Physical Interpretation of Generalized Conservation Laws (*).

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(ricevuto il 5 Luglio 1965)

Generalized conservation laws have been discussed by Lipkin (1), Morgan (2), Fradkin (3), Kibble (4), Fairlie (5), Candlin (6), and the present authors (7,8). We have also shown (8) how all the usual conservation laws for free fields with mass may be derived without the use of Lagrangians or Noether's theorem, by a method similar to that used by Good and Hammer (9) for massless free fields, and Fradkin (3) reached a similar conclusion from a consideration of the equations of motion in the Pauli-Fierz-Dirac form. It is our purpose here to give a clear insight into the physical interpretation of the generalized conservation laws. We will also show that all the usual conservation laws in the number operator formalism may be written down very simply in one equation.

To avoid unnecessary formalism we will consider, as an example, the Dirac field, but it will be obvious that our procedure will apply to all fields. Furthermore, the Dirac field will be useful for a direct generalization to the Bargmann-Wigner (10)

(*) Research sponsored by the U. S. Atomic Energy Commission under contract with the Union Carbide Corporation.

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(1) D. M. Lipkin: *Journ. Math. Phys.*, 5, 696 (1964).

(2) T. A. Morgan: *Journ. Math. Phys.*, 5, 1659 (1964).

(3) D. M. Fradkin: *Journ. Math. Phys.*, 6, 879 (1965).

(4) D. W. B. Kibble: *Conservation Laws for Free Fields*, Imperial College, London, I.C.T.P. 64/65.

(5) D. B. Fairlie: *Nuovo Cimento*, 37, 898 (1965).

(6) D. J. Candlin: *Nuovo Cimento*, 37, 1390 (1965).

(7) R. F. O'Connell and D. R. Tompkins: *Generalized Solutions for Massless Free Fields and Consequent Generalized Conservation Laws*, in *Journ. Math. Phys.* (in press).

(8) R. F. O'Connell and D. R. Tompkins: *Generalized Conservation Laws for Free Fields with Mass*, in *Nuovo Cimento* (in press).

(9) R. H. Good jr.: *Phys. Rev.*, **105**, 1914 (1957); C. L. Hammer and R. H. Good jr.: *Phys. Rev.*, **108**, 882 (1957).

(10) V. Bargmann and E. P. Wigner: *Proc. Nat. Acad. Sci. U. S.*, **34**, 211 (1948).

equation, the latter being the most general relativistic field equation possible. We have already shown (8) that all the usual conservation laws associated with the Dirac field, as well as all the generalized conservation laws, may be summarized in one equation (11)

$$\frac{\partial}{\partial x_\mu}(\psi'^{+}\gamma_4\gamma_\mu\psi'') = 0\,, \tag{1}$$

where $\psi' = V'\psi$, $\psi'' = V''\psi$, V' and V'' are operators so selected that ψ' and ψ'' satisfy the Dirac equation, and ψ is the usual Dirac wave-function.

The particle aspect of these conservation laws is obtained by considering the usual expansion of ψ in plane waves which are eigenstates of momentum and polarization (12):

$$\psi = \frac{1}{\sqrt{V}}\sum_p \sum_{r=1}^{2} \{a_r(p)u_r(p)\exp[ip\cdot x] + b_r^+(p)v_r(p)\exp[-ip\cdot x]\}\,, \tag{2}$$

where the operators a_r and b_r obey the Fermi-Dirac commutator rules. Thus the conserved charge

$$Q = e\int \psi^+\psi\,\mathrm{d}^3x \tag{3}$$

may be written, in a second quantization theory, as (13) the expectation value of the charge operator Q:

$$\langle Q\rangle = e\sum_p \sum_{r=1}^{2}\{N_r^{(+)}(p) - N_r^{(-)}(p)\}\,, \tag{4}$$

where $N_r^{(+)}$ and $N_r^{(-)}$ are the expectation values of the particle number operators

$$\eta_r^{(+)}(p) = a_r^+(p)a_r(p) \tag{5a}$$

and

$$\eta_r^{(-)}(p) = b_r^+(p)b_r(p)\,. \tag{5b}$$

We now wish to generalize. We immediately find (8) that *all* conserved quantities for the Dirac field, which we denote by $\tilde{Q}$, may be simply written as

$$\tilde{Q} = e\int \psi'^{+}\psi''\,\mathrm{d}^3x\,. \tag{6}$$

Going over now to the particle picture and making the *particular* choice of

$$V' = \partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}\,,\qquad V'' = \partial_{\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_m}\,, \tag{7}$$

(11) We have absorbed the operator O which appears in the corresponding equation in ref. (8) into V''.

