

Time transformations in post-Newtonian Lagrangians

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We show that the use of time transformations in post-Newtonian Lagrangians is equivalent to using the lowest-order energy-conservation equation in the highest-order terms of the Lagrangian (complementing Schäfer's observation that the use of coordinate transformations is equivalent to using the lowest-order equations of motion in the highest-order terms of the Lagrangian). We also show how identity coordinate or identity time transformations can add *double-zero* terms to the Lagrangian. Next we use time and coordinate transformations to simplify the Einstein-Infeld-Hoffmann Lagrangian with parametrized-post-Newtonian parameters γ and β .

I. INTRODUCTION

Sometimes, certain terms in a Lagrangian are eliminated by either (a) *substituting* into the highest-order terms in the Lagrangian the lowest-order *equations of motion* which were obtained from this Lagrangian, or (b) *substituting* into the highest-order terms in the Lagrangian the lowest-order *energy-conservation equation* which was obtained from this same Lagrangian. Such procedures need to be justified and are, in fact, wrong in certain circumstances.

We have already pointed out^{1,2} (in connection with electromagnetic theory) that the substitution of the lowest-order equations of motion into the highest-order terms in a Lagrangian, in order to eliminate acceleration terms, is an incorrect procedure in general because it changes the functional form of the Lagrangian and, hence, leads to different equations of motion. In particular, we concluded^{1,2} that the well-known two-body post-post-Newtonian Lagrangian in classical electromagnetic theory given by Golubenkov and Smorodinskii^{3,4} is incorrect for this reason (and our conclusion was correct, as we shall see below, if one requires that the coordinate system be inertial in electromagnetic theory). This led us to introduce a new procedure for eliminating unwanted acceleration terms in a Lagrangian which we called *the method of the double zero*.¹ This method was subsequently used to eliminate the acceleration-dependent terms in the post-post-Newtonian electromagnetic Lagrangian.⁵

When Lifshitz brought our work on the electromagnetic Lagrangian to the attention of Schäfer,⁶ the latter was motivated to investigate the corresponding problem in general relativity. Schäfer concluded that, in contrast to the situation in electromagnetism, it is permissible to use lowest-order equations of motion in the highest-order terms of general relativistic Lagrangians because such a procedure is equivalent to carrying out a coordinate transformation (i.e., during this procedure, an *implicit* coordinate transformation has been made). In electromagnetism, the above procedure would be equivalent to

carrying out a coordinate transformation from inertial (Cartesian) to some noninertial system,⁶ and thus, the procedure is *wrong if one wishes to consider, as one usually does, only inertial coordinates*.

In Sec. II, we extend the work of Schäfer⁶ to include time transformations as well as coordinate transformations, and in particular, we will show that the *use of lowest-order energy conservation in the highest-order terms of a Lagrangian is equivalent to carrying out a time transformation* (i.e., during this procedure, an *implicit* time transformation has been made). In electromagnetism, such a time transformation would generally change space-time from inertial to noninertial, which is usually undesirable.

In Sec. III, we will show how what we call identity coordinate and identity time transformations can add *double-zero* terms to the Lagrangian.

In Sec. IV, we show how time and coordinate transformations can be used to simplify the two-body Einstein-Infeld-Hoffmann (EIH) Lagrangian⁷ with parametrized-post-Newtonian (PPN) parameters γ and β . The use of energy conservation in the EIH Lagrangian has been used by one of us⁸ to simplify this Lagrangian, and our present work justifies this procedure once one understands that a time transformation was implicitly made during this procedure.

In Sec. V, we further discuss a pedagogically useful acceleration-dependent electromagnetic Lagrangian that we have introduced before,^{1,2} and in Sec. VI we present our conclusions.

