down some well-known properties[14-16] of the coherent states $|\alpha\rangle$. They are defined in terms of the number states $|n\rangle$ as follows:

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\hat{a}^n}{(n!)^{1/2}} \exp[-1/2 |\alpha|^2]|n\rangle. \qquad (27)$$

It may be verified that

$$|\langle\alpha|\alpha\rangle|^2 = 1, \tag{28}$$

$$|\langle\alpha|\beta\rangle|^2 = \exp[-|\beta-\alpha|^2], \tag{29}$$

$$\pi^{-1}\int d^2\alpha\ |\alpha\rangle\langle\alpha| = I, \tag{30}$$

and

$$\pi^{-1}\int d^2\beta\ \exp[-|\beta-\alpha|^2] = \pi^{-1}\int d^2\beta\ |\langle\alpha|\beta\rangle|^2 = 1. \tag{31}$$

The Glauber-Sudarshan $P(\alpha)$ distribution function is defined by expanding the density matrix $\hat{\rho}$ as follows:[5-6]

$$\hat{\rho} = \int d^2\alpha\ P(\alpha)|\alpha\rangle\langle\alpha| \tag{32}$$

It then follows that, for an arbitrary operator $\hat{A}$,

$$\begin{aligned}\langle\hat{A}\rangle &= \mathrm{Tr}(\hat{\rho}\hat{A}) \\ &= \int d^2\alpha\ P(\alpha)\langle\alpha|\hat{A}|\alpha\rangle \\ &\equiv \int d^2\alpha\ P(\alpha)\ A(\alpha).\end{aligned} \tag{33}$$

Turning now to $Q(\alpha)$, it is a non-negative function, defined by Glauber[14] and Kano[15] according to the relation

$$Q(\alpha) = \pi^{-1}\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{34}$$

Substituting Eq. (32) into Eq. (34), and using Eq. (29), we obtain a well-known relation between the Glauber-Sudarshan and $Q(\alpha)$ functions:[14]

$$Q(\alpha) = \pi^{-1}\int d^2\beta\ P(\beta)\exp[-|\alpha-\beta|^2]. \tag{35}$$

From Eqs. (35), (31), and (33) it may be verified that

$$\int P(\alpha)d^2\alpha = 1 \tag{36a}$$

$$\int Q(\alpha)d^2\alpha = 1. \tag{36b}$$

Next we wish to prove that $P(\alpha)$ is the same as $P_n(\alpha)$. First of all, we define

$$\hat{C}_n \equiv \exp[i\eta^* \hat{a}^+]\exp[i\eta\hat{a}], \qquad (37)$$

so that, from Eqs. (17) and (33), we may write

$$C_n(\eta,\eta^*) = \langle \hat{C}_n \rangle$$
$$= \int d^2\alpha \, P(\alpha) \, C_n(\alpha) \quad , \qquad (38)$$

i.e. C_n is the Fourier transform of $(1/2)P(\alpha)$. But, by definition, C_n is the Fourier transform of $P_n(\alpha,\alpha^*)$, which from Eq. (10) is the same as $(1/2)P_n(\alpha)$. Thus

$$P(\alpha) = P_n(\alpha), \qquad (39)$$

i.e. the Glauber-Sudarshan and normal distributions are identical.

There also exists a well-known relation between P_w and $P(\alpha)$, which is most easily obtained by starting with the following relation between their corresponding characteristic functions (see Eq. (14))

$$C_w(\eta,\eta^*) = \exp[-\frac{|\eta|^2}{2}] \, C_n(\eta,\eta^*). \qquad (40)$$

First, we take Fourier transforms on both sides of the equation and then we use the Fourier Convolution Theorem i.e.

$$F[f_1(x)] \, F[f_2(x)] = F[\int_{-\infty}^{\infty} f_1(x-y)f_2(y)dy]. \qquad (41)$$

Then using the fact that $P_w(\alpha)$ and $P(\alpha)$ are the Fourier transforms of $C_w(\eta,\eta^*)$ and $C_n(\eta,\eta^*)$, respectively, and the fact that the Fourier transform of a Gaussian is also a Gaussian, we obtain

$$P_w(\alpha) = (2/\pi)\int d^2\beta \, P(\beta)\exp[-2|\alpha-\beta|^2], \qquad (42)$$

a result first obtained by Glauber.[14)]

