Multiple reflection effects in the theory of the Faraday effect and ellipticity for propagation through three distinct media

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Received May 20, 1982

Donovan and Medcalf formulated the theory of the free-carrier Faraday effect in semiconductors for a vacuum–medium–vacuum, taking into account multiple internal reflections of the electromagnetic radiation in the medium. Here, we generalize Donovan and Medcalf's results to the case of three distinct media and also extend their analysis in the sense of obtaining a *unique* decomposition of the multiple pass rotation Θ which gives added physical insight into the various contributions to Θ. The multiple pass ellipticity Δ is similarly treated. We also present an alternate and simpler method of carrying out the calculations.

Donovan et Metcalf ont formulé la théorie de l'effet Faraday dû aux porteurs de charges libres pour un système vide–milieu–vide, en tenant compte des réflexions multiples à l'intérieur du milieu. Nous généralisons ici les résultats de Donovan et Metcalf pour le cas de trois milieux distincts, et nous étendons aussi leur analyse pour obtenir une décomposition *unique* de la rotation due aux passages multiples, désignée par Θ, ce qui donne une meilleure compréhension physique des diverses contributions à Θ. L'ellipticité Δ due aux passages multiples est traitée de façon similaire. Nous présentons aussi une méthode nouvelle et plus simple pour effectuer les calculs.

[Traduit par le journal]

Can. J. Phys., 61, 49 (1983)

I. Introduction

Donovan and Medcalf (1) formulated the theory of the free-carrier Faraday effect and the ellipticity affected by the propagation of plane-polarized electromagnetic radiation through a vacuum–medium–vacuum system, taking into account multiple internal reflections in the medium. Since in many physical situations three distinct media are encountered, we are motivated to generalize the Donovan–Medcalf results to the case of *three distinct media*. In a future publication we will apply our results to the metal oxide semiconductor (MOS) system, which is generating much current interest and for which Faraday rotation measurements have just been obtained.[1] A summary of some of our results has already been presented in another context (2).

In Sect. II we present an analysis of the multiple reflection effects on the Faraday rotation for which we give a *unique* decomposition of the multiple pass rotation Θ into the single pass, boundary, and multiple reflection effects. Section III contains a similar analysis of the multiple pass ellipticity Δ.

We explicitly point out that these results are independent of the model chosen for the dielectric constants.

II. Multiple pass Faraday rotation

Consider the propagation of plane-polarized electromagnetic radiation through a three media system, where the propagation vector is along the z direction, coincident with the direction of an applied magnetic field B. Letting the pair of lower case Latin indices (ij) indicate that the electromagnetic wave is propagating from medium i in the direction of medium j, the Fresnel transmission and reflection coefficients (3) at normal incidence can be compactly written as (see Fig. 1 with $\phi_1 = \phi_2 = \phi_3 = 0$),

$$[1a] \qquad t_{ij\pm} = \frac{2N_{i\pm}}{N_{j\pm} + N_{i\pm}}$$

$$[1b] \qquad r_{ij\pm} \equiv -r_{ji\pm} = \frac{N_{j\pm} - N_{i\pm}}{N_{j\pm} + N_{i\pm}}$$

where

$$[1c] \qquad N_{j\pm} \equiv n_{j\pm} + i\kappa_{j\pm}, \qquad j = 1, 2, 3$$

is the complex refractive index of medium j, and

$$[1d] \qquad t_{ij\pm} + r_{ij\pm} = 1$$

The "±" signs in [1] refer to the left and right circularly polarized components of the plane-polarized electromagnetic radiation.

It is convenient to rewrite [1a] and [1b] in the form

$$[2a] \qquad r_{ij\pm} \equiv |r_{ij\pm}|e^{i\xi_{ij\pm}}$$

$$[2b] \qquad t_{ij\pm} \equiv |t_{ij\pm}|e^{i\eta_{ij\pm}}$$

where

$$[2c] \qquad \tan \xi_{ij\pm} = \frac{2(n_{i\pm}\kappa_{j\pm} - n_{j\pm}\kappa_{i\pm})}{n_{j\pm}^2 - n_{i\pm}^2 + \kappa_{j\pm}^2 - \kappa_{i\pm}^2}$$

[1]H. Piller, private communication.

