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SPIN NUTATION IN BINARY SYSTEMS DUE TO GENERAL RELATIVISIC AND QUADRUPOLE EFFECTS

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ABSTRACT

We consider two spinning compact objects in a binary system where the orbital angular momentum is much greater than the spin angular momenta. The amplitude and frequency of the spin nutation of body 2 (and body 1 by interchanging indices) due to general relativistic and quadrupole effects are given. The spin nutation frequency of either body is found to be equal to the difference in the two spin precession frequencies.

Subject headings: relativity — stars: binaries

I. INTRODUCTION

In a recent paper (Barker, Byrd, and O'Connell 1981) we considered two spinning compact objects in a binary system with the assumption that the spin $\boldsymbol{S}^{(1)}$ of body 1 could be neglected. With $\boldsymbol{S}^{(1)} = 0$, both the spin $\boldsymbol{S}^{(2)}$ of body 2 and the orbital angular momentum $\boldsymbol{L}$ precess about the total angular momentum $\boldsymbol{J}$ at the same constant rate *without* any nutation.

We now wish to consider two spinning compact objects in a binary system where $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}$ (i.e., we do *not* set $\boldsymbol{S}^{(1)} = 0$). Because of the increasing importance in observational astronomy of binary systems in which general relativistic and quadrupole interactions may cause spin precession and nutation, we consider here these effects *in general* rather than for any particular system. In order to obtain the spin nutation result in a very simple form that is easy to derive and easy for observers to understand and use, we shall restrict ourselves to some conditions which fortuitously turn out to be the most likely conditions expected to occur in binary systems of interest. We shall assume (i) that $L \gg S^{(1)}$ and $L \gg S^{(2)}$. We shall also assume (ii) that the angles θ_1 (angle between $\boldsymbol{S}^{(1)}$ and $\boldsymbol{J}$) and θ_2 (angle between $\boldsymbol{S}^{(2)}$ and $\boldsymbol{J}$) are *not* very small [i.e., *not* $\ll 1$ (radian)]. Because of assumption (i), the angle θ (angle between $\boldsymbol{L}$ and $\boldsymbol{J}$) will always be very small and, hence, there is no point in deriving the orbital angular momentum nutation which would be extremely small.

II. PRECESSION AND NUTATION

The two-body secular results (Barker and O'Connell 1975, 1979) for the precession of the spin (of body 2) are given by (a dot denotes d/dt, and av denotes average over an orbital period)

$$\dot{\boldsymbol{S}}^{(2)}_{\text{av}} = \boldsymbol{\Omega}^{(2)}_{\text{av}} \times \boldsymbol{S}^{(2)} , \tag{1}$$

$$\boldsymbol{\Omega}^{(2)}_{\text{av}} = \boldsymbol{\Omega}^{(2)}_{Q2\text{av}} + \boldsymbol{\Omega}^{(2)}_{\text{DSav}} + \boldsymbol{\Omega}^{(2)}_{\text{LTav}} , \tag{2}$$

where $\boldsymbol{\Omega}^{(2)}_{Q2\text{av}}$, $\boldsymbol{\Omega}^{(2)}_{\text{DSav}}$, and $\boldsymbol{\Omega}^{(2)}_{\text{LTav}}$ are the quadrupole, de Sitter (spin-orbit), and Lense-Thirring (spin-spin) contributions. We also have

$$\boldsymbol{\Omega}^{(2)}_{Q2\text{av}} = A^{(2)}_{Q2}[\boldsymbol{n}^{(2)} - 3(\boldsymbol{n} \cdot \boldsymbol{n}^{(2)})\boldsymbol{n}] , \qquad A^{(2)}_{Q2} = \frac{Gm_1 \Delta I^{(2)}}{2S^{(2)}a^3(1-e^2)^{3/2}} , \tag{3}$$

$$\boldsymbol{\Omega}^{(2)}_{\text{DSav}} = A^{(2)}_{\text{DS}}\boldsymbol{n} , \qquad A^{(2)}_{\text{DS}} = \frac{GL(4 + 3m_1/m_2)}{2c^2a^3(1-e^2)^{3/2}} , \tag{4}$$

$$\boldsymbol{\Omega}^{(2)}_{\text{LTav}} = A^{(2)}_{\text{LT}}[\boldsymbol{n}^{(1)} - 3(\boldsymbol{n} \cdot \boldsymbol{n}^{(1)})\boldsymbol{n}] , \qquad A^{(2)}_{\text{LT}} = \frac{GS^{(1)}}{2c^2a^3(1-e^2)^{3/2}} , \tag{5}$$

where $\boldsymbol{n}$, $\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(2)}$ are unit vectors in the $\boldsymbol{L}$, $\boldsymbol{S}^{(1)}$, $\boldsymbol{S}^{(2)}$ directions, respectively, a is the semimajor axis, e is the eccentricity, $\Delta I^{(2)}$ is the moment of inertia about the polar axis minus the moment of inertia about an equatorial axis (of body 2), c is the velocity of light, and m_1 and m_2 are the masses of bodies 1 and 2.

