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Removal of acceleration terms from the two-body Lagrangian to order c^{-4} in electromagnetic theory

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Starting with the acceleration-dependent Lagrangian to order c^{-4} for two charged bodies (with $e_1/m_1 = e_2/m_2$) in electromagnetic theory, we show how to correctly convert the acceleration-dependent terms in this Lagrangian into velocity- and coordinate-dependent terms by a procedure that we call the *method of the double zero*. From the resulting velocity-dependent Lagrangian we easily find the corresponding Hamiltonian and corresponding Hamiltonian and Lagrangian in center-of-mass coordinates. We also give the Ostrogradsky Hamiltonian.

Partant avec le lagrangien dépendant de l'accélération à l'ordre c^{-4} pour deux corps chargés (avec $e_1/m_1 = e_2/m_2$) en théorie électromagnétique, nous montrons comment transformer de façon correcte les termes dépendant de l'accélération dans ce lagrangien en termes dépendant de la vitesse et des coordonnées au moyen d'une procédure que nous appelons la *méthode du double zéro*. Du lagrangien dépendant de la vitesse ainsi obtenu, nous tirons facilement l'hamiltonien correspondant, ainsi que l'hamiltonien et le lagrangien dans le système du centre de masse. Nous donnons aussi l'hamiltonien d'Ostrogradsky.

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I. Introduction

The acceleration-dependent post–post Newtonian (i.e., to order c^{-4}) Lagrangian for n bodies with equal masses and charges was first given by Golubenkov and Smorodinskii (1) with, however, some serious sign misprints. Then, for the two-body identicle particle result, they used the lowest-order equations of motion in the Lagrangian to eliminate the acceleration terms which are all of order c^{-4}. Landau and Lifshitz (2) also give (without misprints) this two-body result. However, using the equations of motion in the Lagrangian (as done in refs. 1 and 2) changes its functional form and, hence, leads to different and, thus, incorrect equations of motion. In another paper (3) we shall give a new and detailed derivation of the acceleration-dependent post–post Newtonian Lagrangian for two charged bodies with the requirement that $e_1/m_1 = e_2/m_2$ in order to postpone dipole radiation from the c^{-3} to the c^{-5} order.

It can easily be seen that if one uses the lowest-order equations of motion in a Lagrangian $\mathscr{L}$ with acceleration terms of order c^{-4} to obtain the Lagrangian $\mathscr{L}^*$ without acceleration terms, then

$$[1] \quad \mathscr{L}^* \equiv \mathscr{L} + Z$$

and Z is of order c^{-4}. Also $Z = 0$ upon use of the lowest-order equations of motion. However, upon use of the lowest-order equations of motion $\partial Z/\partial \boldsymbol{r}_i \neq 0$ and $\partial Z/\partial \boldsymbol{a}_i \neq 0$ (though $\partial Z/\partial \boldsymbol{v}_i = 0$ since the lowest-order equations of motion do not contain $\boldsymbol{v}_i$). The equations of motion for the Lagrangian $\mathscr{L}^*$ are

$$[2] \quad \left[\frac{\partial \mathscr{L}}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{v}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{a}_i}\right)\right] + \left[\frac{\partial Z}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial Z}{\partial \boldsymbol{v}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial Z}{\partial \boldsymbol{a}_i}\right)\right] = 0$$

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which, in general, are clearly *not* the same (to order c^{-4}) as those for the Lagrangian $\mathscr{L}$.

Next, let us consider the Lagrangian $\bar{\mathscr{L}}$ defined as

$$[3] \quad \bar{\mathscr{L}} \equiv \mathscr{L} + Z_1 Z_2$$

where $Z_1 Z_2$ is of order c^{-4} and both $Z_1 = 0$ and $Z_2 = 0$ upon use of the lowest-order equations of motion. The equations of motion for the Lagrangian $\bar{\mathscr{L}}$ are

$$[4] \quad \left[\frac{\partial \mathscr{L}}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{v}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{a}_i}\right)\right] + \left[Z_1 \frac{\partial Z_2}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(Z_1 \frac{\partial Z_2}{\partial \boldsymbol{v}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(Z_1 \frac{\partial Z_2}{\partial \boldsymbol{a}_i}\right)\right] + \left[Z_2 \frac{\partial Z_1}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(Z_2 \frac{\partial Z_1}{\partial \boldsymbol{v}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(Z_2 \frac{\partial Z_1}{\partial \boldsymbol{a}_i}\right)\right] = 0$$

which *are* clearly the same (to order c^{-4}) as those for the Lagrangian $\mathscr{L}$. We conclude that adding a *double-zero* term to a Lagrangian does not change the equations of motion (to the order under consideration).

In Sect. II we give the acceleration-dependent two-body Lagrangian (3) and in Sects. III and IV we eliminate the acceleration-dependent terms by adding double-zero and total time derivative terms. In Sect. V we give the Hamiltonian and in Sects. VI and VII we give, respectively, the Hamiltonian and Lagrangian in center-of-mass coordinates. In Sect. VIII we give the Ostrogradsky Hamiltonian and in Sect. IX we present our conclusions.

