Solid State Communications, Vol. 39, pp. 993–995.
Pergamon Press Ltd. 1981. Printed in Great Britain.

0038–1098/81/330993–03$02.00/0

NULL ELLIPTICITY IN MAGNETO-OPTICS

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(Received 22 *December* 1980 *by A.G. Chynoweth)*

We determine the relationship between the photon, cyclotron, collision and plasma frequencies which ensures a null ellipticity for electromagnetic wave propagation in a free-carrier magnetoplasma. A comparison is made with the condition for null Faraday rotation.

CONSIDER electromagnetic wave propagation in a magnetoplasma. In the case of linearly-polarized radiation propagating along the magnetic field, the Faraday rotation and the ellipticity per unit path length, l, are given by [1]

$$\frac{d\theta}{dl} = \frac{\omega}{2c}(n_- - n_+), \tag{1}$$

and

$$\frac{d\Delta}{dl} = \frac{\omega}{2c}(\kappa_+ - \kappa_-), \tag{2}$$

where c is the speed of light, ω is the photon frequency, n_+, n_- and κ_+, κ_- are the real and the imaginary parts, respectively, of the complex refractive indices of the right and the left circulary-polarized components of the linearly-polarized wave. These quantities, $n_\pm$, $\kappa_\pm$, are obtained from the dielectric constants, $\epsilon_\pm$, as follows:

$$\begin{aligned} \epsilon_\pm &\equiv \epsilon'_\pm + i\epsilon''_\pm \\ &= (n_\pm + i\kappa_\pm)^2, \end{aligned} \tag{3}$$

where the prime and double prime denote real and imaginary parts, respectively, It follows that:

$$n_\pm^2 = \tfrac{1}{2}\{[(\epsilon'_\pm)^2 + (\epsilon''_\pm)^2]^{1/2} + \epsilon'_\pm\}, \tag{4}$$

and

$$\kappa_\pm^2 = \tfrac{1}{2}\{[(\epsilon'_\pm)^2 + (\epsilon''_\pm)^2]^{1/2} - \epsilon'_\pm\}. \tag{5}$$

In a previous paper [2] we have investigated the condition for null Faraday rotation. We now wish to investigate the condition for null ellipticity. It is clear that null Faraday rotation and null ellipticity are obtained whenever

$$n_+ = n_-, \tag{6}$$

and

$$\kappa_+ = \kappa_-, \tag{7}$$

respectively.

Squaring equations (6) and (7) and making use of equations (4) and (5) gives

$$[(\epsilon'_+)^2 + (\epsilon''_+)^2]^{1/2} - [(\epsilon'_-)^2 + (\epsilon''_-)^2]^{1/2} = \epsilon'_+ - \epsilon'_-, \tag{8}$$

and

$$[(\epsilon'_+)^2 + (\epsilon''_+)^2]^{1/2} - [(\epsilon'_-)^2 + (\epsilon''_-)^2]^{1/2} = -(\epsilon'_+ - \epsilon'_-). \tag{9}$$

Removing the radicals by re-arrangement of the terms and two further squarings we obtain the *single* general result

$$\{(\epsilon''_+)^2 - (\epsilon''_-)^2\}^2 = 4(\epsilon'_+ - \epsilon'_-)\{\epsilon'_+(\epsilon''_-)^2 - \epsilon'_-(\epsilon''_+)^2\}. \tag{10}$$

This result has already been obtained [2] to give null Faraday rotation. It is now clear that equation (10) also gives null ellipticity. However, as we shall show explicitly below, the conditions for null θ and Δ correspond to different solutions of equation (10).

We now use a Drude-type classical treatment, i.e. we take

$$\epsilon_\pm = \epsilon_l\left\{1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c + i\nu)}\right\}, \tag{11}$$

where ϵ_l, which is the dielectric constant of the lattice, is real, homogeneous and isotropic, $\omega_c = eB/m^*c$ is the cyclotron frequency, B is the magnetic field, and ν is the collision frequency. In addition

$$\omega_p^2 = 4\pi\frac{ne^2}{m^*\epsilon_l}, \tag{12}$$

is the plasma frequency, where m^* is the effective mass and n is the charge density.