The unit vector $\boldsymbol{n}^{(1)}$ can be expressed as

$$\boldsymbol{n}^{(1)} = \alpha_1 \boldsymbol{n}^{(J)} + \beta_1 \boldsymbol{n}^{(2)} + \gamma_1(\boldsymbol{n}^{(J)} \times \boldsymbol{n}^{(2)})/\sin\theta_2 , \tag{6}$$

where

$$\alpha_1 = [\cos\theta_1 - (\boldsymbol{n}^{(1)} \cdot \boldsymbol{n}^{(2)}) \cos\theta_2]/\sin^2\theta_2 , \tag{7}$$

$$\beta_1 = [\boldsymbol{n}^{(1)} \cdot \boldsymbol{n}^{(2)} - \cos\theta_1 \cos\theta_2]/\sin^2\theta_2 , \tag{8}$$

$$\gamma_1 = \boldsymbol{n}^{(1)} \cdot [(\boldsymbol{n}^{(J)} \times \boldsymbol{n}^{(2)})/\sin\theta_2] , \qquad |\gamma_1| \leq 1 , \tag{9}$$

and $\boldsymbol{n}^{(J)}$ is a unit vector in the $\boldsymbol{J}$ direction. Because we have assumed that θ_1 and θ_2 are *not* very small, $|\alpha_1|$ and $|\beta_1|$ can be considered to be of order unity or smaller. It is clear from equation (1) that $\dot{\boldsymbol{S}}_{\rm av}^{(2)}$ will be unaltered if we replace $\boldsymbol{\Omega}_{\rm av}^{(2)}$ by $\boldsymbol{\Omega}_R^{(2)}$, where these two quantities differ only by a component in the $\boldsymbol{n}^{(2)}$ direction. To obtain $\boldsymbol{\Omega}_R^{(2)}$ from $\boldsymbol{\Omega}_{\rm av}^{(2)}$, first replace $\boldsymbol{n}$ by $(\boldsymbol{J} - \boldsymbol{S}^{(1)} - \boldsymbol{S}^{(2)})/L$, next use equation (6) to replace $\boldsymbol{n}^{(1)}$, and then drop the component in the $\boldsymbol{n}^{(2)}$ direction. We then obtain

$$\boldsymbol{\Omega}_R^{(2)} = \Omega_{\rm prec}^{(2)} \boldsymbol{n}^{(J)} + \dot{\theta}_2(\boldsymbol{n}^{(J)} \times \boldsymbol{n}^{(2)})/\sin\theta_2 , \tag{10}$$

where

$$\Omega_{\rm prec}^{(2)} = \Omega_{Q2{\rm prec}}^{(2)} + \Omega_{\rm DSprec}^{(2)} + \Omega_{\rm LTprec}^{(2)} , \tag{11}$$

$$\dot{\theta}_2 = \dot{\theta}_{2Q2} + \dot{\theta}_{2\rm DS} + \dot{\theta}_{2\rm LT} , \tag{12}$$

and

$$\Omega_{Q2{\rm prec}}^{(2)} = -3(\boldsymbol{n} \cdot \boldsymbol{n}^{(2)}) A_{Q2}^{(2)}(J - \alpha_1 S^{(1)})/L , \tag{13}$$

$$\dot{\theta}_{2Q2} = 3(\boldsymbol{n} \cdot \boldsymbol{n}^{(2)}) A_{Q2}^{(2)}(\gamma_1 S^{(1)})/L , \tag{14}$$

$$\Omega_{\rm DSprec}^{(2)} = A_{\rm DS}^{(2)}(J - \alpha_1 S^{(1)})/L , \tag{15}$$

$$\dot{\theta}_{2\rm DS} = -A_{\rm DS}^{(2)}(\gamma_1 S^{(1)})/L , \tag{16}$$

$$\Omega_{\rm LTprec}^{(2)} = A_{\rm LT}^{(2)}[\alpha_1 - 3(\boldsymbol{n} \cdot \boldsymbol{n}^{(1)})(J - \alpha_1 S^{(1)})/L] , \tag{17}$$

$$\dot{\theta}_{2\rm LT} = A_{\rm LT}^{(2)}[\gamma_1 + 3(\boldsymbol{n} \cdot \boldsymbol{n}^{(1)})(\gamma_1 S^{(1)})/L] . \tag{18}$$