II. COORDINATE AND TIME TRANSFORMATIONS

Let us consider a two-body post-Newtonian Lagrangian (in gravitation theory)

$$\mathcal{L}'(\vec{r}', \vec{v}', \vec{a}') = \mathcal{L}'_0(\vec{r}', \vec{v}') + \mathcal{L}'_1(\vec{r}', \vec{v}', \vec{a}'), \quad (1)$$

where

$$\mathcal{L}'_0(\vec{r}', \vec{v}') = \frac{1}{2} \mu v'^2 + GM\mu/r', \quad (2)$$

$M = m_1 + m_2$, $\mu = m_1 m_2 / M$, $\vec{r}' = \vec{r}'_1 - \vec{r}'_2$, and $\mathcal{L}'_1(\vec{r}', \vec{v}', \vec{a}')$ is a post-Newtonian term. We shall now make the coordinate and time transformations (we use primed quantities for our initial system and reserve the unprimed quantities for the system to which we transform, since we refer to the latter more often)

$$\vec{r}' = \vec{r} + \vec{f}(\vec{r}, \vec{v}, \vec{a}), \quad (3)$$

$$t' = t + \int_0^t u(\vec{r}, \vec{v}, \vec{a}) dt, \quad (4)$$

and obtain the new Lagrangian in the new coordinates

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}, \vec{a}) &= \mathcal{L}_0(\vec{r}, \vec{v}) + \mathcal{L}_1(\vec{r}, \vec{v}, \vec{a}) \\ &= \mathcal{L}_0(\vec{r}, \vec{v}) + \mathcal{L}'_1(\vec{r}, \vec{v}, \vec{a}) \\ &\quad - (\mu \vec{a} + GM\mu\vec{r}/r^3) \cdot \vec{f} + d(\mu\vec{v} \cdot \vec{f})/dt \\ &\quad - (\frac{1}{2}\mu v^2 - GM\mu/r - E_0)u, \end{aligned} \quad (5)$$

where $\mathcal{L}_0(\vec{r}, \vec{v}) = \mathcal{L}'_0(\vec{r}, \vec{v})$, i.e., \mathcal{L}_0 is the same function as \mathcal{L}'_0 , and where E_0 is to be regarded as a constant in the Lagrangian with a value of

$$E_0 \equiv \mu \epsilon_0 = \frac{1}{2}\mu v^2 - GM\mu/r. \quad (6)$$

To show that Eq. (5) is correct (to the post-Newtonian approximation), we verified (see Appendix A) that using the coordinate and time transformations of Eqs. (3) and (4) in the equations of motion obtained from Eq. (1) gave equations of motion that agreed with those obtained from Eq. (5). We have also verified (see Appendix B) that

$$\mathcal{L}(\vec{r}, \vec{v}, \vec{a}) = [\mathcal{L}'(\vec{r}', \vec{v}', \vec{a}')] dt'/dt + E_0(dt'/dt - 1) \quad (7)$$

is in agreement with Eq. (5). Schäfer⁶ derived Eq. (5) without the u term, since he did not make the time transformation of Eq. (4). Schäfer saw from the \vec{f} terms in Eq. (5) that using the lowest-order equations of motion ($\mu \vec{a} + GM\mu\vec{r}/r^3 = 0$) in a highest-order term in the Lagrangian is equivalent to making a coordinate transformation. We can now see from the u term in Eq. (5) that using the lowest-order energy conservation ($\frac{1}{2}\mu v^2 - GM\mu/r - E_0 = 0$) in a highest-order term in the Lagrangian is equivalent to making a time transformation.

III. IDENTITY TRANSFORMATIONS

Let us now consider the identity transformations

$$\begin{aligned} \vec{r}' &= \vec{r} + \alpha_1 \vec{r} \left[\frac{1}{2} \frac{v^2}{c^2} - \frac{GM}{c^2 r} - \frac{\epsilon_0}{c^2} \right] \\ &\quad + \alpha_2 \vec{r} \left[\frac{\vec{a} \cdot \vec{r}}{c^2} + \frac{GM}{c^2 r} \right] + \alpha_3 r^2 \left[\frac{\vec{a}}{c^2} + \frac{GM}{c^2 r^3} \vec{r} \right], \end{aligned} \quad (8)$$

$$\begin{aligned} t' &= t + \int_0^t \left[\alpha_4 \left[\frac{1}{2} \frac{v^2}{c^2} - \frac{GM}{c^2 r} - \frac{\epsilon_0}{c^2} \right] \right. \\ &\quad \left. + \alpha_5 \left[\frac{\vec{a} \cdot \vec{r}}{c^2} + \frac{GM}{c^2 r} \right] \right] dt, \end{aligned} \quad (9)$$

