

# Lecture Notes in Physics

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## Laser Physics

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## DISTRIBUTION FUNCTIONS IN QUANTUM OPTICS

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Perhaps the simplest way of including quantum mechanics, in various problems in quantum optics, is by use of the classical-quantum-correspondence method, by means of which one replaces quantum-mechanical operators by complex numbers. This is carried out by means of quasi-classical distribution functions and here we address the question of which is the best choice of function from the large selection available. Whereas Glauber's  $P(\alpha)$  distribution is useful in many applications, it does not exist as a well-behaved function in many others. In such cases, a more useful function is the generalized  $P$ -representation of Drummond and Gardiner. However, based on simplicity and overall applicability, we conclude that Wigner's function also has a claim to be the optimum choice.

### 1. Introduction

Many problems in quantum optics involve dissipative processes as an essential element. The usual approach to such problems is to consider a system of interest,  $S$  say, coupled to a reservoir ( $R$ ), and interacting with it via a potential  $V$  which results in  $S$  losing energy to  $R$ . From the fluctuation-dissipation theorem [1], we know that the dissipation is related to the fluctuations of the system in equilibrium. Of paramount interest is the time development of the distribution functions for the system. These remarks make it clear that we are in the province of irreversible statistical mechanics and thus--both for the purpose of putting quantum optics problems in perspective and also in the hope of finding the optimum way of solving such problems--we present in Section 2 an overview of the usual approach to traditional problems in irreversible statistical mechanics for comparison with the quantum optics situation.

In particular, we point out in Section 2 that perhaps the simplest way of including quantum mechanics is by means of the so-called classical-quantum correspondence (CQC), a technique originally introduced by Wigner [2], which makes use of quasi-classical distribution functions. In Section 3 we point

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out that there is literally an infinite choice of such functions and thus we are led to make a choice on the basis of what properties we would like our functions to have. This leads us into a discussion of the more widely used functions. For example, we argue that Wigner's choice of distribution function [2] has particularly desirable properties as far as traditional problems are concerned which might suggest that this should also be the choice in the solution of problems in quantum optics. This leads us to examine, in Section 4, a miscellany of topical problems in quantum optics which have been treated by the use of various distribution functions. We conclude that strong claims can be made for the use of either the so-called generalized P-representation or the Wigner distribution for treating problems in quantum optics--our own feeling being that further investigations are required before definitive conclusions can be reached.

## 2. Irreversible Statistical Mechanics/Quantum Optics/...:an Overview

The time evolution of a large system of particles is a common theme in a broad area of studies in non-equilibrium statistical mechanics [3], quantum optics [4-7], and in the many phenomena which are grouped under the common umbrella of synergetics [8,9]. The central equation of classical statistical mechanics is the Liouville equation

$$\frac{\partial P(q,p,t)}{\partial t} = \{H(q,p), P(q,p,t)\}, \quad (1)$$

where  $P$  is a time-dependent distribution function and  $(q,p)$  denotes the phase-space coordinates  $(q_1, \dots, q_N, p_1, \dots, p_N)$  where  $N$  is the number of particles. Also,  $H$  is the Hamiltonian and the curly bracket denotes the Poisson bracket. The state of the system at a given time  $t$  is completely specified by  $P(q,p,t)$ , which is the probability density for finding the particle at the point  $q,p$  in phase space.

In quantum-statistical mechanics, the basic equation is that of von Neumann:

$$\frac{\partial \hat{\rho}(t)}{\partial t} = (i\hbar)^{-1} [\hat{H}, \hat{\rho}(t)] \quad (2)$$

where  $\hat{\rho}(t)$  is the density matrix,  $\hat{H}$  is the Hamiltonian operator and the square bracket denotes the commutator. Since, as we will demonstrate in Section 3, the use of a GQC can be used to convert Eq. (2) into a quantum

Section 3, the use of a CQC can be used to convert Eq. (2) into a quantum Liouville equation i.e. a Liouville equation with quantum corrections taken into account exactly, by the use of c-numbers as distinct from quantum-mechanical operators, the starting point of our discussion will be Eq. (2), as we illustrate in Fig. 1. The rest of this section will be devoted to a discussion of Fig. 1.