II. Acceleration-dependent Lagrangian

The acceleration-dependent two-body (with $e_1/m_1 = e_2/m_2$) Lagrangian to order c^{-4} in electromagnetic theory can be expressed as (1–3)

$$[5] \quad \mathscr{L}'' = \mathscr{L}_{\mathrm{D}}' + \mathscr{L}_{(4)}''$$

where

$$[6] \quad \mathscr{L}_{\mathrm{D}}' = -m_1 c^2 - m_2 c^2 + \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 + \tfrac{1}{8} m_1 v_1^4/c^2 + \tfrac{1}{8} m_2 v_2^4/c^2 - \frac{e_1 e_2}{r}\left[1 - \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{2c^2} - \frac{(\boldsymbol{v}_1 \cdot \boldsymbol{r})(\boldsymbol{v}_2 \cdot \boldsymbol{r})}{2c^2 r^2}\right]$$

is the Darwin Lagrangian, and where

$$[7] \quad \mathscr{L}_{(4)}'' = \mathscr{L}_{(4)b}'' + \mathscr{L}_{(4)a}'' + \mathscr{L}_{(4)aa}''$$

and

$$[8] \quad \mathscr{L}_{(4)b}'' = \frac{e_1 e_2}{8c^4 r}[2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)^2 - v_1^2 v_2^2 + (\boldsymbol{v}_1 \cdot \boldsymbol{r})^2 v_2^2/r^2 + (\boldsymbol{v}_2 \cdot \boldsymbol{r})^2 v_1^2/r^2 - 3(\boldsymbol{v}_1 \cdot \boldsymbol{r})^2(\boldsymbol{v}_2 \cdot \boldsymbol{r})^2/r^4] + \frac{m_1 v_1^6}{16c^4} + \frac{m_2 v_2^6}{16c^4}$$

$$[9] \quad \mathscr{L}_{(4)a}'' = \frac{e_1 e_2}{8c^4 r}[v_1^2(\boldsymbol{a}_2 \cdot \boldsymbol{r}) - 2(\boldsymbol{v}_1 \cdot \boldsymbol{r})(\boldsymbol{v}_1 \cdot \boldsymbol{a}_2) - (\boldsymbol{v}_1 \cdot \boldsymbol{r})^2(\boldsymbol{a}_2 \cdot \boldsymbol{r})/r^2 - v_2^2(\boldsymbol{a}_1 \cdot \boldsymbol{r}) + 2(\boldsymbol{v}_2 \cdot \boldsymbol{r})(\boldsymbol{v}_2 \cdot \boldsymbol{a}_1) + (\boldsymbol{v}_2 \cdot \boldsymbol{r})^2(\boldsymbol{a}_1 \cdot \boldsymbol{r})/r^2]$$

$$[10] \quad \mathscr{L}_{(4)aa}'' = \frac{e_1 e_2}{8c^4 r}[(\boldsymbol{a}_1 \cdot \boldsymbol{r})(\boldsymbol{a}_2 \cdot \boldsymbol{r}) - 3r^2(\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)]$$

We have defined[1] $\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2$ and [9] is correct with this definition of $\boldsymbol{r}$. We have also used the convention of adding a prime to the Lagrangian every time a total time derivative was added.

The lowest-order equations of motion obtained from $\mathscr{L}''$ of [5] are

$$[11] \quad \boldsymbol{a}_1 - e_1 e_2 \boldsymbol{r}/m_1 r^3 = 0$$

$$[12] \quad \boldsymbol{a}_2 + e_1 e_2 \boldsymbol{r}/m_2 r^3 = 0$$

[1]Note that ref. 2 uses $\boldsymbol{R} = \boldsymbol{R}_2 - \boldsymbol{R}_1$. Reference 1 is ambiguous since according to their equations of motion they must be using $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$ while according to their [1] they should be using $\boldsymbol{R} = \boldsymbol{R}_2 - \boldsymbol{R}_1$. The sign of the v^6 terms in [1] and [2] of ref. 1 is in error along with a few other assorted misprints.

III. Removal of Quadratic Acceleration Terms

We now wish to add double-zero terms and total time derivative terms to $\mathscr{L}_{(4)aa}''$ in order to remove the quadratic acceleration terms. Let us put

[13] $$\mathscr{L}_{(4)BB}''' = \mathscr{L}_{(4)aa}'' + \frac{e_1e_2}{8c^4r}[3r^2(\boldsymbol{a}_1 - e_1e_2\boldsymbol{r}/m_1r^3)\cdot(\boldsymbol{a}_2 + e_1e_2\boldsymbol{r}/m_2r^3) - (\boldsymbol{a}_1\cdot\boldsymbol{r} - e_1e_2/m_1r) \times (\boldsymbol{a}_2\cdot\boldsymbol{r} + e_1e_2/m_2r)] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e_1{}^2e_2{}^2}{4c^4r^2}\left(\frac{\boldsymbol{v}_1\cdot\boldsymbol{r}}{m_2} - \frac{\boldsymbol{v}_2\cdot\boldsymbol{r}}{m_1}\right)\right]$$

which upon evaluation gives us

[14] $$\mathscr{L}_{(4)BB}''' = \frac{e_1{}^2e_2{}^2}{8c^4r^2m_2}[2(\boldsymbol{v}_1\cdot\boldsymbol{v}_2) - 2v_1{}^2 + 4(\boldsymbol{v}_1\cdot\boldsymbol{r})^2/r^2 - 4(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})/r^2] + \frac{e_1{}^2e_2{}^2}{8c^4r^2m_1} \times [2(\boldsymbol{v}_1\cdot\boldsymbol{v}_2) - 2v_2{}^2 + 4(\boldsymbol{v}_2\cdot\boldsymbol{r})^2/r^2 - 4(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})/r^2] - \frac{e_1{}^3e_2{}^3}{4c^4r^3m_1m_2}$$