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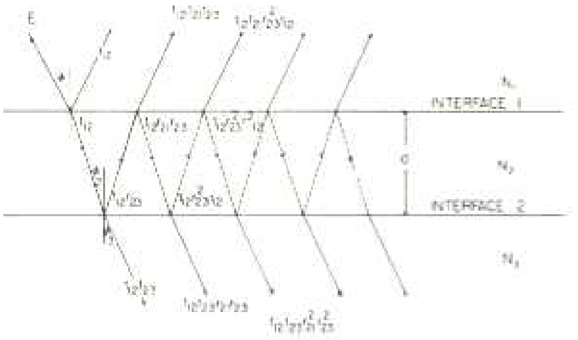

FIG. 1. Multiple internal reflections in a plane-parallel sample of thickness d. The angles ϕ_1, ϕ_2, and ϕ_3 are shown nonzero for clarity.

[2d] $$\tan \gamma_{ij\pm} = \frac{n_{j\pm}\kappa_{i\pm} - n_{i\pm}\kappa_{j\pm}}{n_{i\pm}(n_{i\pm} + n_{j\pm}) + \kappa_{i\pm}(\kappa_{i\pm} + \kappa_{j\pm})}$$

From Fig. 1 we see that the transmission amplitudes $\tau_{n\pm}$ of the $2n$-times reflected beam are

[3] $$\tau_{n\pm} = t_{12\pm} t_{23\pm} r^{n}_{23\pm} r^{n}_{21\pm} e^{i(2n+1)\delta_\pm}$$

where

[4] $$\delta_\pm = \frac{\omega d}{c}(n_{2\pm} + i\kappa_{2\pm}) = (\beta_\pm + i\alpha_\pm/2)d$$

is the change in phase of the corresponding beam introduced by transversing the sample (medium 2), ω is the frequency of the radiation, c is the speed of light, and d is the thickness of the sample.

Summing all the multiple internally reflected amplitudes we obtain

[5] $$\tau_\pm = \sum_{n=0}^{\infty} \tau_{n\pm} = t_{12\pm} t_{23\pm} e^{i\delta_\pm} \sum_{n=0}^{\infty} (r_{23\pm} r_{21\pm} e^{2i\delta_\pm})^n$$

Equation [5] is a geometric series whose common ratio is $r_{23\pm} r_{21\pm} e^{2i\delta_\pm}$, which has a magnitude less than unity. Hence we may write

[6] $$\tau_\pm = T_\pm e^{i\delta_\pm}$$

where

[7] $$T_\pm \equiv \frac{t_{12\pm} t_{23\pm}}{1 - r_{23\pm} r_{21\pm} e^{2i\delta_\pm}}$$

In Appendix A, we present a simpler derivation of the amplitudes $\tau_\pm$ in the sense that the summation technique used in [5] is bypassed. The difference in these two methods is that to obtain [5] we sequentially followed the beam event-by-event while the method used in Appendix A considers the situation at a steady-state time.

As proved in Appendix B, the multiple pass Faraday rotation Θ and the multiple pass ellipticity Δ may be obtained from

[8] $$\frac{\tau_-}{\tau_+} = \left|\frac{\tau_-}{\tau_+}\right| e^{-2i\Theta}$$

and

[9] $$\Delta = \frac{|\tau_+| - |\tau_-|}{|\tau_+| + |\tau_-|}$$

However, to obtain a unique decomposition of Θ and Δ into single pass, boundary, and multiple reflection effects, we proceed as follows.

We define $\eta_\pm$ by

[10] $$(1 - r_{23\pm} r_{21\pm} e^{2i\delta_\pm})^{-1} = |1 - r_{23\pm} r_{21\pm} e^{2i\delta_\pm}|^{-1} e^{i\eta_\pm}$$

Using [2] and [4], we may write

[10a] $$1 - r_{23\pm} r_{21\pm} e^{2i\delta_\pm} = 1 - |r_{23\pm} r_{21\pm}| e^{-\alpha_\pm d} \times e^{i(2\beta_\pm d + \xi_{21\pm} + \xi_{23\pm})}$$

so that

[11] $$\tan \eta_\pm = \frac{|r_{23\pm} r_{21\pm}| e^{-\alpha_\pm d} \sin (2\beta_\pm d + \xi_{21\pm} + \xi_{23\pm})}{1 - |r_{23\pm} r_{21\pm}| e^{-\alpha_\pm d} \cos (2\beta_\pm d + \xi_{21\pm} + \xi_{23\pm})}$$

Hence, from [2], [7], and [10], it follows that

[12] $$T_\pm = \frac{|t_{12\pm} t_{23\pm}| e^{i(\gamma_{12\pm} + \gamma_{23\pm} + \eta_\pm)}}{|1 - r_{23\pm} r_{21\pm} e^{2i\delta_\pm}|}$$

From the equation (see Appendix B, [B.6])

[13] $$\frac{\tau_-}{\tau_+} = \left|\frac{\tau_-}{\tau_+}\right| e^{-2i\Theta}$$

and also using [6], [4], and [12], we obtain for the multiple pass Faraday rotation Θ