In a similar manner, analogous to Eq. (40), we may write

$$C_a(\eta,\eta^*) = \exp[-\frac{|\eta|^2}{2}]\, C_W(\eta,\eta^*), \qquad (43)$$

from which it follows that

$$P_a(\alpha) = (2/\pi) \int d^2\beta\, P_W(\beta)\exp[-2|\alpha-\beta|^2]. \qquad (44)$$

Thus, by comparison with Eq. (25), we see immediately that

$$P_a(\alpha) = P_S(\alpha). \qquad (45)$$

An alternative proof of this relation may be found in Ref. 9. However, here we were also interested in obtaining the relations given by Eqs. (42) and (44), which we now combine together by substituting Eq. (42) into Eq. (44) to obtain

$$\begin{aligned} P_a(\alpha) &= (2/\pi)^2 \int\int d^2\beta\, d^2\gamma\, P(\gamma)\, \exp[-2|\beta-\alpha|^2 - 2|\beta-\gamma|^2] \\ &= (2/\pi)^2 \int\int d^2\beta\, d^2\gamma\, P(\gamma)\exp[-4|\beta|^2 + 2(\gamma^* + \alpha^*)\beta + 2(\gamma+\alpha)\beta^*] \\ &= (2/\pi)^2 \int d^2\gamma\, P(\gamma)\exp[-2|\gamma|^2 - 2|\alpha|^2] \\ &\times \{\int d^2\beta\, \exp[-4|\beta|^2 + 2(\gamma^* + \alpha^*)\beta + 2(\gamma+\alpha)]\}\ . \end{aligned} \qquad (46)$$

The β integration may be carried out to give the result $(\pi/4)\exp(|\gamma+\alpha|^2)$ so finally we obtain

$$P_a(\alpha) = \pi^{-1} \int d^2\gamma\, P(\gamma)\, \exp[-|\gamma-\alpha|^2]. \qquad (47)$$

But this is identical with the expression given for $Q(\alpha)$ in Eq. (35). Combining this result with Eq. (45) we deduce that

$$Q(\alpha) = P_a(\alpha) = P_S(\alpha). \qquad (48)$$

In summary, the $Q(\alpha)$ distribution as defined in Eq. (34), the anti-normal distribution, and the Husimi (smoothed) distribution are identical. In addition, the Glauber-Sudarshan distribution, as defined in Eq. (32), is identical to the normal distribution.

V. A NEW CLASS OF GENERALIZED DISTRIBUTION FUNCTIONS

We define a new class of generalized distribution functions $P_G(q,p,b)$, where b is a parameter, by replacing $C_w(\sigma,\tau)$ in Eq. (2) by $C_G(\sigma,\tau)$, defined as follows

$$C_G(\sigma,\tau) \equiv \langle\psi|\exp[-\frac{ib\sigma\tau}{2\hbar}+\frac{i}{\hbar}(\sigma\hat{q}+\tau\hat{p})]|\psi\rangle. \qquad (49)$$

Again using the Baker-Hausdorff theorem, and the fact that $[\hat{q},\hat{p}] = i\hbar$, we obtain

$$C_{ST}(\sigma,\tau) = \langle\psi|\exp[\frac{i}{\hbar}\sigma\hat{q}]\exp[\frac{i}{\hbar}\tau\hat{p}]|\psi\rangle, \qquad (50)$$

$$C_{AS}(\sigma,\tau) = \langle\psi|\exp[\frac{i}{\hbar}\tau\hat{P}]\exp[\frac{i}{\hbar}\sigma\hat{q}]|\psi\rangle, \qquad (51)$$

where C_{ST} and C_{AS} are the Fourier transforms of the so-called standard and anti-standard distributions, as discussed by Agarwal and Wolf,[13] corresponding to taking b=1 and b=-1, respectively. Hence

$$P_G(q,p) = \exp[\frac{i\hbar}{2} b \frac{\partial^2}{\partial q\partial p}] P_w(q,p). \qquad (52)$$