where the α 's are constant. Because the additions to \vec{r} and t are single-zero terms, it follows that there will be *double-zero* terms on the right-hand side of Eq. (5). We have already shown that^{1,5} *double-zero* terms in a Lagrangian do not change the equations of motion to the order required, which is consistent with the obvious fact that an identity transformation also cannot change the equations of motion to order required. In general, the equations of motion obtained from a Lagrangian with *double-zero terms added* and those obtained from the original Lagrangian can be put into the same form by using the lowest-order equations of motion (and/or lowest-order energy-conservation equation) in the highest-order terms of the equations of motion. As already mentioned, the method of the *double-zero* is a convenient method for eliminating unwanted terms in a Lagrangian, and the above further demonstrates that the coordinates and time remain the same to the order required. This is why we found this method to be an excellent technique for eliminating the acceleration-dependent terms in the post-post-Newtonian electromagnetic Lagrangian.⁵

IV. EIH LAGRANGIAN WITH PPN PARAMETERS γ AND β

Let us next consider the two-body EIH Lagrangian with PPN parameters γ and β which can be written as⁷

$$\begin{aligned} \mathcal{L}^E(\vec{r}_E, \vec{v}_E) &= \frac{1}{2}\mu v_E^2 + GM\mu/r_E \\ &\quad + \frac{1}{8}\bar{k}_1 \mu v_E^4/c^2 + \frac{3}{2}\bar{k}_2 GM\mu v_E^2/c^2 r_E \\ &\quad + k'_2 GM\mu(\vec{v}_E \cdot \vec{r}_E)^2/c^2 r_E^3 \\ &\quad - \frac{1}{2}\bar{k}_3 G^2 M^2 \mu/c^2 r_E^2, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \bar{k}_1 &= 1 - \frac{3\mu}{M}, \quad \bar{k}_2 = \frac{1}{3} + \frac{2\gamma}{3} + \frac{\mu}{3M}, \\ k'_2 &= \frac{\mu}{2M}, \quad \bar{k}_3 = 2\beta - 1. \end{aligned} \quad (11)$$

We now make the coordinate and time transformations

$$\vec{r}_E = \vec{r} \left[1 - \alpha \frac{GM}{c^2 r} + \alpha_0 \frac{\epsilon_0}{c^2} \right], \quad (12)$$

$$t_E = t + \int_0^t \left[\alpha_I \frac{v^2}{c^2} + \alpha_{II} \frac{GM}{c^2 r} + \alpha_{III} \frac{\epsilon_0}{c^2} \right] dt. \quad (13)$$

For our present application, it is convenient to write Eq. (5) in the form

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}) &= \mathcal{L}^E(\vec{r}, \vec{v}) + \left[\mu \vec{v} \cdot \frac{d\vec{f}}{dt} - \frac{GM\mu}{r^3} \vec{r} \cdot \vec{f} \right] \\ &\quad - (\frac{1}{2}\mu v^2 - GM\mu/r - \mu \epsilon_0)u, \end{aligned} \quad (14)$$

where

$$\vec{f} = -\alpha \frac{GM}{c^2 r} \vec{r} + \alpha_0 \frac{\epsilon_0}{c^2} \vec{r}, \quad (15)$$

$$u = \alpha_I \frac{v^2}{c^2} + \alpha_{II} \frac{GM}{c^2 r} + \alpha_{III} \frac{\epsilon_0}{c^2}, \quad (16)$$

as can be seen from Eqs. (3), (4), (12), and (13). Using (15) and (16) in (14) gives us

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}) = & \mathcal{L}^E(\vec{r}, \vec{v}) + \alpha_0 \frac{\epsilon_0}{c^2} \left[\mu v^2 - \frac{GM\mu}{r} \right] \\ & + \alpha \left[\frac{GM\mu}{r} \left[\frac{(\vec{v} \cdot \vec{r})^2}{c^2 r^2} - \frac{v^2}{c^2} \right] + \frac{G^2 M^2 \mu}{c^2 r^2} \right] \end{aligned}$$

$$\begin{aligned} & - \left[\frac{1}{2} \mu v^2 - \frac{GM\mu}{r} - \mu \epsilon_0 \right] \\ & \times \left[\alpha_I \frac{v^2}{c^2} + \alpha_{II} \frac{GM}{c^2 r} + \alpha_{III} \frac{\epsilon_0}{c^2} \right]. \quad (17) \end{aligned}$$