[14] $$\Theta = \theta + \theta_T + \theta_{MR}$$

where

$$\theta = (\beta_+ - \beta_-)d/2$$

is the single pass Faraday rotation,

[15] $$\theta_T \equiv \tfrac{1}{2}(\gamma_{12+} + \gamma_{23+} - \gamma_{12-} - \gamma_{23-})$$

is due purely to boundary effects and is independent of the sample thickness d, and

[16] $$\theta_{MR} \equiv \tfrac{1}{2}(\eta_+ - \eta_-)$$

is the contribution due purely to multiple reflection effects. We recall that the various β, γ, and η terms may be calculated from [4], [2d], and [11], respectively.

We now define the single pass Faraday rotation *with boundary effects* Θ_T as

$$[17] \qquad \Theta_T \equiv \theta + \theta_T$$

We explicitly point out that the decomposition of Θ given in [14] is independent of the model used for the dielectric constants $\epsilon_\pm$.

III. Multiple pass ellipticity

The multiple pass ellipticity Δ, accounting for multiple reflections, is (from Appendix B),

$$[18] \qquad \Delta = \frac{|\tau_+| - |\tau_-|}{|\tau_+| + |\tau_-|}$$

To obtain a decomposition of Δ, similar to that given for Θ in [14], we make use of [4] and [6] to write

$$|\tau_\pm| = |T_\pm| e^{-\alpha_\pm d/2} = e^{-\alpha_\pm d/2 + \ln|T_\pm|}$$

Thus, we have for the ellipticity Δ

$$[19] \qquad \Delta = \tanh(\delta_S + \delta_T + \delta_{MR})$$

where

$$[20] \qquad \delta_S \equiv \tanh^{-1}\delta = \frac{\omega d}{2c}(\kappa_- - \kappa_+)$$

with $\delta = \tanh \delta_S$ as the single pass ellipticity,

$$[21] \qquad \delta_T \equiv \frac{1}{2}\ln\left|\frac{t_{12-}t_{23-}}{t_{12+}t_{23+}}\right|$$

is due purely to boundary effects and is independent of the sample thickness d, and

$$[22] \qquad \delta_{MR} \equiv \frac{1}{2}\ln\left|\frac{1 - r_{23+}r_{12+}e^{2ih_+}}{1 - r_{23-}r_{12-}e^{2ih_-}}\right|$$

is due purely to multiple reflection effects.

Similar to the definition Θ_T we now define the single pass ellipticity *with boundary effects* Δ_T as

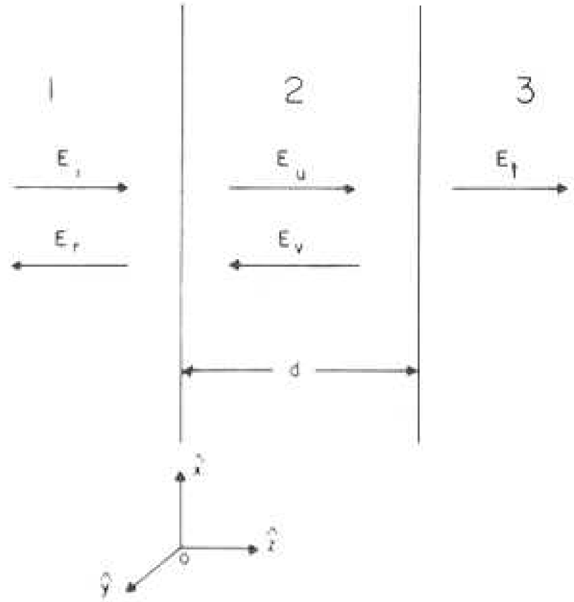

FIG. 2. Geometry used for an alternate derivation of the magneto-transmittance for three distinct media.

$$[23] \qquad \Delta_T \equiv \tanh(\delta_S + \delta_T)$$

In a future publication, these results will be applied to the case of a MOS structure.

IV. Acknowledgement

This research was partially supported by the Department of Energy, Division of Materials Science, under Contract No. DE-AS05-79ER10459.

1. B. DONOVAN and T. MEDCALF. Br. J. Appl. Phys. **15**, 1139 (1964).
2. R. F. O'CONNELL and G. L. WALLACE. Phys. Lett. A, **86**, 283 (1981).
3. M. BORN and E. WOLF. Principles of optics. 4th ed. Pergamon Press, Inc., New York, NY. 1970.