Let us also put

[15] $$\mathscr{L}''' = \mathscr{L}_{\mathrm{D}}' + \mathscr{L}_{(4)}'''$$

where

[16] $$\mathscr{L}_{(4)}''' = \mathscr{L}_{(4)b}'' + \mathscr{L}_{(4)a}'' + \mathscr{L}_{(4)BB}'''$$

IV. Removal of Linear Acceleration Terms

In order to remove the linear acceleration terms we shall first change to coordinates $\boldsymbol{r}_{\mathrm{CM}}$ and $\boldsymbol{r}$ by the transformation

[17] $$\boldsymbol{r}_{\mathrm{CM}} = (m_1/M)\boldsymbol{r}_1 + (m_2/M)\boldsymbol{r}_2, \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$$

which has the inverse form of

[18] $$\boldsymbol{r}_1 = \boldsymbol{r}_{\mathrm{CM}} + (m_2/M)\boldsymbol{r}, \qquad \boldsymbol{r}_2 = \boldsymbol{r}_{\mathrm{CM}} - (m_1/M)\boldsymbol{r}$$

We are also using

[19] $$M = m_1 + m_2, \qquad \mu = m_1m_2/(m_1 + m_2)$$

for the total mass and reduced mass, respectively. After some lengthy manipulations $\mathscr{L}_{(4)a}''$ (see [9]) can be put into the form

[20] $$\mathscr{L}_{(4)a}'' = \frac{e_1e_2}{8c^4}\Big\{2[\boldsymbol{v}_{\mathrm{CM}} \times (\boldsymbol{r} \times \boldsymbol{v})]\cdot\boldsymbol{a}_{\mathrm{CM}}/r - 2(\mu/M)(\tfrac{1}{2}v^2 + e_1e_2/\mu r)(\boldsymbol{a}\cdot\boldsymbol{r})/r + 2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{a})/r + 4[(m_1 - m_2)/M](\tfrac{1}{2}v^2 + e_1e_2/\mu r)(\boldsymbol{a}_{\mathrm{CM}}\cdot\boldsymbol{r})/r - 2[(m_1 - m_2)/M][\boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{v}) + e_1e_2\boldsymbol{r}/\mu r]\cdot\boldsymbol{a}_{\mathrm{CM}}/r + 2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2(\boldsymbol{a}\cdot\boldsymbol{r})/r^3 - (\mu/M)[v^2r^2 - (\boldsymbol{v}\cdot\boldsymbol{r})^2](\boldsymbol{a}\cdot\boldsymbol{r})/r^3 - [(m_1 - m_2)/M][v^2r^2 - (\boldsymbol{v}\cdot\boldsymbol{r})^2] \times (\boldsymbol{a}_{\mathrm{CM}}\cdot\boldsymbol{r})/r^3 + v_{\mathrm{CM}}{}^2[v^2/r - (\boldsymbol{v}\cdot\boldsymbol{r})^2/r^3] - (\mu/M)[v^4/r - v^2(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^3] + 2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{r})/r^3 + (\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2[v^2/r^3 - 3(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^5] - 2(\mu/M)(e_1e_2/\mu r)[v^2/r - 2(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^3] + 2[(m_1 - m_2)/M] \times (e_1e_2/\mu r)[(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{v})/r - 2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}\cdot\boldsymbol{r})/r^3] + \frac{\mathrm{d}}{\mathrm{d}t}\Big[-v_{\mathrm{CM}}{}^2(\boldsymbol{v}\cdot\boldsymbol{r})/r + (\mu/M)v^2(\boldsymbol{v}\cdot\boldsymbol{r})/r - (\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2(\boldsymbol{v}\cdot\boldsymbol{r})/r^3 + 2(e_1e_2/\mu r)\{(\mu/M)(\boldsymbol{v}\cdot\boldsymbol{r})/r - [(m_1 - m_2)/M](\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})/r\}\Big]\Big\}$$

The lowest-order equations of motion can now be written as

[21] $$\boldsymbol{a}_{\mathrm{CM}} = 0$$

[22] $$\boldsymbol{a} - e_1e_2\boldsymbol{r}/\mu r^3 = 0$$

We also have (correct to lowest order) the constants of the motion:

[23] $\boldsymbol{v}_0 = \boldsymbol{v}_{\mathrm{CM}}$

[24] $\varepsilon_0 = \frac{1}{2}v^2 + e_1e_2/\mu r$

[25] $\boldsymbol{l}_0 = \boldsymbol{r} \times \boldsymbol{v}$

[26] $l_0{}^2 = r^2v^2 - (\boldsymbol{r}\cdot\boldsymbol{v})^2$

[27] $\boldsymbol{\alpha}_0 = \boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{v}) + e_1e_2\boldsymbol{r}/\mu r$

Note that ε_0, $\boldsymbol{l}_0$, and $\boldsymbol{\alpha}_0$ are related as

[28] $\alpha_0{}^2 = 2\varepsilon_0 l_0{}^2 + e_1{}^2e_2{}^2/\mu$

We now add double-zero terms and total time derivative terms to $\mathscr{L}_{(4)a}{}''$ in order to remove the linear acceleration terms. Let us put