Substituting equation (11) into equation (10) and simplifying, gives the condition for null Faraday rotation and null ellipticity in the form of a quintic equation, previously obtained [2], for $x = (\omega/\Omega)^2$:

$$\begin{aligned} f(x) \equiv{}& 4x^5 + \{8[2(\nu/\Omega)^2 - 1] - 3(\omega_P/\Omega)^2\}x^4 \\ &- 8(\omega_P/\Omega)^2[2(\nu/\Omega)^2 - 1]x^3 - 2\{4[2(\nu/\Omega)^2 - 1] \\ &+ 3(\omega_P/\Omega)^2\}x^2 - 4x + (\omega_P/\Omega)^2 = 0, \end{aligned} \tag{13}$$

where

$$\Omega = (\omega_c^2 + \nu^2)^{1/2}.$$

In general, as we have already pointed out [2], inspection of equation (13) gives the following:

$$f(\pm\infty) = \pm\infty, \quad (14)$$

$$f(0) = (\omega_P/\Omega)^2 > 0, \quad (15)$$

$$f(1) = -16(\nu/\Omega)^2(\omega_P/\Omega)^2 < 0, \quad (16)$$

$$f(-1) = -16(\omega_c/\Omega)^2(\omega_P/\Omega)^2 < 0, \quad (17)$$

and

$$f'(0) = -4 < 0, \quad (18)$$

where the prime denotes differentiation with respect to x.

For fixed ν, ω_P and Ω, equation (13) has, for $x = (\omega/\Omega)^2$, two complex roots, one negative root in the interval $(-1, 0)$, and two positive roots, one in each of the intervals $(0,1)$ and $(1, \infty)$.

Our previous paper [2] dealt only with the larger positive root with the following conclusions:

(1) Only the larger positive root gives $\theta = 0$ when inserted into equation (1),

(2) By inspecting equations (14)–(18), the desired root for $\theta = 0$ always occurs for $\omega > \Omega$ (or $x > 1$),

(3) For $\omega_p/\Omega \ll 1; x \approx 1 + (\omega_p^2/2\Omega^2)$, so that $\theta = 0$ when $\omega = \Omega[1 + (\omega_p^2/4\Omega^2)] \approx \Omega$,

(4) For $\omega_P/\Omega \gg 1: x \approx \frac{3}{4}(\omega_p/\Omega)^2$, so that $\theta = 0$ when $\omega \approx 0.866\omega_p \gg \Omega$.

By inspection of equations (8) and (9) we see that equations (6) and (7) are not satisfied simultaneously, since this would imply that $\epsilon'_+ = \epsilon'_-$ which can only occur [2] for $\omega = \Omega$ (or $x = 1$), contradicting equation (16). Thus we expect the smaller positive root of equation (13) to give null ellipticity. This expectation has been borne out by a detailed and exact numerical investigation.

The results corresponding to the smaller positive root of equation (13) are as follows:

(1) Only the smaller positive root gives $\Delta = 0$ when inserted into equation (2),

(2) By inspecting equations (14)–(18), the desired root for $\Delta = 0$ always occurs for $0 < \omega < \Omega$ (or $0 < x < 1$),

(3) For $\omega_p/\Omega \ll 1: x \approx \frac{1}{4}(\omega_p/\Omega)^2$, so that $\Delta = 0$ when $\omega \approx \frac{1}{2}\omega_p \ll \Omega$,

(4) For $\omega_p/\Omega \gg 1$: Consider the two functions $y = f(x, \omega_p)$ and $y = f(x, \omega'_p)$ for $\omega_p \neq \omega'_p \neq 0$. Using equation (13) we deduce that the intersection points of these curves are given by the real solutions of