Appendix A

We consider the propagation of linearly polarized electromagnetic radiation of frequency ω through the three layer system, in the presence of an applied external dc magnetic field $B = B\hat{z}$, $B > 0$. The electric fields, in circularly polarized coordinates, in the three regions of Fig. 2 are

$$[A.1a] \quad z < 0: \qquad E_{i\pm} = E_{i\pm}e^{i(k_{1\pm}z - \omega t)}\hat{\epsilon}_\pm, \qquad E_{r\pm} = E_{r\pm}e^{-i(k_{1\pm}z - \omega t)}\hat{\epsilon}_\pm$$

$$[A.1b] \quad 0 < z < d: \qquad E_{u\pm} = E_{u\pm}e^{i(k_{2\pm}z - \omega t)}\hat{\epsilon}_\pm, \qquad E_{v\pm} = E_{v\pm}e^{-i(k_{2\pm}z + \omega t)}\hat{\epsilon}_\pm$$

$$[A.1c] \quad z > d: \qquad E_{t\pm} = E_{t\pm}e^{i(k_{3\pm}(z-d) - \omega t)}\hat{\epsilon}_\pm$$

so that $E_i = E_{i+} + E_{i-}$, $E_r = E_{r+} + E_{r-}$, etc., k is the wave number of the corresponding medium given by

$$[A.2a] \quad k_{i\pm} = \frac{\omega}{c}N_{i\pm}, \qquad i = 1, 2, 3$$

and

[A.2*b*] $\hat{\epsilon}_{\pm} = (\hat{x} \mp i\hat{y})/\sqrt{2}$

The boundary conditions at $z = 0$ and $z = d$ are (*a*) continuity of the tangential component of the ac electric field and (*b*) equality of the discontinuity of the ac magnetic field and $(4\pi/c)j$, which in the case we are considering is identically zero. These conditions imply, respectively, the following results (suppressing the "±" sign):

[A.3*a*] $z = 0: \quad E_i + E_r = E_u + E_v$

[A.3*b*] $k_1(E_i - E_r) - k_2(E_u - E_v) = 0$

[A.3*c*] $z = d: \quad E_u e^{ik_2 d} + E_v e^{-ik_2 d} = E_t$

[A.3*d*] $k_2(E_u e^{ik_2 d} - E_v e^{-ik_2 d}) - k_3 E_t = 0$

After some algebra, we find that the solution of [A.3] for the transmission coefficient $t \equiv E_t/E_i$, in circularly polarized coordinates, is

[A.4] $$t_{\pm} = \frac{4k_{1\pm}k_{2\pm}}{(k_{1\pm} - k_{2\pm})(k_{2\pm} - k_{3\pm})e^{ik_{2\pm}d} + (k_{1\pm} + k_{\pm})(k_{2\pm} + k_{3\pm})e^{-ik_{2\pm}d}}$$

Using [A.2], [1], and [4] we have

[A.5] $$t_{\pm} = \frac{t_{12\pm}t_{23\pm}e^{i\delta_{\pm}}}{1 - r_{23\pm}r_{21\pm}e^{2i\delta_{\pm}}}$$

which agrees with [6] and [7].

Appendix B

Consider a monochromatic plane wave of frequency ω linearly polarized in the x direction and propagating in the z direction

[B.1] $\boldsymbol{E}^i = E_{\omega}e^{i\tau}\hat{x}$

where $\tau \equiv \boldsymbol{k}\cdot\boldsymbol{r} - \omega t$ and for convenience we assume E_{ω} to be real. The left and right circularly polarized components of $\boldsymbol{E}^i$ are

[B.2] $\boldsymbol{E}^i_{\pm} = (\hat{\epsilon}^*_{\pm}\cdot\boldsymbol{E}^i)\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}E_{\omega}e^{i\tau}\hat{\epsilon}_{\pm}$

where $\hat{\epsilon}_{\pm}$ is defined in [A.2*b*] and satisfies the orthogonality relations

$$\hat{\epsilon}^*_{\pm}\cdot\hat{\epsilon}_{\pm} = 1$$
$$\hat{\epsilon}^*_{\pm}\cdot\hat{\epsilon}_{\mp} = 0$$
$$\hat{\epsilon}_{\pm}\cdot\hat{z} = 0$$

so that

[B.3] $\boldsymbol{E}^i = \boldsymbol{E}^i_+ + \boldsymbol{E}^i_-$

We will now prove that if the circularly polarized components of the transmitted waves are

[B.4] $\boldsymbol{E}^t_{\pm} = \tau_{\pm}\boldsymbol{E}^i_{\pm}$

then the total transmitted wave

[B.5] $\boldsymbol{E}^t = \boldsymbol{E}^t_+ + \boldsymbol{E}^t_-$

is elliptically polarized, with ellipticity Δ, and the semi-major axis is rotated by an angle Θ from the positive x axis. Θ may be obtained from the ratio