[29]
$$\mathscr{L}_{(4)B}{}''' = \mathscr{L}_{(4)a}{}'' + \frac{e_1e_2}{8c^4}\Big\{-2[\boldsymbol{v}_{\mathrm{CM}} \times (\boldsymbol{r} \times \boldsymbol{v}) - \boldsymbol{v}_0 \times \boldsymbol{l}_0]\cdot\boldsymbol{a}_{\mathrm{CM}}/r + 2(\mu/M)[\tfrac{1}{2}v^2 + e_1e_2/\mu r - \varepsilon_0]$$
$$\times [(\boldsymbol{a} - e_1e_2\boldsymbol{r}/\mu r^3)\cdot\boldsymbol{r}]/r - 2[(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})\boldsymbol{v}_{\mathrm{CM}} - (\boldsymbol{v}_0\cdot\boldsymbol{r})\boldsymbol{v}_0]\cdot(\boldsymbol{a} - e_1e_2\boldsymbol{r}/\mu r^3)/r - 4[(m_1 - m_2)/M]$$
$$\times [\tfrac{1}{2}v^2 + e_1e_2/\mu r - \varepsilon_0](\boldsymbol{a}_{\mathrm{CM}}\cdot\boldsymbol{r})/r + 2[(m_1 - m_2)/M][\boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{v}) + e_1e_2\boldsymbol{r}/\mu r - \boldsymbol{\alpha}_0]\cdot\boldsymbol{a}_{\mathrm{CM}}/r$$
$$- 2[(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2 - (\boldsymbol{v}_0\cdot\boldsymbol{r})^2][(\boldsymbol{a} - e_1e_2\boldsymbol{r}/\mu r^3)\cdot\boldsymbol{r}]/r^3 + (\mu/M)[v^2r^2 - (\boldsymbol{v}\cdot\boldsymbol{r})^2 - l_0{}^2]$$
$$\times [(\boldsymbol{a} - e_1e_2\boldsymbol{r}/\mu r^3)\cdot\boldsymbol{r}]/r^3 + [(m_1 - m_2)/M][v^2r^2 - (\boldsymbol{v}\cdot\boldsymbol{r})^2 - l_0{}^2](\boldsymbol{a}_{\mathrm{CM}}\cdot\boldsymbol{r})/r^3$$
$$- \frac{\mathrm{d}}{\mathrm{d}t}\Big[-v_{\mathrm{CM}}{}^2(\boldsymbol{v}\cdot\boldsymbol{r})/r + (\mu/M)v^2(\boldsymbol{v}\cdot\boldsymbol{r})/r - (\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2(\boldsymbol{v}\cdot\boldsymbol{r})/r^3 + 2(e_1e_2/\mu r)((\mu/M)(\boldsymbol{v}\cdot\boldsymbol{r})/r$$
$$- [(m_1 - m_2)/M](\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})/r) + 2(\boldsymbol{v}_0 \times \boldsymbol{l}_0)\cdot\boldsymbol{v}_{\mathrm{CM}}/r - 2(\mu/M)\varepsilon_0(\boldsymbol{v}\cdot\boldsymbol{r})/r + 2(\boldsymbol{v}_0\cdot\boldsymbol{r})(\boldsymbol{v}_0\cdot\boldsymbol{v})/r$$
$$+ 4[(m_1 - m_2)/M]\varepsilon_0(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})/r - 2[(m_1 - m_2)/M](\boldsymbol{\alpha}_0\cdot\boldsymbol{v}_{\mathrm{CM}})/r + 2(\boldsymbol{v}_0\cdot\boldsymbol{r})^2(\boldsymbol{v}\cdot\boldsymbol{r})/r^3$$
$$- (\mu/M)l_0{}^2(\boldsymbol{v}\cdot\boldsymbol{r})/r^3 - [(m_1 - m_2)/M]l_0{}^2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})/r^3\Big]\Big\}$$

which upon evaluation gives us

[30]
$$\mathscr{L}_{(4)B}{}''' = \frac{e_1e_2}{8c^4r}\Big[v_{\mathrm{CM}}{}^2[v^2 - (\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2] + (\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2[v^2/r^2 - 3(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^4] + 2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{r})/r^2$$
$$- (\mu/M)[v^4 - v^2(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2] + 2[(\boldsymbol{v}_0 \times \boldsymbol{l}_0)\cdot\boldsymbol{v}_{\mathrm{CM}}](\boldsymbol{v}\cdot\boldsymbol{r})/r^2 - 2[(m_1 - m_2)/M](\boldsymbol{\alpha}_0\cdot\boldsymbol{v}_{\mathrm{CM}})(\boldsymbol{v}\cdot\boldsymbol{r})/r^2$$
$$- 2(\boldsymbol{v}_0\cdot\boldsymbol{v})^2 - 2(\boldsymbol{v}_0\cdot\boldsymbol{r})(\boldsymbol{v}_0\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{r})/r^2 - 2(\boldsymbol{v}_0\cdot\boldsymbol{r})^2[v^2/r^2 - 3(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^4] + 2(\mu/M)\varepsilon_0[v^2 - (\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2]$$
$$- 4[(m_1 - m_2)/M]\varepsilon_0[\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{v} - (\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}\cdot\boldsymbol{r})/r^2] + (\mu/M)l_0{}^2[v^2/r^2 - 3(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^4]$$
$$+ [(m_1 - m_2)/M]l_0{}^2[(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{v})/r^2 - 3(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}\cdot\boldsymbol{r})/r^4]\Big] + \frac{e_1{}^2e_2{}^2}{8c^4r^2\mu}\Big[4(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})^2/r^2 - (\mu/M)$$
$$\times [4v^2 - 5(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2] + 2[(m_1 - m_2)/M][\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{v} - 2(\boldsymbol{v}_{\mathrm{CM}}\cdot\boldsymbol{r})(\boldsymbol{v}\cdot\boldsymbol{r})/r^2] - 4(\boldsymbol{v}_0\cdot\boldsymbol{r})^2/r^2$$
$$+ (\mu/M)[2\varepsilon_0 + l_0{}^2/r^2]\Big] - \frac{e_1{}^3e_2{}^3}{4c^4r^3m_1m_2}$$

