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# INTERNAL CONVERSION COEFFICIENTS FOR HIGH-ENERGY TRANSITIONS

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**Abstract:** For transition energies $\omega \gg mc^2$ (rest-energy of the electron) our previously derived exact analytical results for the internal conversion coefficients (ICC) for all shells and for all transition energies (point-nucleus, no screening model) simplify considerably. We find to lowest order in $1/\omega$ that the ICC are proportional to $1/\omega$ and that for a particular $Z$ and shell designation, the electric and magnetic ICC for all multipoles are equal. The corresponding numerical results, are presented for the K, L and M subshells.

In a previous publication [1]) we derived exact analytical results for the internal conversion coefficients (hereafter referred to as ICC), for all shells and for all transition energies (point nucleus, no screening model). We have also examined in detail the ICC at threshold [1,2]). It is further of interest to investigate the other end of the spectrum, i.e., the ICC for high transition energies. Actually, the highest energy [3]) at which ICC can be measured †† corresponds to $\omega \approx 5$ (or 2.5 MeV). Notations introduced without explanation are the same as in ref. [1]) and equation numbers such as (I.26) refer to the corresponding formulae in ref. [1]). It will be convenient to introduce the symbols

$$X \equiv 2ip/(\lambda + ip - i\omega), \tag{1}$$

$$y \equiv \gamma + \gamma' + t - k. \tag{2}$$

In the high-energy limit we have

$$p \to W \to \omega \gg 1, \tag{3}$$

$$v \to \rho, \quad (\lambda + ip - i\omega) \to \lambda + iW_B, \tag{4}$$

$$X \to 2\omega \exp\{i \operatorname{arctg}(\lambda/W_B)\}. \tag{5}$$

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†† In our units $\hbar = c = m = 1$, where $m$ denotes the mass of the electron.

Of course, the condition $\omega R \ll 1$ (where $R_{\max} < \frac{1}{50}$) is always understood. Since $X$ is of order $\omega$ we can expand the hypergeometric functions [4]) appearing in eq. (I.26) in a power series in $1/\omega$. For example

$$
\begin{aligned}
{}_2F_1(\gamma+i\nu, y; 2\gamma+1; X) = {} & \frac{\Gamma(2\gamma+1)\Gamma(\gamma'+t-k-i\rho)}{\Gamma(\gamma+\gamma'+t-k)\Gamma(\gamma+1-i\rho)}(-X)^{-\gamma-i\rho} \\
& + \frac{\Gamma(2\gamma+1)\Gamma(-\gamma'-t+k+i\rho)}{\Gamma(\gamma+i\rho)\Gamma(\gamma+1-\gamma'-t+k)}(-X)^{-\gamma-\gamma'-t+k} \\
& + \text{higher powers of } (1/X).
\end{aligned} \tag{6}
$$

Thus

$$
\begin{aligned}
(-X)^{-k}c^{\pm} \to {} & \omega^{\frac{1}{2}}e^{-i\eta}\Gamma(2\gamma'+1)\frac{n'!}{t!(n'-t)!}\frac{(-2\lambda)^t\Gamma(y)\Gamma(2\gamma+1)}{\Gamma(2\gamma'+t+1)(\lambda+iW_B)^y}(-X)^{-\gamma} \\
& \times\left\{-\frac{\kappa\Gamma(\gamma'+t-k-1-i\rho)}{\Gamma(y)\Gamma(\gamma-i\rho)}(-X)^{-1-i\rho-k} - \frac{\kappa\Gamma(1+i\rho-\gamma'-t+k)}{\Gamma(\gamma+1+i\rho)\Gamma(\gamma+1-\gamma'-t+k)}(-X)^{-\gamma'-t}\right. \\
& \left.\pm\frac{\Gamma(\gamma'+t-k-i\rho)}{\Gamma(y)\Gamma(\lambda-i\rho)}(-X)^{-i\rho-k} \pm \frac{(\gamma-i\rho)\Gamma(k-\gamma'-t+i\rho)}{\Gamma(\gamma+i\rho)\Gamma(\gamma+1-\gamma'-t+k)}(-X)^{-\gamma'-t}\right\}.
\end{aligned} \tag{7}
$$