Using $L \gg S^{(1)}$, $L \gg S^{(2)}$, and noting that

$$A_{\rm LT}^{(2)} = A_{\rm DS}^{(2)} \frac{(S^{(1)}/L)}{(4 + 3m_1/m_2)} , \tag{19}$$

we obtain the following approximations:

$$\Omega_{\rm prec}^{(2)} \approx -3A_{Q2}^{(2)} \cos\theta_2 + A_{\rm DS}^{(2)} , \tag{20}$$

$$\dot{\theta}_2 \approx [(3A_{Q2}^{(2)} \cos\theta_2 - A_{\rm DS}^{(2)})(S^{(1)}/L) + A_{\rm LT}^{(2)}]\gamma_1 . \tag{21}$$

Let us also define $\Omega_{\rm nu}$ (it will turn out to be the nutational frequency) as

$$\Omega_{\rm nu} = |\Omega_{\rm prec}^{(1)} - \Omega_{\rm prec}^{(2)}| \approx |A_{\rm DS}^{(1)} - 3A_{Q1}^{(1)} \cos\theta_1 - A_{\rm DS}^{(2)} + 3A_{Q2}^{(2)} \cos\theta_2| , \tag{22}$$

where terms referring to body 1 can be obtained from the corresponding term of body 2 by interchanging indices 1 and 2. Since $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$ precess about $\boldsymbol{n}^{(J)}$ at the rates of $\Omega_{\rm prec}^{(1)}$ and $\Omega_{\rm prec}^{(2)}$, rspectively, we find that γ_1 of equation (9) can be expressed as

$$\gamma_1 = k_1 \sin\theta_1 \cos\left[\int_{t_0}^{t} \Omega_{\rm nu}\, dt\right] , \tag{23}$$

where $\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(J)}$, and $(\boldsymbol{n}^{(J)} \times \boldsymbol{n}^{(2)})/\sin\theta_2$ are coplanar with γ_1 positive at time t_0. We define $k_1 = 1$ and $k_2 = -1$, and the indices 1 and 2 are to be treated as all other indices 1 and 2 when interchanging indices to obtain additional equations. Because $\Omega_{\rm nu}$ is essentially constant, we can put

$$\gamma_1 \approx k_1 \sin\theta_1 \cos[\Omega_{\rm nu}(t - t_0)] . \tag{24}$$

Using equation (24) in (21), we obtain our final result for the nutation of body 2 as

$$\theta_2 \approx k_1 A_{\rm nu}^{(2)} \sin[\Omega_{\rm nu}(t - t_0)] + \theta_{02} , \tag{25}$$

$$A_{\rm nu}^{(2)} \approx \sin\theta_1[(3A_{Q2}^{(2)} \cos\theta_2 - A_{\rm DS}^{(2)})(S^{(1)}/L) + A_{\rm LT}^{(2)}]/\Omega_{\rm nu} . \tag{26}$$

Noting that $\dot{\phi}_2 = \Omega_{\text{prec}}^{(2)}$, where ϕ_2 is the azimuthal angle of $\boldsymbol{S}^{(2)}$ with respect to $\boldsymbol{J}$, we obtain our final result for the precession of body 2 as

$$\phi_2 \approx \Omega_{\text{prec}}^{(2)}(t - t_0) + \phi_{02} \, . \tag{27}$$

Equations (25)–(26) will not be valid for degenerate cases where $\Omega_{\text{nu}} = 0$ or is sufficiently small so that $A_{\text{nu}}^{(2)}$ is not very small.

Since we are assuming that both nutation amplitudes $A_{\text{nu}}^{(1)}$ and $A_{\text{nu}}^{(2)}$ are very small [i.e., $\ll 1$ (radian)], the factors $\cos \theta_1$ and $\cos \theta_2$ in equation (22) will not change by much and, thus, the integration of equation (23) to obtain equation (24) is justified. Integration of equation (21) to obtain equations (25) and (26) also assumes very small nutation amplitudes. However, assuming the nutation amplitudes are very small does *not* imply that they are negligibly small. For example, very small nutation amplitudes of the order of $\frac{1}{20}$ radian $\approx 3°$ would *not* be negligibly small in that they would be observable.

III. CONCLUSION

We have developed a simple mathematical procedure based on equation (10) that leads in a straightforward manner to the easy-to-use spin nutation result of equations (25) and (26). For an entirely different mathematical approach to the precession and nutation problem in compact binary systems, see the recent preprint of Hamilton and Sarazin (1980). The Hamilton-Sarazin general case contains special mathematical functions and, hence, is more difficult to understand and use than our treatment which covers the case most likely to occur in compact binary systems of interest.

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