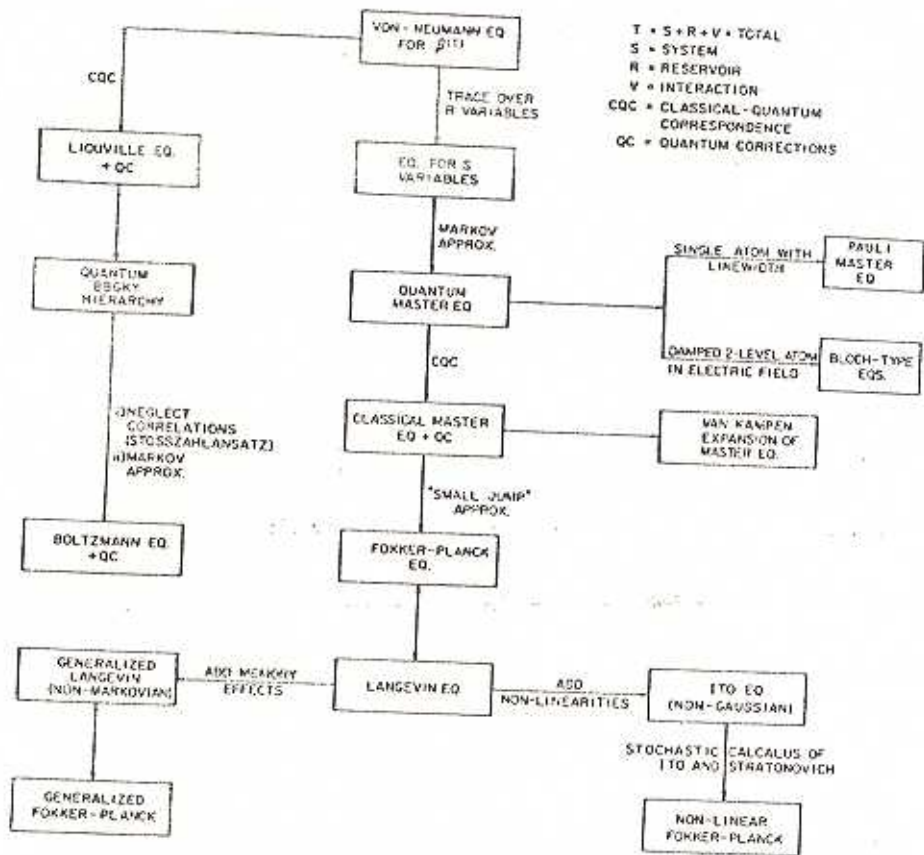


Fig. 1 Irreversible Statistical Mechanics/Quantum Optics/...: an Overview

The left-hand chain of Fig. 1 illustrates a route towards the traditional kinetic equations of statistical mechanics. First of all, the generalized Liouville equation is rewritten exactly in terms of the BBGKY hierarchy [3,10], which is a set of  $N$  equations for the reduced distribution functions: the rate of change of the distribution function,  $P_M$  say, for  $M$  particles ( $M \leq N-1$ ) depends on  $P_{M+1}$  as well as  $P_M$ . However, because of the complexity of these exact equations, various approximations are usually made to reduce them to kinetic equations i.e. equations (generally non-linear) for the time-evolution of the one-particle reduced distribution function. The most famous of such equations is the celebrated Boltzmann equation [3,11,12,13]

$$\frac{\partial P}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial P}{\partial \vec{q}} + \vec{F} \cdot \frac{\partial P}{\partial \vec{p}} = C(P) \quad (3)$$

where  $\vec{q}$  and  $\vec{p}$  now refer to the coordinates and momenta of a single particle (in contrast to the case of the  $N$  particles considered in Eq. (1)), where  $\vec{F}$  is the external force acting on the particle, and  $C(P)$  denotes the rate of change of  $P$  due to collisions with the other particles.

