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Relativistic quadrupole moment

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The relativistic quadrupole-moment contribution to the spin precession (and orbit precession) of a binary system consisting of two rotating bodies with arbitrary masses and quadrupole moments is examined.

I. INTRODUCTION

The relativistic spin precession of either body in a binary system (as well as the relativistic orbit precession) was derived by Barker and O'Connell[1] using quantum-theoretical techniques. The spin precession of body 1 was found to be[1]

$$\dot{\vec{S}}^{(1)}_{av}=\vec{\Omega}^{(1)}_{av}\times\vec{S}^{(1)}, \quad (1)$$

with

$$\vec{\Omega}^{(1)}_{av}=\vec{\Omega}^{(1)}_{DS\,av}+\vec{\Omega}^{(1)}_{LT\,av}+\vec{\Omega}^{(1)}_{Q1\,av}, \quad (2)$$

while the oribit precession was found to be[1]

$$\dot{\vec{L}}_{av}=\vec{\Omega}^{*}\times\vec{L}, \quad \dot{\vec{A}}_{av}=\vec{\Omega}^{*}\times\vec{A}, \quad (3)$$

with

$$\vec{\Omega}^{*}=\vec{\Omega}^{*(E)}+\vec{\Omega}^{*(1)}+\vec{\Omega}^{*(2)}+\vec{\Omega}^{*(1,2)}+\vec{\Omega}^{*(Q1)}+\vec{\Omega}^{*(Q2)}, \quad (4)$$

where $\vec{S}^{(1)}$ is the spin of body 1, $\vec{L}$ is the orbital angular momentum, and $\vec{A}$ is the Runge-Lenz vector. The quadrupole moment of body 1 gives rise to the terms $\vec{\Omega}^{(1)}_{Q1\,av}$ and $\vec{\Omega}^{*(Q1)}$ [see Eqs. (5) and (6)] which are the terms of interest for this paper. The other terms on the right-hand side of Eqs. (2) and (4) can be found in Ref. 1. Classical verifications of the above equations were subsequently given by a number of authors.[2-4]

Recently, the spin and orbit precession for a two-body system was derived from the general relativistic equations of motion by McCrea and O'Brien[5] using an approximation method developed by Synge.[6] While Barker and O'Connell[1,7] consider elastic spheres which become distorted when spinning, McCrea and O'Brien consider "rigid" bodies which preserve their spherical shape (in a system where the center of mass is at rest). Clearly, the quadrupole results of Barker and O'Connell[1] are strictly Newtonian effects while those of McCrea and O'Brien[5] are some sort of relativistic quadrupole effect.

II. QUADRUPOLE-MOMENT EFFECTS

The quadrupole-moment contributions (due to body 1) given by Barker and O'Connell[1] are

$$\vec{\Omega}^{(1)}_{Q1\,av}=\frac{Gm_2\Delta I^{(1)}}{2S^{(1)}a^3(1-e^2)^{3/2}}\times[\vec{n}^{(1)}-3(\vec{n}\cdot\vec{n}^{(1)})\vec{n}], \quad (5)$$

$$\vec{\Omega}^{*(Q1)}=-\frac{3Gm_2\Delta I^{(1)}}{4La^3(1-e^2)^{3/2}}\times\{2(\vec{n}\cdot\vec{n}^{(1)})\vec{n}^{(1)}+[1-5(\vec{n}\cdot\vec{n}^{(1)})^2]\vec{n}\}, \quad (6)$$

where a is the semimajor axis, e is the eccentricity, $\vec{n}$ is a unit vector in the $\vec{L}$ direction, $\vec{n}^{(1)}$ is a unit vector in the $\vec{S}^{(1)}$ direction, and m_2 is the mass of body 2. The quadrupole moment $D^{(1)}$ of body 1 is equal to $2\Delta I^{(1)}$ where $\Delta I^{(1)}$ is the moment of inertia about the polar axis minus the moment of inertia about an equatorial axis[1,7] and

$$\Delta I^{(1)}=\frac{1}{2}\int dV'(r'^2-3z'^2)\rho_{N1}(\vec{r}'), \quad (7)$$

where ρ_{N1} is the Newtonian (i.e., nonrelativistic) mass density of body 1 and the integration is over the *distorted* sphere of body 1 with the z' axis in the $\vec{n}^{(1)}$ direction and $\vec{r}'$ denoting the *strained* position from the center of mass.[1,7]

