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New Interpretation of the Scalar Product in Hilbert Space

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A new interpretation of the scalar product in Hilbert space was presented recently by Aharonov, Albert, and Au. Here the essential equivalence between such a viewpoint and Wigner's exact reformulation of nonrelativistic quantum mechanics using distribution functions is pointed out.

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Aharonov, Albert, and Au¹ have recently presented a new interpretation of the scalar product of two states in Hilbert space in terms of a relationship between two or more distinct physical systems. In particular, whereas the conventional viewpoint is that the scalar product gives the projection of one state on another in *Hilbert space*, in the work of Aharonov, Albert, and Au it appears as a measure of the proximity of the two states in *phase space*.

Our purpose here is to point out that such a viewpoint is closely related to the work of Wigner,² who introduced his now-famous quantum mechanical distribution function by means of which one obtains an exact reformulation of nonrelativistic quantum mechanics.

The distribution functions $P(q, p)$ are functions of position and momentum coordinates q and p . In the classical limit, $P(q, p)$ is the phase-space distribution function which gives the probability that the coordinates and momenta have the values q and p . In general, $P(q, p)$ depends on \hbar and may assume negative values,^{2,3} which accounts for the frequent description of this quantity as a quasi-classical distribution function.

In one dimension the distribution function may

be written in the form

$$\begin{aligned} P(q, p) &= (\pi\hbar)^{-1} \int \langle q-y | \rho | q+y \rangle e^{2ipy/\hbar} dy \\ &= (\pi\hbar)^{-1} \int \psi^*(q+y) \psi(q-y) e^{2ipy/\hbar} dy, \end{aligned} \quad (1)$$

where ρ is the density matrix. As first shown in Ref. 3, and discussed at length by O'Connell and Wigner,^{4,5} this function (as distinct from other distribution functions which have been proposed⁶) has the property that the transition probability between two states ψ and φ is given, in terms of the corresponding distribution functions, P_ψ and P_φ say, as follows:

$$\begin{aligned} &(2\pi\hbar)^{-1} \left| \int \psi(x)^* \varphi(x) dx \right|^2 \\ &= \iint P_\psi(q, p) P_\varphi(q, p) dq dp. \end{aligned} \quad (2)$$

In other words, the scalar product in Hilbert space has been replaced by an integration in phase space. Further, since $P_\psi(p, q)$ can be interpreted as the probability that a particle in the state ψ has momentum p and coordinate q (even though such a probability cannot be directly measured), it follows that $P_\psi(p, q)P_\varphi(p, q)$ is the probability that two particles, one in the state ψ and the other in the state φ , have the same specific momenta p and the same specific coordinate

q . Hence, carrying out the phase-space integration over all p and q , we reach the conclusion that the left-hand side of (2), i.e., $(2\pi\hbar)^{-1} |\langle \psi | \varphi \rangle|^2$ is, in the words of Aharonov, Albert, and Au,¹ "... a propensity of two particles, one in the state $|\varphi\rangle$ and the other in the state $|\psi\rangle$, to have

$$J = (\pi\hbar)^{-1} \int \int dy dq \psi^*(q+y) \psi(q-y) \varphi^*(q+\beta-y) \varphi(q+\beta+y) e^{2i\alpha y/\hbar}. \quad (4)$$

Changing variables by setting $q+y=u$ and $q-y=v$, we obtain

$$J = (2\pi\hbar)^{-1} \int \int du dv \psi^*(u) \psi(v) \varphi^*(v+\beta) \varphi(u+\beta) e^{i\alpha(u-v)/\hbar} = (2\pi\hbar)^{-1} \left| \int \psi^*(x) e^{i\alpha x/\hbar} \varphi(x+\beta) dx \right|^2. \quad (5)$$

Now using the fact that

$$e^{i\alpha x/\hbar} \varphi(x+\beta) = e^{i\alpha(\hat{x}-\beta)/\hbar} \varphi(x+\beta), \quad (6)$$

and

$$\varphi(x+\beta) = e^{-i\beta\hat{p}/\hbar} \varphi(x), \quad (7)$$

where \hat{x} and \hat{p} are operators, we obtain

$$\begin{aligned} J &= (2\pi\hbar)^{-1} \left| \psi^*(x) e^{-i\alpha\beta/\hbar} e^{i\alpha\hat{x}/\hbar} e^{-i\beta\hat{p}/\hbar} \varphi(x) \right|^2 \\ &= (2\pi\hbar)^{-1} \left| \psi^*(x) e^{i(\alpha\hat{x}-\beta\hat{p})/\hbar} \varphi(x) \right|^2. \end{aligned} \quad (8)$$

Thus, we conclude that

$$\begin{aligned} \int \int P_\psi(q, p) P_\varphi(q+\beta, p+\alpha) dp dq \\ = (2\pi\hbar)^{-1} |\langle \psi | \alpha, \beta \rangle_\varphi|^2, \end{aligned} \quad (9)$$

where, in the notation of Aharonov, Albert, and Au,¹

$$|\alpha, \beta\rangle_\varphi = e^{i(\alpha\hat{x}-\beta\hat{p})/\hbar} |\varphi\rangle, \quad (10)$$

i.e., $|\varphi\rangle$ is translated through a distance β in coordinate space and α in momentum space. Hence we can conclude that $(2\pi\hbar)^{-1} |\langle \psi | \alpha, \beta \rangle_\varphi|^2$ is a propensity of two particles, one in the state $|\varphi\rangle$ and the other in the state $|\psi\rangle$ to have positions and momenta differing by amounts β and α , respectively.⁶ This corresponds to a result obtained by Aharonov, Albert, and Au¹ [see their Eq. (4)].

In summary, we have demonstrated that the new interpretation given by Aharonov, Albert, and Au¹ to the scalar product of two states in Hilbert space is essentially equivalent to the exact reformulation of nonrelativistic quantum

the same positions and the same momenta."

We turn now to a generalization of Eq. (2) by considering

$$J \equiv \int \int P_\psi(q, p) P_\varphi(q+\beta, p+\alpha) dp dq. \quad (3)$$

Using Eq. (1) in Eq. (3) and carrying out the p and one of the y integrations, we obtain

mechanics in terms of distribution functions, which was pioneered by Wigner nearly one-half century ago.²

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