[B.6] $$\frac{\tau_-}{\tau_+} = \left|\frac{\tau_-}{\tau_+}\right|e^{-2i\Theta}$$

and

[B.7] $$\Delta = \frac{|\tau_+| - |\tau_-|}{|\tau_+| + |\tau_-|}$$

or, more simply

[B.8] $$\frac{\tau_-}{\tau_+} = \frac{1 - \Delta}{1 + \Delta}e^{-2i\Theta}$$

The total transmitted wave is

[B.9] $\boldsymbol{E}^t = \boldsymbol{E}^t_+ + \boldsymbol{E}^t_- = \tilde{E}^t_x\hat{x} + \tilde{E}^t_y\hat{y}$

where the complex x and y components $\tilde{E}^t_x$, $\tilde{E}^t_y$ are (using [B.2] and [B.4])

[B.10] $$\tilde{E}^t_x = \frac{1}{\sqrt{2}}(E^t_+ + E^t_-)e^{i\tau} = \frac{1}{\sqrt{2}}(|\tau_+|e^{i\zeta_+} + |\tau_-|e^{i\zeta_-})E_{\omega}e^{i\tau}$$

[B.11] $$\tilde{E}^t_y = -\frac{i}{\sqrt{2}}(E^t_+ - E^t_-)e^{i\tau} = -\frac{i}{\sqrt{2}}(|\tau_+|e^{i\zeta_+} - |\tau_-|e^{i\zeta_-})E_{\omega}e^{i\tau}$$

where $\zeta_\pm$ is defined by

$$\tau_\pm \equiv |\tau_\pm| e^{i\zeta_\pm}$$

Defining the quantities

[B.12] $\zeta \equiv (\zeta_+ + \zeta_-)/2$

and

[B.13] $\Theta \equiv (\zeta_+ - \zeta_-)/2$

$\tilde{E}'_x$ and $\tilde{E}'_y$ may be written as

[B.14] $\tilde{E}'_x = \frac{1}{\sqrt{2}}(|\tau_+| e^{i\Theta} + |\tau_-| e^{-i\Theta}) E_\omega e^{i(\tau+\zeta)}$

[B.15] $\tilde{E}'_y = \frac{-i}{\sqrt{2}}(|\tau_+| e^{i\Theta} - |\tau_-| e^{-i\Theta}) E_\omega e^{i(\tau+\zeta)}$

Expanding the complex exponentials and taking the real parts to obtain the physical wave, we have

[B.16] $E_x \equiv \mathrm{Re}\,\{\tilde{E}'_x\}$

$$= \frac{1}{\sqrt{2}}\{(|\tau_+| + |\tau_-|)\cos\Theta\cos(\tau+\zeta) - (|\tau_+| - |\tau_-|)\sin\Theta\sin(\tau+\zeta)\}E_\omega$$

[B.17] $E_y \equiv \mathrm{Re}\,\{\tilde{E}'_y\}$

$$= \frac{1}{\sqrt{2}}\{(|\tau_+| - |\tau_-|)\cos\Theta\sin(\tau+\zeta) + (|\tau_+| + |\tau_-|)\sin\Theta\cos(\tau+\zeta)\}E_\omega$$

Defining

[B.18] $a \equiv \frac{1}{\sqrt{2}}(|\tau_+| + |\tau_-|)E_\omega$

[B.19] $b \equiv \frac{1}{\sqrt{2}}(|\tau_+| - |\tau_-|)E_\omega$

so that $(b/a) = \Delta$, we can write

[B.20] $E_x = a\cos\Theta\cos(\tau+\zeta) - b\sin\Theta\sin(\tau+\zeta)$

[B.21] $E_y = a\sin\Theta\cos(\tau+\zeta) + b\cos\Theta\sin(\tau+\zeta)$

We now define E'_x, E'_y by

[B.22] $E'_x \equiv E_x\cos\Theta + E_y\sin\Theta$

[B.23] $E'_y \equiv -E_x\sin\Theta + E_y\cos\Theta$

Hence

[B.24] $E'_x = a\cos(\tau+\zeta)$

[B.25] $E'_y = b\sin(\tau+\zeta)$

and therefore

[B.26] $\frac{E'^2_x}{a^2} + \frac{E'^2_y}{b^2} = 1$

Thus, the transmitted wave is elliptically polarized with semimajor axis a, semiminor axis b, and inclined at an angle Θ with respect to the positive x-axis with Θ positive in the counterclockwise direction.