Let us also put

[31] $\mathscr{L}'''' = \mathscr{L}_{\mathrm{D}}{}' + \mathscr{L}_{(4)}{}''''$

where

[32] $\mathscr{L}_{(4)}{}'''' = \mathscr{L}_{(4)b}{}'' + \mathscr{L}_{(4)B}{}''' + \mathscr{L}_{(4)BB}{}'''$

The Lagrangian $\mathscr{L}''''$ should be regarded as a function of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ (or $\boldsymbol{r}_{CM}$, $\boldsymbol{r}$, $\boldsymbol{v}_{CM}$, $\boldsymbol{v}$) and in taking partial derivatives with respect to these variables the quantities $\boldsymbol{v}_0$, ε_0, $\boldsymbol{l}_0$, l_0^2, $\boldsymbol{\alpha}_0$ must be regarded as constants.

V. The Hamiltonian

Let us consider the two-body Lagrangian (to order c^{-4})

$$[33]\quad \mathscr{L} = -\sum_{i=1}^{2} m_i c^2 + \sum_{i=1}^{2} \tfrac{1}{2} m_i v_i^2 - V_{(0)}(\boldsymbol{r}) + \mathscr{L}_{(2)}(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2) + \mathscr{L}_{(4)}(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2)$$

where the subscript (n) means the term is of order c^{-n}. The canonical momentum is

$$[34]\quad \boldsymbol{P}_i = m_i \boldsymbol{v}_i + \boldsymbol{P}_{i(2)} + \boldsymbol{P}_{i(4)}$$

where

$$[35]\quad \boldsymbol{P}_{i(2)} = \partial \mathscr{L}_{(2)}/\partial \boldsymbol{v}_i, \qquad \boldsymbol{P}_{i(4)} = \partial \mathscr{L}_{(4)}/\partial \boldsymbol{v}_i$$

and the Hamiltonian is

$$[36]\quad \mathscr{H} = \sum_{i=1}^{2} \boldsymbol{P}_i \cdot \boldsymbol{v}_i - \mathscr{L}$$

It can then be easily shown that

$$[37]\quad \mathscr{H} = \sum_{i=1}^{2} m_i c^2 + \sum_{i=1}^{2} P_i^2/2m_i + V_{(0)} - \mathscr{L}_{(2)} - \sum_{i=1}^{2} P_{i(2)}^2/2m_i - \mathscr{L}_{(4)} - \sum_{i=1}^{2} (2\boldsymbol{P}_{i(2)} \cdot \boldsymbol{P}_{i(4)} + P_{i(4)}^2)/2m_i$$

and [37] is exact if [33] is considered to be exact. As we are working to order c^{-4} we can write $\mathscr{H}$ as

$$[38]\quad \mathscr{H} = \sum_{i=1}^{2} m_i c^2 + \sum_{i=1}^{2} P_i^2/2m_i + V_{(0)} - \mathscr{L}_{(2)} - \sum_{i=1}^{2} P_{i(2)}^2/2m_i - \mathscr{L}_{(4)}$$

In the $\mathscr{L}_{(2)}$ term we use $\boldsymbol{v}_i = \boldsymbol{P}_i/m_i +$ (term of order c^{-2}) which is obtained by inverting [34] to order c^{-2}. In the $P_{i(2)}^2$ term and in the $\mathscr{L}_{(4)}$ term we use $\boldsymbol{v}_i = \boldsymbol{P}_i/m_i$. This procedure for getting the Hamiltonian in terms of $\boldsymbol{P}_i$ is much easier than using [36] directly and inverting [34] to order c^{-4}.

Using the above procedure[2] we find the Hamiltonian corresponding to $\mathscr{L}''''$ of [31] is

$$[39]\quad \mathscr{H} = \mathscr{H}_D + \mathscr{H}_{(4)}$$

where

$$[40]\quad \mathscr{H}_D = m_1 c^2 + m_2 c^2 + P_1^2/2m_1 + P_2^2/2m_2 - P_1^4/8m_1^3c^2 - P_2^4/8m_2^3c^2 + \frac{e_1 e_2}{r} \times \left[1 - \frac{\boldsymbol{P}_1 \cdot \boldsymbol{P}_2}{2m_1 m_2 c^2} - \frac{(\boldsymbol{P}_1 \cdot \boldsymbol{r})(\boldsymbol{P}_2 \cdot \boldsymbol{r})}{2m_1 m_2 c^2 r^2}\right]$$