From Eqs. (10) and (17), we obtain

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}) = & \alpha_{III} \mu \epsilon_0^2 / c^2 + \frac{1}{2} \mu v^2 + GM\mu / r + (2\alpha_0 - \alpha_{III} + 2\alpha_I) (\frac{1}{2} \mu v^2) \epsilon_0 / c^2 \\ & + (-\alpha_0 + \alpha_{III} + \alpha_{II}) (GM\mu / r) \epsilon_0 / c^2 + (\frac{1}{8} \bar{k}_1 - \frac{1}{2} \alpha_I) \mu v^4 / c^2 + (\frac{3}{2} \bar{k}_2 - \alpha + \alpha_I - \frac{1}{2} \alpha_{II}) GM\mu v^2 / c^2 r \\ & + (k'_2 + \alpha) GM\mu (\vec{v} \cdot \vec{r})^2 / c^2 r^3 + (\alpha - \frac{1}{2} \bar{k}_3 + \alpha_{II}) G^2 M^2 \mu / c^2 r^2. \quad (18) \end{aligned}$$

We next set

$$2\alpha_0 - \alpha_{III} + 2\alpha_I = 0, \quad (19)$$

$$-\alpha_0 + \alpha_{III} + \alpha_{II} = 0, \quad (20)$$

$$\frac{1}{8} \bar{k}_1 - \frac{1}{2} \alpha_I = 0, \quad (21)$$

$$\frac{3}{2} \bar{k}_2 - \alpha + \alpha_I - \frac{1}{2} \alpha_{II} = 0, \quad (22)$$

$$k'_2 + \alpha = 0. \quad (23)$$

The solution to the above five equations is easily found to be

$$\alpha = -\mu / 2M, \quad \alpha_0 = -2 - 2\gamma + \mu / M, \quad (24)$$

$$\alpha_I = \frac{1}{4} - \frac{3\mu}{4M}, \quad \alpha_{II} = \frac{3}{2} + 2\gamma + \frac{\mu}{2M}, \quad (25)$$

$$\alpha_{III} = -\frac{7}{2} - 4\gamma + \frac{\mu}{2M},$$

and thus Eq. (18) reduces to

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}) = & - \left[\frac{7}{2} + 4\gamma - \frac{\mu}{2M} \right] \frac{\mu \epsilon_0^2}{c^2} + \frac{1}{2} \mu v^2 + GM\mu / r \\ & + (2 + 2\gamma - \beta) G^2 M^2 \mu / c^2 r^2. \quad (26) \end{aligned}$$

The above Lagrangian (apart from the constant term which does not affect the equations of motion) was first given by one of us⁸ using lowest-order energy conservation in the highest-order terms of the Lagrangian, thus not showing all the coordinate and time transformations which we have explicitly given above. As mentioned in our previous work,⁸ this Lagrangian leads easily to the correct periastron precession.

V. ACCELERATION-DEPENDENT ELECTROMAGNETIC LAGRANGIAN

Let us now consider the pedagogically useful acceleration-dependent electromagnetic Lagrangian⁹ that we have introduced before,^{1,2}

$$\mathcal{L}^I(\vec{r}_I, \vec{v}_I) = \frac{1}{2} \mu v_I^2 - e_1 e_2 / r_I + (e_1 e_2 / c^2) \vec{a}_I \cdot \vec{n}_0, \quad (27)$$

where \vec{n}_0 is a constant unit vector, $\vec{r}_I = \vec{r}_{1I} - \vec{r}_{2I}$, and the I indicates an inertial coordinate system. The last term in Eq. (27) can be written as a total time derivative, and thus does not affect the equations of motion which are

$$\mu \vec{a}_I = e_1 e_2 \vec{r}_I / r_I^3. \quad (28)$$

Let us now make the coordinate transformation

$$\vec{r}_I = \vec{r} + \vec{n}_0 e_1 e_2 / \mu c^2, \quad (29)$$

and obtain from Eq. (5) (with $GM\mu$ replaced by $-e_1 e_2$)