Inherent in a kinetic description is the use of a Markovian approximation i.e. the rate of change of  $P$  at time  $t$  depends only on its value at that time and not on its previous history (no memory effects). Another assumption underlying the derivation of the Boltzmann equation is the neglect of correlations caused by the interactions. This is referred to as Boltzmann's Stosszahlansatz and, in essence, it means that the time duration of a collision (i.e. the time during which the particle trajectories differ significantly from a straight line) is much smaller than the time between collisions. As a result, the interaction process is confined to two-body collisions and the Boltzmann equation is only applicable to a dilute gas. Another point of note is the time reversible character of the BBGKY hierarchy vis-à-vis the irreversible character of the Boltzmann equation, a necessary but not sufficient condition for the latter being the presence of interactions [3,13].

There is, of course, a host of other kinetic equations [13], such as the Landau equation and the Vlasov equation but the Boltzmann equation is a prototype for displaying all the general features of kinetic equations relevant to our present purposes. All of these kinetic equations treat a "system" of one particle interacting via collisions with a reservoir of  $N-1$  particles. This observation brings us to the second branch of Fig. 1 where

the total system of  $N$  particles is considered as consisting of a system  $S$  and a reservoir  $R$  interacting via a potential  $V$ . Since our interest lies in the system and not in the reservoir, a trace is carried out over the  $R$  variables, resulting in an equation for the  $S$  variables. This is the starting point of many investigators in a variety of disciplines such as, for example, Zwanzig, Nakajima, Mori and Prigogine [14] in the general area of statistical mechanics; Lax [5], Louisell [4], and Haken [7-9] in quantum optics and synergetics; and Barker and Ferry [15] in their investigations of transport properties in small devices.

Next, applying a Markov approximation leads to a quantum master equation, particular examples of same being the original master equation derived by Pauli for a single atom with line-width and the Bloch-type equations (familiar from NMR studies) for the case of a damped two-level atom in an electric field. The quantum master equation is essentially the differential form of the Chapman-Kolmogorov equation for Markov processes [16] and it amounts to having carried out a coarse-grained averaging in time. It is at this stage that many authors use the CQC to get a c-number equation with c-number quantum corrections. Whereas a general expansion of the master equation is possible [16], the next reduction is more commonly that of making the so-called "small-jump" approximation to obtain the Fokker-Planck equation, from which the stochastic Langevin equation may be derived and vice versa. The Fokker-Planck and/or Langevin equations have been applied not only to a large selection of problems in quantum optics--which constitutes our present interest--but also to areas as diverse as condensed matter physics [17] and nuclear physics [18]. It should be emphasized that both methods should lead to the same result, as was found, for example, in investigations of rotational Brownian motion [19]. It is of interest to note that a conclusion arising from the latter work is that the solution of the Langevin stochastic equation presented less mathematical difficulties than the corresponding Fokker-Planck equation but that the latter allowed results to be calculated to a higher order of approximation. Also, Agarwal [20] has concluded that Langevin equations are easier to interpret than the master equation.

Generalizations of the Langevin equation (and hence the Fokker-Planck equation) include the addition of memory (non-Markovian effects) and nonlinearities, the latter addition resulting in a complete new stochastic calculus due to Ito and Stratonovich [21].

The Langevin and Fokker-Planck equations and their generalizations have found wide application in quantum optics, some examples of which we will discuss in Section 4 but before doing so, we will first of all discuss (in

Section 3) the method of the CQC which leads to such equations.

In closing this section, we should emphasize that the scenario outlined in the block diagram of Fig. 1 is not the only route to the "Holy Grail". However, our purpose will be fulfilled if it enables the non-expert to obtain some insight into the interconnection of the many different approaches to a myriad of problems which possess, however, certain unifying aspects.