$$\begin{aligned}[41]\quad \mathscr{H}_{(4)} = {} & \frac{P_1^6}{16m_1^5c^4} + \frac{P_2^6}{16m_2^5c^4} + \frac{e_1 e_2}{8c^4 r m_1^2 m_2^2}\Big[2(m_2/m_1)[P_1^2(\boldsymbol{P}_1 \cdot \boldsymbol{P}_2) + P_1^2(\boldsymbol{P}_1 \cdot \boldsymbol{r})(\boldsymbol{P}_2 \cdot \boldsymbol{r})/r^2] \\
& + 2(m_1/m_2)[P_2^2(\boldsymbol{P}_1 \cdot \boldsymbol{P}_2) + P_2^2(\boldsymbol{P}_1 \cdot \boldsymbol{r})(\boldsymbol{P}_2 \cdot \boldsymbol{r})/r^2] - 2(\boldsymbol{P}_1 \cdot \boldsymbol{P}_2)^2 + P_1^2 P_2^2 - (\boldsymbol{P}_1 \cdot \boldsymbol{r})^2 P_2^2/r^2 \\
& - (\boldsymbol{P}_2 \cdot \boldsymbol{r})^2 P_1^2/r^2 + 3(\boldsymbol{P}_1 \cdot \boldsymbol{r})^2(\boldsymbol{P}_2 \cdot \boldsymbol{r})^2/r^4 - P_{CM}^2 P^2 + P_{CM}^2(\boldsymbol{P} \cdot \boldsymbol{r})^2/r^2 - (\boldsymbol{P}_{CM} \cdot \boldsymbol{r})^2 P^2/r^2 \\
& + 3(\boldsymbol{P}_{CM} \cdot \boldsymbol{r})^2(\boldsymbol{P} \cdot \boldsymbol{r})^2/r^4 - 2(\boldsymbol{P}_{CM} \cdot \boldsymbol{r})(\boldsymbol{P}_{CM} \cdot \boldsymbol{P})(\boldsymbol{P} \cdot \boldsymbol{r})/r^2 + (M/\mu)[P^4 - P^2(\boldsymbol{P} \cdot \boldsymbol{r})^2/r^2] \\
& - 2m_1 m_2[(\boldsymbol{v}_0 \times \boldsymbol{l}_0) \cdot \boldsymbol{P}_{CM}](\boldsymbol{P} \cdot \boldsymbol{r})/r^2 + 2\mu(m_1 - m_2)(\boldsymbol{\alpha}_0 \cdot \boldsymbol{P}_{CM})(\boldsymbol{P} \cdot \boldsymbol{r})/r^2 + 2M^2[(\boldsymbol{v}_0 \cdot \boldsymbol{P})^2 \\
& + (\boldsymbol{v}_0 \cdot \boldsymbol{r})(\boldsymbol{v}_0 \cdot \boldsymbol{P})(\boldsymbol{P} \cdot \boldsymbol{r})/r^2 + (\boldsymbol{v}_0 \cdot \boldsymbol{r})^2 P^2/r^2 - 3(\boldsymbol{v}_0 \cdot \boldsymbol{r})^2(\boldsymbol{P} \cdot \boldsymbol{r})^2/r^4] - 2m_1 m_2 \varepsilon_0[P^2 - (\boldsymbol{P} \cdot \boldsymbol{r})^2/r^2] \\
& + 4\mu(m_1 - m_2)\varepsilon_0[\boldsymbol{P}_{CM} \cdot \boldsymbol{P} - (\boldsymbol{P}_{CM} \cdot \boldsymbol{r})(\boldsymbol{P} \cdot \boldsymbol{r})/r^2] - m_1 m_2 l_0^2[P^2/r^2 - 3(\boldsymbol{P} \cdot \boldsymbol{r})^2/r^4]
\end{aligned}$$

[2]In the $\mathscr{L}_{(4)}''''$ term we also use $\boldsymbol{v}_{CM} = \boldsymbol{P}_{CM}/M$ and $\boldsymbol{v} = \boldsymbol{P}/\mu$.

$$- \mu(m_1 - m_2)l_0^2[(\boldsymbol{P}_{CM}\cdot\boldsymbol{P})/r^2 - 3(\boldsymbol{P}_{CM}\cdot\boldsymbol{r})(\boldsymbol{P}\cdot\boldsymbol{r})/r^4]\Big] + \frac{e_1^2e_2^2}{8c^4r^2m_1^2m_2^2}\Big[3m_2P_1^2 - 2m_1(\boldsymbol{P}_1\cdot\boldsymbol{P}_2)$$

$$- m_2(\boldsymbol{P}_1\cdot\boldsymbol{r})^2/r^2 + 4m_1(\boldsymbol{P}_1\cdot\boldsymbol{r})(\boldsymbol{P}_2\cdot\boldsymbol{r})/r^2 + 3m_1P_2^2 - 2m_2(\boldsymbol{P}_1\cdot\boldsymbol{P}_2) - m_1(\boldsymbol{P}_2\cdot\boldsymbol{r})^2/r^2$$

$$+ 4m_2(\boldsymbol{P}_1\cdot\boldsymbol{r})(\boldsymbol{P}_2\cdot\boldsymbol{r})/r^2 - 4\mu(\boldsymbol{P}_{CM}\cdot\boldsymbol{r})^2/r^2 + 4MP^2 - 5M(\boldsymbol{P}\cdot\boldsymbol{r})^2/r^2 - 2(m_1 - m_2)(\boldsymbol{P}_{CM}\cdot\boldsymbol{P})$$

$$+ 4(m_1 - m_2)(\boldsymbol{P}_{CM}\cdot\boldsymbol{r})(\boldsymbol{P}\cdot\boldsymbol{r})/r^2 + 4M^2\mu(\boldsymbol{v}_0\cdot\boldsymbol{r})^2/r^2 - 2M\mu^2\varepsilon_0 - M\mu^2l_0^2/r^2\Big] + \frac{e_1^3e_2^3}{2c^4r^3m_1m_2}$$