We have considered $X^{-k}c^{\pm}$ due to the fact that we have a factor $\omega^{-k}$ appearing in our eq. (I.19) for $a$. Examination of the other terms contributing leads to the result

$$
\omega\beta_L^{(\lambda)} = A + B\omega^{-\gamma'} + C\omega^{-1} + \ldots, \tag{8}
$$

where $A$, $B$ and $C$ are quantities independent of $\omega$. The only contribution to $A$ will come from the third term in our eq. (7) for $\omega^{-k}c^{\pm}$ (and then only for $k = 0$); the second and fourth terms (with $t = 0$) will contribute to $B$ and the first term (with $k = 0$) will contribute to $C$. Now $C$ will be more difficult to compute because the third term will also contribute to it (with $k = 1$) as well as higher order terms arising from our expansion in eq. (6).

We here concern ourselves with the evaluation of the lowest order term $A$. To this order, we obtain (omitting factors which give unity when we take the absolute value squared to get $\beta_L^{(\lambda)}$)

$$
\begin{aligned}
d_1 ac^{\pm} = {} & \pm\frac{1}{\omega}(2\pi\lambda)^{\frac{1}{2}}(1+W_B)^{\frac{1}{2}}\lambda^{\gamma'+1}2^{\gamma'} \\
& \times\exp\left[-\rho\,\mathrm{arctg}(W_B/\lambda)\right]\left[\frac{\Gamma(2\gamma'+n'+1)}{n'!\rho(\rho-\lambda\kappa')}\right]^{\frac{1}{2}} g(t),
\end{aligned} \tag{9}
$$

where

$$
g(t) = \frac{n'!}{(n'-t)!t!}\left(\frac{-2\lambda}{\lambda+iW_B}\right)^t\frac{\Gamma(\gamma'+t-i\rho)}{\Gamma(2\gamma'+t+1)}. \tag{10}
$$

Further manipulation enables us to evaluate analytically the summation over $t$. We obtain

$$\sum_{t=0}^{n'} g(t) = \frac{\Gamma(\gamma'-i\rho)}{\Gamma(2\gamma'+1)}\, {}_2F_1\left[(\gamma'-i\rho),\, -n';\, 2\gamma'+1;\, \frac{2\lambda}{\lambda+iW_B}\right], \tag{11}$$

$$\sum_{t=0}^{n'} (n'-t)g(t) = \frac{n'\Gamma(\gamma'-i\rho)}{\Gamma(2\gamma'+1)}\, {}_2F_1\left[(\gamma'-i\rho),\, 1-n';\, 2\gamma'+1;\, \frac{2\lambda}{\lambda+iW_B}\right]. \tag{12}$$

Thus, to this order, the results for the radial integrals contain no $t$ or $k$ summations. We also find that

$$R_2 = -iR_6, \tag{13a}$$

$$R_3 = R_1, \tag{13b}$$

$$R_4 = iR_6, \tag{13c}$$

$$R_5 = -iR_1. \tag{13d}$$

Then using the relation [3])

$$C_{-\kappa,\kappa'} = B_{\kappa\kappa'}/(\kappa+\kappa')^2 \tag{14}$$

we conclude that

$$\beta_L^{(0)} = \beta_L^{(1)} = \frac{\alpha\omega}{16\pi}\, \frac{1}{L(L+1)(2L+1)} \sum_{\kappa} B_{\kappa\kappa'} |R_1 + iR_6|^2, \tag{15}$$

i.e., the corresponding electric and magnetic ICC are equal. Furthermore we notice that the radial integrals have the remarkable property that they are actually independent of $\kappa$. Thus $|R_1+iR_6|^2$ may be taken outside the $\kappa$ summation sign. An explicit evaluation of $\sum_\kappa B_{\kappa\kappa'}$ for $|\kappa'| = 1$, 2, or 3 gives