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}) = & \frac{1}{2} \mu v^2 - e_1 e_2 / r + \frac{e_1^2 e_2^2}{\mu c^2} \frac{\vec{r} \cdot \vec{n}_0}{r^3} \\ & + \frac{d}{dt} \left[\frac{e_1 e_2}{c^2} \vec{v} \cdot \vec{n}_0 \right], \quad (30) \end{aligned}$$

which gives us the equations of motion

$$\mu \vec{a} = \frac{e_1 e_2 \vec{r}}{r^3} + \frac{e_1^2 e_2^2}{\mu c^2} \left[\frac{\vec{n}_0}{r^3} - \frac{3(\vec{r} \cdot \vec{n}_0) \vec{r}}{r^5} \right] \quad (31)$$

in the *noninertial curvilinear coordinate system*.

If we now use Eq. (28) in (27) to eliminate the acceleration term, we obtain the new Lagrangian

$$\mathcal{L}^*(\vec{r}_I, \vec{v}_I) = \frac{1}{2} \mu v_I^2 - e_1 e_2 / r_I + \frac{e_1^2 e_2^2}{\mu c^2} \frac{\vec{r}_I \cdot \vec{n}_0}{r_I^3}, \quad (32)$$

which (except for the total time derivative term) is of the same form as Eq. (30). Clearly, Eq. (32) is *incorrect* because it is in terms of the *wrong coordinates*.

We conclude that when one uses the lowest-order equations of motion in the highest-order terms of a Lagrangian one must also replace the old coordinate labels with new coordinate labels because an implicit coordinate transformation has taken place. But even after doing this, the situation will not be completely satisfactory because one would still not know the coordinate transformation between the old and new coordinates. However, one can find the coordinate transformation from Eq. (5) which is

how we found Eq. (29). (Everything said above in this paragraph regarding lowest-order equations of motion also applies to lowest-order energy conservation if one replaces implicit coordinate transformation with implicit time transformation.)

VI. CONCLUSIONS

Our primary conclusion is that the use of lowest-order energy conservation in the highest-order terms in a Lagrangian is equivalent to a time transformation (complementing Schäfer's observation that the use of lowest-order equations of motion in the highest-order terms in a Lagrangian is equivalent to a coordinate transformation). We also showed that *double-zero* terms may be added to a Lagrangian by the use of an identity transformation. We then showed that the combined use of coordinate and time transformations provide the simplest and most rigorous method for simplifying the two-body EIH Lagrangian with PPN parameters γ and β .

Finally, we conclude that the Golubenkov-Smorodinskii treatment of the acceleration terms in the post-post-Newtonian electromagnetic Lagrangian is equivalent to a coordinate transformation to a noninertial coordinate system—a not-very-useful end result, since in electromagnetic theory one prefers to work with inertial coordinates (which is what we used when employing the method of the double zero⁵). Because the acceleration terms are post-post-Newtonian, the $\vec{f}(\vec{r}, \vec{v}, \vec{a})$ of Eq. (3) must be of order rv^4/c^4 . The pedagogically useful acceleration-dependent post-Newtonian electromagnetic Lagrangian that we introduced in Sec. V was used to illustrate many points in a simple way that would have been much more cumbersome to do with the post-post-Newtonian electromagnetic Lagrangian.

APPENDIX A

Let us define $\delta\mathcal{L}/\delta r$ in the usual way as

$$\frac{\delta\mathcal{L}}{\delta\vec{r}} \equiv \frac{\partial\mathcal{L}}{\partial\vec{r}} - \frac{d}{dt} \left[\frac{\partial\mathcal{L}}{\partial\vec{v}} \right] + \frac{d^2}{dt^2} \left[\frac{\partial\mathcal{L}}{\partial\vec{a}} \right], \quad (\text{A1})$$

where $\mathcal{L} = \mathcal{L}(\vec{r}, \vec{v}, \vec{a})$. The equations of motion are then $\delta\mathcal{L}/\delta\vec{r} = 0$. We now want to find the equations of motion for the Lagrangian of Eq. (5). To this end, we note that