The purpose of the present paper is to *directly* calculate a relativistic replacement for Eq. (7) which can then be used in Eqs. (5) and (6) to obtain final results that can be compared to the corresponding results of McCrea and O'Brien.[5] For general relativity in the *weak-field limit* Poisson's equation[8] for the gravitational potential ϕ is $\Delta\phi=4\pi G\rho_e$ where the effective mass density is given by (note $\Delta\phi=4\pi G\rho_N$ for Newtonian theory)

$$\rho_e=(T_{00}+T_{ii})/c^2, \quad (8)$$

and $T_{\mu\nu}$ is the matter energy-momentum tensor. We are using a notation where $x_\mu=(x_i,x_4)$ and $x_4=ix_0=ict$. We shall neglect all gravitational binding effects and assume that in the absence of rotation we have a sphere of uniform density held together by elastic forces. Thus, the special-relativity version of Eq. (8) was given where $g_{\mu\nu}=\delta_{\mu\nu}$. The relativistic replacement for Eq. (7) is thus (since in Poisson's equation ρ_N is replaced by ρ_e)

$$\Delta I^{(1)}=\tfrac{1}{2}\int dV'(r'^2-3z'^2)\rho_{e1}(\vec{r}\,'). \tag{9}$$

The energy-momentum tensor for the elastic continua is given by[9]

$$T_{\mu\nu}=\rho_{00}U_\mu U_\nu+S_{\mu\nu}, \tag{10}$$

where $U_\mu=dx_\mu/d\tau$ and

$$\begin{aligned} S_{ik}&=t_{ik}+t_{il}U_lU_k/c^2,\\ S_{i4}&=t_{il}U_lU_4/c^2,\quad S_{44}=-t_{lm}U_lU_m/c^2. \end{aligned} \tag{11}$$

The rest mass of a rotating body is

$$m_0=\int\rho_{00}dV_0=\int\rho_* dV, \tag{12}$$

where

$$\begin{aligned} \rho_*&=\rho_{00}(1-v^2/c^2)^{-1/2},\\ dV&=dV_0(1-v^2/c^2)^{1/2}. \end{aligned} \tag{13}$$

Thus, the conserved mass density ρ_* for a spherically symmetric (in ρ_*) rigid body does not depend on its angular velocity (in a system where the center of mass is at rest). It is easy to see that

$$T_{00}=\rho_*c^2+\tfrac{1}{2}\rho_*v^2+O(c^{-2}), \tag{14}$$

$$T_{ii}=\rho_*v^2+t_{ii}+O(c^{-2}). \tag{15}$$

We need to calculate t_{ii} to first order only (i.e., the Newtonian approximation). We find that (see Appendix)

$$t_{ii}=-\left(\frac{1+\sigma}{7+5\sigma}\right)\rho_*v^2+t_{ii}^{(SS)}+O(c^{-2}), \tag{16}$$

where σ is Poisson's ratio and $t_{ii}^{(SS)}$ is a spherically symmetric term which cannot contribute to the quadrupole moment. It then follows that

$$\rho_e=\rho_*+\left[\frac{3}{2}-\left(\frac{1+\sigma}{7+5\sigma}\right)\right]\rho_*v^2/c^2 +t_{ii}^{(SS)}/c^2+O(c^{-4}). \tag{17}$$

Using the result of Eq. (17) in Eq. (9) we obtain

$$\Delta I^{(1)}=\Delta I_N^{(1)}+\Delta I_R^{(1)}, \tag{18}$$

where

$$\Delta I_N^{(1)}=\tfrac{1}{2}\int dV'(r'^2-3z'^2)\rho_{*1}(\vec{r}\,') =\tfrac{1}{2}\int dV(r'^2-3z'^2)\rho_{*1}(\vec{r}), \tag{19}$$

is the Newtonian contribution and $\vec{r}$ denotes the *unstrained* position from the center of mass and $\int dV$ is over the sphere[7] of body 1. The result $\Delta I_N^{(1)}$ for *uniform* $\rho_*(\vec{r})$ was found to be[7]

$$\Delta I_N^{(1)}=\frac{5}{14}\left[\frac{(1+\sigma)(13+9\sigma)}{7+5\sigma}\right]\frac{(I^{(1)}\omega^{(1)})^2}{YV_1}, \tag{20}$$

where Y is Young's modulus and V_1 is the volume of body 1 (in the system where the center of mass is at rest). The relativistic contribution $\Delta I_R^{(1)}$ for *uniform* $\rho_*(\vec{r})$ is