$$\sum_{\kappa} B_{\kappa\kappa'} = 2|\kappa'|L(L+1)(2L+1). \tag{16}$$

Thus we get the simple result

$$\beta_L^{(0)} = \beta_L^{(1)} = \frac{\alpha\omega}{8\pi} |\kappa'||R_1 + iR_6|^2, \tag{17}$$

i.e., all other parameters being equal, the ICC are the same for all $L$ values. Putting in explicit values for the radial integrals we obtain

$$\beta_L^{(\lambda)} = \frac{1}{\omega}\, \frac{|\kappa'|\lambda^{3+2\gamma'}(1+W_B)4^{\gamma'-1}e^{-2\rho\,\mathrm{arctg}(W_B/\lambda)}}{Zn'!(\rho-\lambda\kappa')[\Gamma(2\gamma'+1)]^2}\, |\Gamma(\gamma'-i\rho)|^2\Gamma(2\gamma'+n'+1)|M|^2, \tag{18}$$

where

$$M = n'\left\{1+\frac{i(1-W_B)}{\lambda}\right\} {}_2F_1\left[(\gamma'-i\rho),\, 1-n';\, 2\gamma'+1;\, \frac{2\lambda}{\lambda+iW_B}\right]$$
$$+ \left(\kappa'-\frac{\rho}{\lambda}\right)\left\{1-\frac{i(1-W_B)}{\lambda}\right\} {}_2F_1\left[(\gamma'-i\rho),\, -n';\, 2\gamma'+1;\, \frac{2\lambda}{\lambda+iW_B}\right]. \tag{19}$$

Eq. (16) is thus a relatively simple closed-form expression giving the lowest order term (i.e., $A/\omega$) in the result for the ICC for high transition energies $\omega$. Thus it is a simple matter to calculate $A$ for all $Z$ and for all shells. The results are presented in table 1.

TABLE 1

Values of the coefficient $A$ (used in calculating internal conversion coefficients at very high energies) for the K, L and M shells

| $Z$ | K | $L_I$ | $L_{II}$ | $L_{III}$ |
|---|---|---|---|---|
| 5 | 6.36(−7) | 7.95(−8) | 2.21(−12) | 1.76(−11) |
| 10 | 4.60(−6) | 5.76(−7) | 6.40(−11) | 5.06(−10) |
| 15 | 1.41(−5) | 1.77(−6) | 4.44(−10) | 3.47(−9) |
| 20 | 3.06(−5) | 3.86(−6) | 1.72(−9) | 1.32(−8) |
| 25 | 5.50(−5) | 6.96(−6) | 4.86(−9) | 3.67(−8) |
| 30 | 8.76(−5) | 1.12(−5) | 1.13(−8) | 8.32(−8) |
| 35 | 1.29(−4) | 1.65(−5) | 2.28(−8) | 1.64(−7) |
| 40 | 1.78(−4) | 2.31(−5) | 4.17(−8) | 2.93(−7) |
| 45 | 2.36(−4) | 3.09(−5) | 7.09(−8) | 4.84(−7) |
| 50 | 3.03(−4) | 3.99(−5) | 1.14(−7) | 7.54(−7) |
| 55 | 3.77(−4) | 5.03(−5) | 1.75(−7) | 1.12(−6) |
| 60 | 4.58(−4) | 6.21(−5) | 2.59(−7) | 1.59(−6) |
| 65 | 5.47(−4) | 7.52(−5) | 3.71(−7) | 2.19(−6) |
| 70 | 6.43(−4) | 8.99(−5) | 5.18(−7) | 2.94(−6) |
| 75 | 7.45(−4) | 1.06(−4) | 7.10(−7) | 3.85(−6) |
| 80 | 8.53(−4) | 1.24(−4) | 9.55(−7) | 4.93(−6) |
| 85 | 9.67(−4) | 1.44(−4) | 1.27(−6) | 6.20(−6) |
| 90 | 1.09(−3) | 1.66(−4) | 1.66(−6) | 7.67(−6) |
| 95 | 1.21(−3) | 1.91(−4) | 2.15(−6) | 9.36(−6) |