$$\begin{aligned} \frac{\delta}{\delta\vec{r}} [(\mu\vec{a} + GM\mu\vec{r}/r^3) \cdot \vec{f}] &= -\frac{3GM\mu}{r^5} \vec{r}(\vec{r} \cdot \vec{f}) \\ &+ \frac{GM\mu}{r^3} \vec{f} + \mu \frac{d^2\vec{f}}{dt^2}, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \frac{\delta}{\delta\vec{r}} [(\frac{1}{2}\mu v^2 - GM\mu/r - E_0)u] &= (GM\mu\vec{r}/r^3 - \mu\vec{a})u \\ &- \mu\vec{v} du/dt. \end{aligned} \quad (\text{A3})$$

We can always use $\mu\vec{a} + GM\mu\vec{r}/r^3 = 0$ and $\frac{1}{2}\mu v^2 - GM\mu/r - E_0 = 0$ in the highest-order terms of the equations of motion, and we have done so in obtaining the right-hand side of Eqs. (A2) and (A3). Thus, it turns out if $\vec{f} = \vec{f}(\vec{r}, \vec{v}, \vec{a}, \dot{\vec{a}}, \dots)$ and $u = u(\vec{r}, \vec{v}, \vec{a}, \dot{\vec{a}}, \dots)$, rather

than stopping at acceleration dependence, that Eq. (A2) and (A3) *would still be the same*, even though extra terms would have to be added to Eq. (A1). Hence, the above generalization could be made throughout this paper. Our equations of motion are

$$\begin{aligned} &-(\mu\vec{a} + GM\mu\vec{r}/r^3) + \delta\mathcal{L}'_1(\vec{r}, \vec{v}, \vec{a})/\delta\vec{r} \\ &- \left[-\frac{3GM\mu}{r^5} \vec{r}(\vec{r} \cdot \vec{f}) + \frac{GM\mu}{r^3} \vec{f} + \mu \frac{d^2\vec{f}}{dt^2} \right] \\ &-(GM\mu\vec{r}/r^3 - \mu\vec{a})u + \mu\vec{v} du/dt = 0. \end{aligned} \quad (\text{A4})$$

The equations of motion for the Lagrangian of Eq. (1) are

$$-(\mu\vec{a}' + GM\mu\vec{r}'/r'^3) + \delta\mathcal{L}'_1(\vec{r}', \vec{v}', \vec{a}')/\delta\vec{r}' = 0. \quad (\text{A5})$$

We next make the coordinate and time transformations of Eqs. (3) and (4) and from Eq. (4) we obtain

$$dt'/dt = 1 + u. \quad (\text{A6})$$

From Eq. (3) we obtain (to order needed)

$$\frac{\vec{r}'}{r'^3} = \frac{\vec{r}}{r^3} - \frac{3\vec{r}(\vec{r} \cdot \vec{f})}{r^5} + \frac{\vec{f}}{r^3}. \quad (\text{A7})$$

From Eqs. (3) and (A6) we obtain (to order needed)

$$\vec{v}' \equiv \frac{d\vec{r}'}{dt'} = \frac{d\vec{r}'}{dt} \frac{dt}{dt'} = \vec{v} + \frac{d\vec{f}}{dt} - \vec{v}u, \quad (\text{A8})$$

$$\vec{a}' \equiv \frac{d\vec{v}'}{dt'} = \frac{d\vec{v}'}{dt} \frac{dt}{dt'} = \vec{a}(1-2u) - \vec{v} \frac{du}{dt} + \frac{d^2\vec{f}}{dt^2}. \quad (\text{A9})$$

Also (to order needed)

$$\delta\mathcal{L}'_1(\vec{r}', \vec{v}', \vec{a}')/\delta\vec{r}' = \delta\mathcal{L}'_1(\vec{r}, \vec{v}, \vec{a})/\delta\vec{r}. \quad (\text{A10})$$

Using Eqs. (A7), (A9), and (A10) in Eq. (A5) gives us

$$\begin{aligned} &-(\mu\vec{a} + GM\mu\vec{r}/r^3) + \delta\mathcal{L}'_1(\vec{r}, \vec{v}, \vec{a})/\delta\vec{r} \\ &- \left[-\frac{3GM\mu}{r^5} \vec{r}(\vec{r} \cdot \vec{f}) + \frac{GM\mu}{r^3} \vec{f} + \mu \frac{d^2\vec{f}}{dt^2} \right] \\ &- (-2\mu\vec{a})u + \mu\vec{v} du/dt = 0. \end{aligned} \quad (\text{A11})$$

Multiplying Eq. (A11) by $(1+u)$ gives us Eq. (A4) (to order needed). We have thus proved that Eq. (5) is correct.