The Hamiltonian of [39] should be regarded as a function of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ (or $\boldsymbol{r}_{CM}$, $\boldsymbol{r}$, $\boldsymbol{P}_{CM}$, $\boldsymbol{P}$) and in taking partial derivatives with respect to these variables the quantities $\boldsymbol{v}_0$, ε_0, $\boldsymbol{l}_0$, l_0^2, $\boldsymbol{\alpha}_0$ must be regarded as constants. We also have

$$[42] \quad \boldsymbol{P}_{CM} = \boldsymbol{P}_1 + \boldsymbol{P}_2, \qquad \boldsymbol{P} = (m_2/M)\boldsymbol{P}_1 - (m_1/M)\boldsymbol{P}_2$$

which has the inverse form of

$$[43] \quad \boldsymbol{P}_1 = (m_1/M)\boldsymbol{P}_{CM} + \boldsymbol{P}, \qquad \boldsymbol{P}_2 = (m_2/M)\boldsymbol{P}_{CM} - \boldsymbol{P}$$

VI. Hamiltonian in Center-of-mass Coordinates

We shall now set $\boldsymbol{P}_{CM} = 0$ in [39]. We also must set $\boldsymbol{v}_0 = 0$ and let $\boldsymbol{P}_1 \to \boldsymbol{P}$ and $\boldsymbol{P}_2 \to -\boldsymbol{P}$. The Hamiltonian in center-of-mass coordinates is thus

$$[44] \quad \mathscr{H} = \mathscr{H}_D + \mathscr{H}_{(4)}$$

where

$$[45] \quad \mathscr{H}_D = Mc^2 + \frac{P^2}{2\mu} - \left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right)\frac{P^4}{8c^4} + \frac{e_1e_2}{r}\left[1 + \frac{P^2}{2m_1m_2c^2} + \frac{(\boldsymbol{P}\cdot\boldsymbol{r})^2}{2m_1m_2c^2r^2}\right]$$

$$[46] \quad \mathscr{H}_{(4)} = \left(\frac{1}{m_1^5} + \frac{1}{m_2^5}\right)\frac{P^6}{16c^4} + \frac{e_1e_2}{8c^4rm_1^2m_2^2}[(3 - M/\mu)P^4 + (2 - 3M/\mu)P^2(\boldsymbol{P}\cdot\boldsymbol{r})^2/r^2 + 3(\boldsymbol{P}\cdot\boldsymbol{r})^4/r^4$$

$$- m_1m_2(2\varepsilon_0 + l_0^2/r^2)P^2 + m_1m_2(2\varepsilon_0 + 3l_0^2/r^2)(\boldsymbol{P}\cdot\boldsymbol{r})^2/r^2] + \frac{e_1^2e_2^2}{8c^4r^2m_1^2m_2^2}$$

$$\times [9MP^2 - 10M(\boldsymbol{P}\cdot\boldsymbol{r})^2/r^2 - M\mu^2(2\varepsilon_0 + l_0^2/r^2)] + \frac{e_1^3e_2^3}{2c^4r^3m_1m_2}$$

VII. Lagrangian in Center-of-mass Coordinates

Let us consider the Hamiltonian (to order c^{-4})

$$[47] \quad \mathscr{H} = Mc^2 + P^2/2\mu + V_{(0)}(\boldsymbol{r}) + \mathscr{H}_{(2)}(\boldsymbol{r}, \boldsymbol{P}) + \mathscr{H}_{(4)}(\boldsymbol{r}, \boldsymbol{P})$$

The velocity is

$$[48] \quad \boldsymbol{v} = \boldsymbol{P}/\mu + \boldsymbol{v}_{(2)} + \boldsymbol{v}_{(4)}$$

where

$$[49] \quad \boldsymbol{v}_{(2)} = \partial\mathscr{H}_{(2)}/\partial\boldsymbol{P}, \qquad \boldsymbol{v}_{(4)} = \partial\mathscr{H}_{(4)}/\partial\boldsymbol{P}$$

and the Lagrangian is

$$[50] \quad \mathscr{L} = \boldsymbol{P}\cdot\boldsymbol{v} - \mathscr{H}$$

It can easily be shown that

$$[51] \quad \mathscr{L} = -Mc^2 + \tfrac{1}{2}\mu v^2 - V_{(0)} - \mathscr{H}_{(2)} - \tfrac{1}{2}\mu v_{(2)}^2 - \mathscr{H}_{(4)} - \tfrac{1}{2}\mu(2\boldsymbol{v}_{(2)}\cdot\boldsymbol{v}_{(4)} + v_{(4)}^2)$$

and [51] is exact if [47] is considered to be exact. As we are working to order c^{-4} we can write $\mathscr{L}$ as

$$[52] \quad \mathscr{L} = -Mc^2 + \tfrac{1}{2}\mu v^2 - V_{(0)} - \mathscr{H}_{(2)} - \tfrac{1}{2}\mu v_{(2)}^2 - \mathscr{H}_{(4)}$$

In the $\mathscr{H}_{(2)}$ term we use $\boldsymbol{P} = \mu\boldsymbol{v}$ + (term of order c^{-2}) which is obtained by inverting [48] to order c^{-2}. In the $v_{(2)}{}^2$ term and in the $\mathscr{H}_{(4)}$ term we use $\boldsymbol{P} = \mu\boldsymbol{v}$.