| $Z$ | $M_I$ | $M_{II}$ | $M_{III}$ | $M_{IV}$ | $M_V$ |
|---|---|---|---|---|---|
| 15 | 5.26(−7) | 1.56(−10) | 1.22(−9) | | |
| 20 | 1.14(−6) | 6.04(−10) | 4.66(−9) | 1.38(−13) | 3.27(−12) |
| 25 | 2.06(−6) | 1.71(−9) | 1.29(−8) | 5.98(−13) | 1.41(−11) |
| 30 | 3.30(−6) | 3.95(−9) | 2.94(−8) | 1.96(−12) | 4.57(−11) |
| 35 | 4.88(−6) | 7.98(−9) | 5.81(−8) | 5.26(−12) | 1.22(−10) |
| 40 | 6.81(−6) | 1.46(−8) | 1.04(−7) | 1.23(−11) | 2.83(−10) |
| 45 | 9.10(−6) | 2.49(−8) | 1.72(−7) | 2.58(−11) | 5.88(−10) |
| 50 | 1.18(−5) | 3.99(−8) | 2.69(−7) | 4.97(−11) | 1.12(−9) |
| 55 | 1.48(−5) | 6.12(−8) | 4.00(−7) | 8.95(−11) | 2.00(−9) |
| 60 | 1.82(−5) | 9.04(−8) | 5.72(−7) | 1.52(−10) | 3.36(−9) |
| 65 | 2.21(−5) | 1.30(−7) | 7.91(−7) | 2.47(−10) | 5.40(−9) |
| 70 | 2.63(−5) | 1.81(−7) | 1.06(−6) | 3.86(−10) | 8.32(−9) |
| 75 | 3.10(−5) | 2.48(−7) | 1.40(−6) | 5.82(−10) | 1.24(−8) |
| 80 | 3.62(−5) | 3.33(−7) | 1.80(−6) | 8.53(−10) | 1.79(−8) |
| 85 | 4.19(−5) | 4.41(−7) | 2.28(−6) | 1.22(−9) | 2.52(−8) |
| 90 | 4.82(−5) | 5.76(−7) | 2.83(−6) | 1.70(−9) | 3.47(−8) |
| 95 | 5.51(−5) | 7.46(−7) | 3.48(−6) | 2.32(−9) | 4.67(−8) |

The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

To lowest order in $1/\omega$ we have obtained the striking result that for a given shell and $Z$ values, the electric and magnetic ICC are equal for all $L$ values. This result, of course, should not be too surprising when we recall that in the wave zone ($\omega r \gg 1$) the electric and magnetic fields are transverse and all multipoles look alike [5]). Of course, if we consider other terms in our expansion (eq. (8)), we see that this conclusion will no longer hold. However, it certainly gives us an indication of the type of results to be expected at high energies.

Because the fact that $\gamma' = (\kappa'^2 - \alpha^2 Z^2)^{\frac{1}{2}}$ we see that our results should be very accurate for $|\kappa|' \neq 1$. However, for $|\kappa'| = 1$ the coefficient $B$ in eq. (8) is certainly important, particularly for large $Z$ values, and we hope to compute it in the near future along with the coefficient $C$. This should then provide us with very accurate high energy ICC. It is of interest to compare our numerical results to the K shell results of Rose [3,6]) at $\omega = 5$ which corresponds to the highest energy considered by Rose. We find that the results of Rose are in all cases larger than those we have obtained, as they should be (for example, at $Z = 10$, Rose quotes a lowest value of $1.224 \times 10^{-6}$ which should be compared to our value of $0.92 \times 10^{-6}$).

As a final remark, we note that the equality of the electric and magnetic multipoles for high transition energies was known for particular cases by Casimir [7]) (K shell, electric dipole, non-relativistic calculation) and by Dancoff and Morrison [8]) (K shell with neglect of binding energy).

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