In addition, making use of Eq. (22), it follows that

$$P_G(\alpha,\alpha^*) = \exp[-\frac{b}{4}(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\alpha^{*2}})] P_w(\alpha,\alpha^*). \qquad (53)$$

From either Eqs. (52) or (53) it is apparent that

$$P_{ST} = P^*_{AS} . \qquad (54)$$

We turn now to a demonstration of the equivalence of P_{AS} and P_K, where P_K is the Kirkwood distribution function[16] given by

$$P_k(q,p) = (\pi\hbar)^{-1}\int\langle q|\hat{\rho}|q+2y\rangle e^{2ipy/\hbar}dy, \qquad (55)$$

which for the case of a pure state reduces to

$$P_k(q,p) = (\pi\hbar)^{-1}\int\psi^*(q+2y)\ \psi(q)\ e^{2ipy/\hbar}dy . \qquad (56)$$

If we now change variables from y to q' according to the relation $2y = q'-q$ then Eq. (56) may be re-written as

$$P_k(q,p) = (2\pi\hbar)^{-1}\int\psi^*(q')\psi(q)e^{ip(q'-q)/\hbar}dq' \;, \qquad (57)$$

which is the form originally presented by Kirkwood.[16] Defining the wave-function in momentum space

$$\phi(p) = (2\pi\hbar)^{-1}\int\psi(q)e^{-iqp/\hbar}dq \;, \qquad (58)$$

it is clear that Eq. (57) may also be re-expressed as

$$P_k(q,p) = \phi^*(p)\;\psi(q)e^{-ipq/\hbar}. \qquad (59)$$

Still another form for $P_K(q,p)$ is obtained if in Eq. (55) we write $\hat{\rho} = |\psi\rangle\langle\psi|$. Then using the fact that

$$|q+2y\rangle = e^{-2iy\hat{p}/\hbar}|q\rangle, \qquad (60)$$

and

$$|q\rangle\langle q| = (2\pi\hbar)^{-1}\int_{-\infty}^{\infty} d\sigma\, \exp[\tfrac{i}{\hbar}\,\sigma(\hat{q} - q)], \qquad (61)$$

and replacing y by $(-\tau/2)$, we obtain

$$P_k(q,p) = (2\pi\hbar)^{-2}\iint d\sigma d\tau\, \exp[-\tfrac{i}{\hbar}(\sigma q+\tau p)]\langle\psi|\exp[\tfrac{i}{\hbar}\,\tau\hat{p}]\exp[\tfrac{i}{\hbar}\,\sigma\hat{q}]|\psi\rangle \qquad (62)$$

Comparing Eq. (51), for the Fourier transform of P_{AS}, with this result, we conclude that

$$P_k(q,p) = P_{AS}(q,p),$$

i.e. the Kirkwood and anti-standard functions are equivalent.

This concludes our discussion of the inter-relationship between the most commonly used quantum distribution functions used in non-equilibrium statistical mechanics, with the exception of distribution functions on four-dimensional phase space.[8] It is clear that they are all encompassed by the generalized functions P_g and P_G defined in Eqs. (18) and (52), respectively. Regarding the properties and use of these generalized functions, we have already obtained[9] an analytic result for the time development of P_g in the case of a non-zero potential V(q). The analogous result for P_G is presently under study as well as the corresponding classical equivalents of the Bloch equation for both P_g and P_G.[17] In addition, we remark that the definition of the Wigner distribution function has been extended recently to the case of spin one-half particles.[18] A similar extension for the generalized distribution functions considered here is readily obtainable.

VI. DISCUSSION OF OUR RESULTS

We have presented a detailed treatment of the relationships between the more commonly used distribution functions in quantum optics. In addition, we have introduced a new class of generalized distribution functions.

The relationships between the functions are of two types--integral and differential. In general, the emphasis in the literature has been on integral relations and these have been used particularly to derive Fokker-Planck equations for, say, the Wigner distribution from Fokker-Planck equations for the Glauber-Sudarshan or other distributions.[19] In our view, the latter goal may be achieved more easily with the use of the differential relationships (it's easier to differentiate than integrate!) and we plan to illustrate this point in a future publication.

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