$$g(x) \equiv 3x^4 + 8[2(\nu/\Omega)^2 - 1]x^3 + 6x^2 - 1 = 0. \quad (19)$$

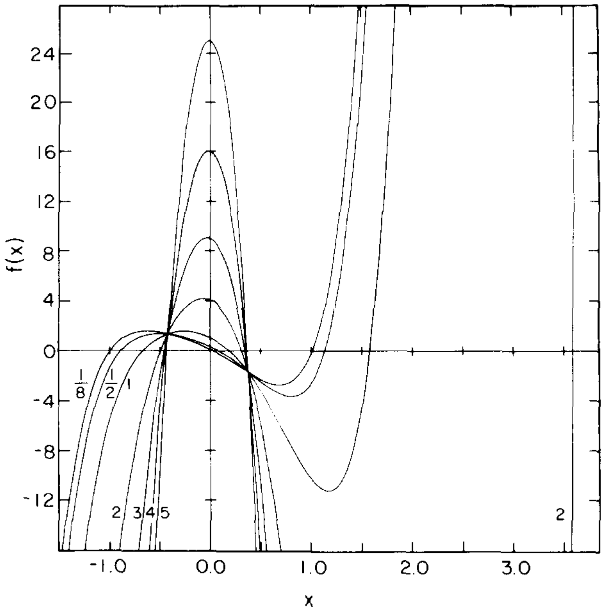

Fig. 1. Plot of $f(x)$ vs x for $(\omega_c/\Omega) = (3/5)$, $(\nu/\Omega) = (4/5)$ and for (ω_p/Ω) values as indicated on the curves.

Note that this is independent of both ω_p and ω'_p. Hence, for *all* values of ω_p, the curves $y = f(x, \omega_p)$, given by equation (13), intersect at the real solutions of equation (19).

The discriminant, D, of $g(x)$ is given by

$$D = -6912(\nu/\Omega)^4[1 - (\nu/\Omega)^2]^2 < 0, \quad (20)$$

since $0 < (\nu/\Omega)^2 < 1$. Therefore, equation (19) has two complex roots and hence two real roots.

For the sake of completeness, we give the exact solution for the two real roots of equation (19):

$$x = \tfrac{1}{3}\{-[a - (a^2 + 9b - 3)^{1/2}] \pm [[a - (a^2 + 9b - 3)^{1/2}]^2 - 3[1 + 6b - 2(4b^2 + 3b + 1)^{1/2}]]^{1/2}\} \quad (21)$$

where

$$a = 2[2(\nu/\Omega)^2 - 1],$$

and

$$b = \tfrac{1}{3}\{4(\nu/\Omega)^2[1 - (\nu/\Omega)^2]\}^{1/3}.$$

Inspection of equation (21) shows that equation (19) has one positive root and one negative root.

If the smaller positive root of equation (13), corresponding to $\Delta = 0$, is denoted by $h(\nu, \Omega, \omega_p)$, it is easy to show that equation (21) may be written as

$$x = \lim_{\omega_p/\Omega \to \infty} h(\nu, \Omega, \omega_p) \quad (22)$$

Thus, equation (21) gives a strict upper bound for the smaller positive root of equation (13). In Fig. 1 we

Table 1. Values of ω/Ω for which null ellipticity is obtained, for various values of ω_c, ν and ω_p

ω_c/Ω	ν/Ω	ω_p/Ω	ω/Ω
3/5	4/5	10^{-2}	$4.999\,965 \times 10^{-3}$
		10^{-1}	$4.996\,415 \times 10^{-2}$
		1	$4.242\,507 \times 10^{-1}$
		10	$6.071\,442 \times 10^{-1}$
		10^2	$6.094\,757 \times 10^{-1}$
4/5	3/5	10^{-2}	$5.000\,035 \times 10^{-3}$
		10^{-1}	$5.003\,414 \times 10^{-2}$
		1	$4.595\,332 \times 10^{-1}$
		10	$6.482\,656 \times 10^{-1}$
		10^2	$6.503\,503 \times 10^{-1}$

present typical plots of $f(x)$ vs x. Table 1 is a list of values of (ω/Ω), for which null ellipticity is achieved, for various values of ν, Ω and ω_p.

In conclusion, the photon frequencies for which the null Faraday rotation and the null ellipticity conditions are satisfied are always distinct but fundamentally related by being solutions of equation (10).

Acknowledgements – This research was partially supported by the Department of Energy under Contract No. DE-AS05-79ER10459.

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