$$\Delta I_R^{(1)}=\frac{1}{2}\left[\frac{3}{2}-\left(\frac{1+\sigma}{7+5\sigma}\right)\right](\omega^{(1)})^2c^{-2} \times\int dV(r^2-3z^2)(r^2-z^2)\rho_{*1}(\vec{r}), \tag{21}$$

and to the order of approximation being considered we need integrate only over the sphere using the unstrained distances. We can put Eq. (21) in the form[10]

$$\Delta I_R^{(1)}=\frac{5}{14}\left[\frac{3}{2}-\left(\frac{1+\sigma}{7+5\sigma}\right)\right]\frac{(I^{(1)}\omega^{(1)})^2}{m_{01}c^2} \tag{22}$$

or

$$\Delta I_R^{(1)}=\frac{1}{4}\left[\frac{3}{2}-\left(\frac{1+\sigma}{7+5\sigma}\right)\right](\omega^{(1)})^2J^{(1)}/c^2, \tag{23}$$

where we define

$$I^{(1)}\equiv\int dV(r^2-z^2)\rho_{*1}=\tfrac{2}{3}\int\rho_{*1}r^2dV\equiv 2I^A, \tag{24}$$

$$J^{(1)}\equiv\int dV(r^2-z^2)^2\rho_{*1}=\tfrac{8}{15}\int\rho_{*1}r^4dV\equiv 8J^A. \tag{25}$$

Body 1 rotates with an angular velocity $\vec{\omega}^{(1)}$ and it should be noted that $\vec{S}^{(1)}=I^{(1)}\vec{\omega}^{(1)}$ plus higher-order terms.

For a numerical example involving a real physical object (of uniform mass density ρ_* and held together by elastic forces) we shall choose the gyro[7] of the relativity gyroscope experiment. This fused quartz gyro has a rest radius $r_0=2$ cm and will spin at $\omega=400\pi$ rad/sec. We also have $\rho_*=2.2$ g/cm^3, $Y=7\times10^{11}$ dyn/cm^2, and $\sigma=0.16$ for fused quartz at 2°K. Noting that $I=\frac{8}{15}\pi\rho_*r_0^5=118$ g cm^2 along with Eqs. (20) and (22), we find that $\Delta I_N/I=6.09\times10^{-6}$ and $\Delta I_R/I=1.36\times10^{-15}$.

McCrea and O'Brien[5] using a different physical model for their rotating spheres (with *spherically symmetric* mass density ρ) give final results that are the same as one would obtain by using

$$\Delta I_R^{(1)}=2(\omega^{(1)})^2J^A/c^2 \tag{26}$$

in Eqs. (5) and (6). It should also be noted that their I^A and J^A are defined by Eqs. (24) and (25) with a spherically symmetric $\rho=T_{00}/c^2$. From Eqs. (8) and (15) we find that

$$\rho_e = \rho + \rho v^2/c^2 + t_{ii}/c^2 + O(c^{-4}) . \quad (27)$$

Noting that the t_{ii} used by McCrea and O'Brien[5] is spherically symmetric the result of Eq. (26) follows from Eq. (27).

Finally let us note that the quadrupole moment of a black hole is a *relativistic* quadrupole moment. If body 1 is a black hole one should use[11]

$$\Delta I^{(1)}_{\rm BH} = (S^{(1)})^2/m_1 c^2 \quad (28)$$

in Eqs. (5) and (6).

III. CONCLUSION

We assert that relativistic expressions for $\Delta I^{(1)}$ such as Eqs. (9), (26), and (28) can be correctly used in Eqs. (5) and (6) which were first derived using nonrelativistic physics. We shall assume that the two bodies are at sufficient distance so that post-Newtonian approximations hold. We shall give three arguments for the above assertion.

(i) The effect of $\Delta I^{(1)}$ on body 2 should depend on its magnitude and not on how the magnitude of $\Delta I^{(1)}$ splits between Newtonian and relativistic parts since the gravitational potential ϕ produced in part by $\Delta I^{(1)}$ and acting on body 2 does not depend on that split. Thus, the corresponding equal and opposite reaction of body 2 (due to $\Delta I^{(1)}$) on body 1 should depend only on the magnitude of $\Delta I^{(1)}$ and not on the nature of the split.