APPENDIX B

We wish to show that Eq. (7) reduces to Eq. (5). We will be working to the post-Newtonian approximation. Using Eqs. (1) and (A6), we obtain from Eq. (7) (note $\mathcal{L}_0 \equiv \mathcal{L}'_0$)

$$\begin{aligned} \mathcal{L}(\vec{r}, \vec{v}, a) &= \mathcal{L}_0(\vec{r}', \vec{v}') + \mathcal{L}_0(\vec{r}, \vec{v})u \\ &+ \mathcal{L}'_1(\vec{r}, \vec{v}, \vec{a}) + E_0u. \end{aligned} \quad (\text{B1})$$

Using Eqs. (3) and (A8) along with a Taylor expansion, we obtain

$$\begin{aligned} \mathcal{L}_0(\vec{r}', \vec{v}') &= \mathcal{L}_0(\vec{r}, \vec{v}) + \frac{\partial \mathcal{L}_0(\vec{r}, \vec{v})}{\partial \vec{r}} \cdot \vec{f} + \frac{\partial \mathcal{L}_0(\vec{r}, \vec{v})}{\partial \vec{v}} \cdot \left[\frac{d\vec{f}}{dt} - \vec{v}u \right] \\ &= \mathcal{L}_0(\vec{r}, \vec{v}) + \frac{d}{dt} \left[\frac{\partial \mathcal{L}_0(\vec{r}, \vec{v})}{\partial \vec{v}} \cdot \vec{f} \right] + \left[\frac{\partial \mathcal{L}_0(\vec{r}, \vec{v})}{\partial \vec{r}} - \frac{d}{dt} \left[\frac{\partial \mathcal{L}_0(\vec{r}, \vec{v})}{\partial \vec{v}} \right] \right] \cdot \vec{f} - \frac{\partial \mathcal{L}_0(\vec{r}, \vec{v})}{\partial \vec{v}} \cdot \vec{v}u . \end{aligned} \quad (\text{B2})$$

Using [see Eq. (2)]

$$\mathcal{L}_0(\vec{r}, \vec{v}) = \frac{1}{2}\mu v^2 + GM\mu/r \quad (\text{B3})$$

in Eq. (B2) gives us

$$\begin{aligned} \mathcal{L}_0(\vec{r}', \vec{v}') &= \mathcal{L}_0(\vec{r}, \vec{v}) + d(\mu \vec{v} \cdot \vec{f})/dt \\ &\quad - (\mu \vec{a} + GM\mu \vec{r}/r^3) \cdot \vec{f} - \mu v^2 u . \end{aligned} \quad (\text{B4})$$

Finally, using Eqs. (B3) and (B4) in (B1) gives us Eq. (5).

¹B. M. Barker and R. F. O'Connell, Phys. Lett. **78A**, 231 (1980).

²B. M. Barker and R. F. O'Connell, Ann. Phys. (N.Y.) **129**, 358 (1980).

³V. N. Golubenkov and I. A. Smorodinskii, Zh. Eksp. Teor. Fiz. **31**, 40 (1956) [Sov. Phys. JETP **4**, 55 (1957)].

⁴L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), 4th revised English edition.

⁵B. M. Barker and R. F. O'Connell, Can. J. Phys. **58**, 1659 (1980).

⁶G. Schäfer, Phys. Lett. **100A**, 128 (1984).

⁷B. M. Barker and R. F. O'Connell, Phys. Rev. D **14**, 861 (1976).

⁸R. F. O'Connell, Gen. Relativ. Gravit. **7**, 805 (1976).

⁹See Eq. (3) of Ref. 1. The $\vec{r}, \vec{v}, \vec{a}$ of Ref. 1 are $\vec{r}_I, \vec{v}_I, \vec{a}_I$ in the notation of the present paper.