### 3. Classical-Quantum Correspondence (CQC)

In classical mechanics the average of any function of  $q$  and  $p$ ,  $A(q,p)$  say, may be written as

$$\langle A \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp A(q,p) P_{cl}(q,p) \quad (4)$$

where for clarity we have added a subscript "cl" to  $P(q,p)$ . The latter quantity will now be used to denote the so-called quasi-classical distribution function and, in general, it depends on  $\hbar$ .

In 1932, Wigner [2] presented an exact reformulation of non-relativistic quantum mechanics in terms of classical concepts. In particular, he showed that the ensemble average of a function of the position and momentum operators,  $\hat{A}(\hat{q},\hat{p})$  say, may be written in a form similar to Eq. (4), as follows:

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp A(q,p) P(q,p) \quad (5)$$

The next question to be considered relates to the choice of  $P$ . As it turns out, there is literally an infinite choice of such functions. However, if one demands some compelling properties then the choice considerably narrows. In particular, Wigner [2] chose the function

$$P_W(q,p) = (\pi\hbar)^{-1} \int_{-\infty}^{\infty} dy \psi^*(q+y) \psi(q-y) e^{2ipy/\hbar} \quad (6)$$

where we have assumed for simplicity that the system is in a pure state

$\psi(q)$ . Since the properties of  $P_W$  have been discussed in detail elsewhere [22,23] we focus our attention on the property most relevant to our present considerations viz. that in the force-free case the equation of motion is the classical one i.e.

$$\frac{\partial P_W}{\partial t} = - \frac{p}{m} \frac{\partial P_W}{\partial q}, \quad (7)$$

for a one-dimensional configurational problem (to which we will confine ourselves from henceforth since generalization is straightforward).

No other distribution function that we are aware of enjoys this property. For example, the non-negative distribution function of Husimi [24] contains an additional  $\hbar^2$  term which is not of quantum origin [23]. In addition, when an external potential is included [23], the quantum corrections which result are much simpler if one uses  $P_W$  in contrast to Husimi's function.

Let us turn now to some distribution functions which, in addition to  $P_W$ , are those most commonly used in quantum optics. A convenient way of introducing these functions is by the use of characteristic functions [25,4], which are simply the Fourier transforms of the respective distribution functions. Thus, for  $P_W(q,p)$ , the corresponding characteristic function is

$$C_W(\sigma, \tau) = \langle \psi | \exp \left[ \frac{i}{\hbar} (\sigma \hat{q} + \tau \hat{p}) \right] | \psi \rangle, \quad (8)$$

or, equivalently, in terms of creation and annihilation operators  $\hat{a}$  and  $\hat{a}^+$  (defined in terms of  $\hat{p}$  and  $\hat{q}$  in the usual way):

$$C_W(\eta, \eta^*) = \langle \psi | \exp \left[ \frac{i}{\hbar} (\eta \hat{a} + \eta^* \hat{a}^+) \right] | \psi \rangle. \quad (9)$$

In quantum optics, other common choices are the normal and anti-normal functions, denoted by  $C_n$  and  $C_a$  respectively, where

$$C_n(\eta, \eta^*) = \langle \psi | \exp \left[ \frac{i}{\hbar} \eta^* \hat{a}^+ \right] \exp \left[ \frac{i}{\hbar} \eta \hat{a} \right] | \psi \rangle, \quad (10)$$

and  $C_a$  is the same as  $C_n$  with the order of the exponential factors interchanged. We remark that the distribution function corresponding to  $C_n, P_n$  say, is the same as Glauber's widely-used  $P(\alpha)$  distribution [4] if, in the evaluation of the expectation value of an arbitrary operator, we first write

note!



it in normal ordering sequence (creation operators precede annihilation operators) prior to carrying out the CQC.