(12) A. I. AKHIEZER and V. B. BERESTETSKY: *Quantum Electrodynamics* (United States Atomic Energy Commission translation 2876), p. 176.

(13) We have omitted the infinite constant charge associated with the vacuum as it is unimportant for our discussion and furthermore it may be eliminated by use of normal products in eq. (3).

we obtain

$$\langle \tilde{Q} \rangle = e \sum_{p} \sum_{r=1}^{2} \{ \tilde{N}_r^{(+)}(p) - \tilde{N}_r^{(-)}(p) \} , \tag{8}$$

where $\tilde{N}_r^{(+)}$ and $\tilde{N}_r^{(-)}$ are the expectation values of the operators

$$\tilde{\eta}_r^{(+)}(p) = a_r'^{+}(p) a_r''(p) \tag{9a}$$

and

$$\tilde{\eta}_r^{(-)}(p) = b_r''^{+}(p) b_r'(p) , \tag{9b}$$

and where

$$a_r'^{+}(p) = (-i)^n p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} a_r^{+}(p) , \tag{10a}$$

$$b_r'(p) \ = i^n p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} b_r(p) , \tag{10b}$$

$$a_r''(p) \ = i^m p_{\beta_1} p_{\beta_2} \cdots p_{\beta_m} a_r(p) \tag{10c}$$

and

$$b_r''^{+}(p) = (-i)^m p_{\beta_1} p_{\beta_2} \cdots p_{\beta_m} b_r^{+}(p) . \tag{10d}$$

Thus

$$\tilde{\eta}_r^{(+)}(p) = (i)^{m-n} p_{\alpha_1} \cdots p_{\alpha_n} p_{\beta_1} \cdots p_{\beta_m} \eta_r^{(+)}(p) \tag{11a}$$

and

$$\tilde{\eta}_r^{(-)}(p) = (-i)^{m-n} p_{\alpha_1} \cdots p_{\alpha_n} p_{\beta_1} \cdots p_{\beta_m} \eta_r^{(-)}(p) . \tag{11b}$$

Therefore

$$\langle \tilde{Q} \rangle = (i)^{m-n} e \sum_{p} \sum_{r=1}^{2} \{ N_r^{(+)}(p) + (-1)^{m+n+1} N_r^{(-)}(p) \} p_{\alpha_1} \cdots p_{\alpha_n} p_{\beta_1} \cdots p_{\beta_m} . \tag{12}$$

Simply by redefinition we see that the most general conserved quantity is

$$\langle \tilde{Q} \rangle = e \sum_{p} \sum_{r=1}^{2} \{ N_r^{(+)}(p) + (-1)^{n+1} N_r^{(-)}(p) \} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} . \tag{13}$$

We obtain the usual charge conservation law with $n = 0$, the usual energy-momentum conservation law with $n = 1$, Lipkin's «zilch» conservation law with $n = 2$, and other more generalized conservation laws for higher n values. We wish to re-emphasize that our results are obtained without the use of Lagrangian theory. MORGAN and JOSEPH [14] have obtained a similar expression by use of tensor Lagrangians and Noether's theorem.

Our expression for $\langle \tilde{Q} \rangle$ makes clear the origin of the generalized conservation laws; they are due to the fact that $N_r^{(+)}(p)$ and $N_r^{(-)}(p)$ are separately constant in

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[14] T. A. MORGAN and D. W. JOSEPH: *Tensor Lagrangians and Generalized Conservation Laws for Free Fields* (preprint).

time for any p and any r. Thus any linear combination of these quantities must also be constant in time. In other words, we can write $\langle \tilde{Q} \rangle$ in a still more general form, *viz.*

$$\langle \tilde{Q} \rangle = e \sum_{p} \sum_{r=1}^{2} \{ N^{(+)}(p) p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_m} \pm N^{(-)}(p) p_{\beta_1} p_{\beta_2} \cdots p_{\beta_m} \} . \tag{14}$$

Our general formalism of course does not restrict V to be a linear operator; the only criterion is that $V\psi$ be a solution of the relevant field equation. Thus for the Dirac equation we could choose the charge-conjugate solution $\psi' = \gamma_2 \psi^*$ (which is Fradkin's [3] ψ^{ch}) or more generally still we could take $\psi' = \partial_{\alpha_1} \dots \partial_{\alpha_n} \gamma_2 \psi^*$. Other nonlinear transformations are associated with space and time reflections. However, examination of all these transformations will not lead to any more general conserved quantity than the $\langle \tilde{Q} \rangle$ given by eq. (14) above.

In summary, we see that the existence of an infinite number of conserved quantities is dependent on the fact that the fields are noninteracting. These so-called generalized conservation laws do not actually constitute new nontrivial conservation laws because they are in fact derivable from the usual basic conservation laws.