(ii) Using the methods of this paper and the McCrea-O'Brien[5] model for rotating spheres we obtained $\Delta I^{(1)}_R$ of Eq. (26) which when used in Eqs. (5) and (6) gave the same results as directly obtained by McCrea and O'Brien[5] using relativistic physics.

(iii) Using the black-hole result $\Delta I^{(1)}_{\rm BH}$ of Eq. (28) in Eqs. (5) and (6) gives the same results[11] as directly obtained by D'Eath[4] using relativistic physics.

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APPENDIX

Let us consider a body at rest to be a perfect sphere of radius r_0 with mass m_0 and uniform mass density ρ_*. We shall denote the coordinates of the point occupied by a particle in the unstrained state of the body by $\vec{r}$ and the coordinates of the point occupied by the same particle in the strained state by $\vec{r}\,' = \vec{r} + \vec{u}$. If the body rotates with an angular velocity ω about the z axis the complete expressions for the displacements are given by[7,12]

$$u_I = \tfrac{1}{3}\rho_* \omega^2 (A r_0^2 + B x_J^2 + C x_3^2) x_I , \quad (A1)$$

$$u_3 = \tfrac{1}{3}\rho_* \omega^2 [\alpha r_0^2 + \beta x_J^2 + \gamma x_3^2] x_3 , \quad (A2)$$

where capital indices (I, J, K, etc.) take on values of 1 and 2, and

$$A = \frac{1}{5(\lambda+2\mu)}\left(\frac{5\lambda+6\mu}{3\lambda+2\mu}\right) + \frac{(4\lambda+3\mu)}{\mu(19\lambda+14\mu)} , \quad (A3)$$

$$B = \frac{-1}{5(\lambda+2\mu)} - \frac{\frac{1}{2}(3\lambda+2\mu)}{\mu(19\lambda+14\mu)} , \quad (A4)$$

$$C = \frac{-1}{5(\lambda+2\mu)} - \frac{\frac{1}{2}(9\lambda+8\mu)}{\mu(19\lambda+14\mu)} , \quad (A5)$$

$$\alpha = \frac{1}{5(\lambda+2\mu)}\left(\frac{5\lambda+6\mu}{3\lambda+2\mu}\right) - \frac{(8\lambda+6\mu)}{\mu(19\lambda+14\mu)} , \quad (A6)$$

$$\beta = \frac{-1}{5(\lambda+2\mu)} + \frac{(6\lambda+5\mu)}{\mu(19\lambda+14\mu)} , \quad (A7)$$

$$\gamma = \frac{-1}{5(\lambda+2\mu)} + \frac{(3\lambda+2\mu)}{\mu(19\lambda+14\mu)} . \quad (A8)$$

We also have

$$\lambda = \frac{Y\sigma}{(1+\sigma)(1-2\sigma)} , \quad \mu = \frac{Y}{2(1+\sigma)} . \quad (A9)$$

The strain tensor e_{ij} is given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) , \quad (A10)$$

and the stress tensor σ_{ij} (note $\sigma_{ij} \equiv -t_{ij}$) is given by

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} , \quad (A11)$$

and hence

$$t_{ii} = -\sigma_{ii} = -(3\lambda + 2\mu) e_{ii} . \quad (A12)$$

Evaluating Eq. (A12) we find that

$$t_{ii} = -\tfrac{1}{3}\rho_* \omega^2 (3\lambda+2\mu)[(4B + \beta - 2C - 3\gamma) x_I^2] + t_{ii}^{(SS)}$$

$$= -\left(\frac{1+\sigma}{7+5\sigma}\right)\rho_* \omega^2 x_I^2 + t_{ii}^{(SS)} , \quad (A13)$$

where

$$t_{ii}^{(SS)} = -\tfrac{1}{3}\rho_* \omega^2 (3\lambda+2\mu)[(2A+\alpha) r_0^2 + (2C + 3\gamma) r^2] . \quad (A14)$$

The force density κ_i is given by

$$\kappa_i = \rho_* a_i = \partial \sigma_{ij}/\partial x_j , \quad (A15)$$

where $\vec{a}$ is the acceleration. Evaluating Eq. (A15) we obtain the result

$$a_I = -\omega^2 x_I , \quad a_3 = 0 . \quad (A16)$$

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[10]For spherically symmetric ρ_* we have $\int \rho_* r^4 dV = 3\int \rho_* r^2 z^2 dV = 5\int \rho_* z^4 dV$.

[11]See second paper of Ref. 1.

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