If we now use the Baker-Hausdorff theorem and then convert back from the  $\hat{a}, \hat{a}^\dagger$  language to the  $\hat{q}, \hat{p}$  language, we find that the corresponding distribution functions may be written in terms of  $P_W$  as follows:

$$P_{a,n}(q,p) = \exp \left( \pm \frac{1}{2} q_0^2 \frac{\partial^2}{\partial q^2} \pm \frac{1}{2} p_0^2 \frac{\partial^2}{\partial p^2} \right) P_W, \quad (11)$$

where  $q_0^2 = \hbar/2m\omega$  and  $p_0^2 = m\hbar\omega/2$ . Next, using Eq. (7), it immediately follows that the time derivative of  $P_{a,n}$  is not as simple as that of  $P_W$  since it contains also second derivative terms. Thus, quantum corrections to Eq. (3), the Boltzmann equation, are simpler in form if one chooses  $P_W$ . In other words, if we solve a problem via the left-hand route of Fig. 1 it is clearly best to choose  $P_W$ . Thus, we might expect the same to hold if we choose the right-hand route--the route of quantum optical investigations. In the next section, we examine briefly some specific applications to enable us to judge what happens in practice.

#### 4. Examples from Quantum Optics

Glauber's  $P(\alpha)$  function ( $P_a$ ) has been the most widely used distribution function in quantum optics, primarily because of its convenience in averaging the normally ordered operator products that often arise in problems in this area. In particular, it was applied to the laser by Weidlich et al. [26], to dispersive optical bistability by Drummond and Walls [27], to the damped harmonic oscillator [28], and to a variety of other problems. However, it was long recognized [4,5,7] that the corresponding Fokker-Planck equations often have non-positive-definite diffusion coefficients. Nevertheless, it was only recently that this problem was taken seriously, principally in the papers of Drummond, Gardiner, and Walls [29,30]. This problem basically arises in dealing with intrinsically non-classical effects, such as photon anti-bunching which occurs in atomic fluorescence experiments [31]. For such nonclassical photon fields,  $P_a$  does not exist as a well-behaved function. In order to avoid such problems, Drummond and Gardiner [29], introduced a class of generalized  $P$ -representations, which include the complex  $P$  representation, in which  $(\alpha, \alpha^*)$  are replaced by the independent complex variables  $(\alpha, \beta)$ . Such

generalized representations were applied successfully to non-linear problems in quantum optics (two-photon absorption; dispersive bistability; degenerate parametric amplifier) and chemical reaction theory [29,30,32].

However, calculations using the generalized P representations are relatively complicated. Since they were designed in effect to handle situations for which the Glauber distribution does not exist the following question naturally arises: why not use the Wigner distribution since it always exists and its equation of motion is simple in that it possesses the desirable property discussed in the last section? While we were pondering such a question in the course of completing the present paper, a paper by Lugiato, Casagrande, and Pizzuto [33] just appeared in which the Wigner distribution was used in the consideration of a system of  $N$  two-level atoms interacting with a resonant mode radiation field and coupled to suitable reservoirs. The presence of an external CW coherent field injected into the cavity is also included, which allows for the possibility of treating optical bistability as well as a laser with injected signal. They then carry out very detailed calculations to obtain Fokker-Planck equations, which they compare with the corresponding ones obtained using the Glauber function, and conclude that the use of the Wigner function is preferable to the Glauber function.

In the problems considered by Lugiato et al., the existence of a smallness parameter  $N_s^{-1/2}$  (where  $N_s$  is the saturation photon number) enabled them to truncate at the second order, which is the essence of the Fokker-Planck approximation. On the other hand, Walls and Milburn [32] conclude that, in the case of dispersive bistability and two-photon absorption, the use of the generalized P representation is preferable to the use of the Wigner function because the latter gives rise to equations containing third order derivatives. It is thus clear that further investigations are required before definitive conclusions can be reached. In particular, does the flexibility which occurs by the use of a complex phase space give the generalized P representation some unique and desirable property which accounts for its success in applications?

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