Using the above procedure we find the Lagrangian in center-of-mass coordinates corresponding to the Hamiltonian of [44] in center-of-mass coordinates is

[53] $\mathscr{L} = \mathscr{L}_{\mathrm{D}} + \mathscr{L}_{(4)}$

where

[54] $$\mathscr{L}_{\mathrm{D}} = -Mc^2 + \tfrac{1}{2}\mu v^2 + \tfrac{1}{8}(1 - 3\mu/M)\mu v^4/c^2 - \frac{e_1e_2}{r}\left[1 + \frac{\mu}{2M}\left(\frac{v^2}{c^2} + \frac{(\boldsymbol{v}\cdot\boldsymbol{r})^2}{c^2r^2}\right)\right]$$

[55] $$\mathscr{L}_{(4)} = \tfrac{1}{16}(1 - 7\mu/M + 13\mu^2/M^2)\mu v^6/c^4 + \frac{e_1e_2}{8rc^4}\left(\frac{\mu}{M}\right)[(-3 + 9\mu/M)v^4 + (-1 + 10\mu/M)v^2(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2$$
$$- 3(\mu/M)(\boldsymbol{v}\cdot\boldsymbol{r})^4/r^4 + (2\varepsilon_0 + l_0{}^2/r^2)v^2 - (2\varepsilon_0 + 3l_0{}^2/r^2)(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2] + \frac{e_1{}^2e_2{}^2}{8r^2c^4}\left(\frac{1}{M}\right)$$
$$\times [(-9 + 4\mu/M)v^2 + (10 + 12\mu/M)(\boldsymbol{v}\cdot\boldsymbol{r})^2/r^2 + (2\varepsilon_0 + l_0{}^2/r^2)] - \frac{e_1{}^3e_2{}^3}{2r^3c^4m_1m_2}$$

VIII. Ostrogradsky Hamiltonian

For an acceleration-dependent two-body Lagrangian such as $\mathscr{L}''$ of [5] or $\mathscr{L}'''$ of [15] the conserved energy and momentum are given, respectively, by (3)

[56] $\mathscr{E} = \boldsymbol{\Pi}_1\cdot\boldsymbol{v}_1 + \boldsymbol{\Pi}_2\cdot\boldsymbol{v}_2 + \boldsymbol{\Phi}_1\cdot\boldsymbol{a}_1 + \boldsymbol{\Phi}_2\cdot\boldsymbol{a}_2 - \mathscr{L}$

[57] $\boldsymbol{\Pi}_{\mathrm{CM}} = \boldsymbol{\Pi}_1 + \boldsymbol{\Pi}_2$

where

[58] $\boldsymbol{\Pi}_i = \boldsymbol{P}_i - \dot{\boldsymbol{\Phi}}_i$

and

[59] $\boldsymbol{P}_i = \partial\mathscr{L}/\partial\boldsymbol{v}_i$

[60] $\boldsymbol{\Phi}_i = \partial\mathscr{L}/\partial\boldsymbol{a}_i$

The Ostrogradsky Hamiltonian (4–7)

[61] $\mathscr{H}_{\mathrm{OS}} = \mathscr{H}_{\mathrm{OS}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\Pi}_1, \boldsymbol{\Pi}_2; \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2)$

is obtained by eliminating $\boldsymbol{a}_i$ from [56] by means of inverting [60] to obtain $\boldsymbol{a}_i$ as a function of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$.

For the Lagrangian $\mathscr{L}''$ of [5] we obtain

[62] $\boldsymbol{\Phi}_1 = (e_1e_2/8rc^4)[-3r^2\boldsymbol{a}_2 + (\boldsymbol{a}_2\cdot\boldsymbol{r})\boldsymbol{r} + 2(\boldsymbol{v}_2\cdot\boldsymbol{r})\boldsymbol{v}_2 - v_2{}^2\boldsymbol{r} + (\boldsymbol{v}_2\cdot\boldsymbol{r})^2\boldsymbol{r}/r^2]$

[63] $\boldsymbol{\Phi}_2 = (e_1e_2/8rc^4)[-3r^2\boldsymbol{a}_1 + (\boldsymbol{a}_1\cdot\boldsymbol{r})\boldsymbol{r} - 2(\boldsymbol{v}_1\cdot\boldsymbol{r})\boldsymbol{v}_1 + v_1{}^2\boldsymbol{r} - (\boldsymbol{v}_1\cdot\boldsymbol{r})^2\boldsymbol{r}/r^2]$

which can be inverted to give us

[64] $\boldsymbol{a}_2 = (8rc^4/e_1e_2)[-\tfrac{1}{3}\boldsymbol{\Phi}_1/r^2 - \tfrac{1}{6}(\boldsymbol{\Phi}_1\cdot\boldsymbol{r})\boldsymbol{r}/r^4] + \tfrac{2}{3}(\boldsymbol{v}_2\cdot\boldsymbol{r})\boldsymbol{v}_2/r^2 - \tfrac{1}{2}v_2{}^2\boldsymbol{r}/r^2 - \tfrac{5}{6}(\boldsymbol{v}_2\cdot\boldsymbol{r})^2\boldsymbol{r}/r^4$

[65] $\boldsymbol{a}_1 = (8rc^4/e_1e_2)[-\tfrac{1}{3}\boldsymbol{\Phi}_2/r^2 - \tfrac{1}{6}(\boldsymbol{\Phi}_2\cdot\boldsymbol{r})\boldsymbol{r}/r^4] - \tfrac{2}{3}(\boldsymbol{v}_1\cdot\boldsymbol{r})\boldsymbol{v}_1/r^2 + \tfrac{1}{2}v_1{}^2\boldsymbol{r}/r^2 - \tfrac{5}{6}(\boldsymbol{v}_1\cdot\boldsymbol{r})^2\boldsymbol{r}/r^4$

Using [62] and [63] in [56] for $\mathscr{L}''$ we obtain

[66] $$\mathscr{E} = \boldsymbol{\Pi}_1\cdot\boldsymbol{v}_1 + \boldsymbol{\Pi}_2\cdot\boldsymbol{v}_2 - \mathscr{L}_{\mathrm{D}}' - \mathscr{L}_{(4)b}'' + \frac{e_1e_2}{8rc^4}[-3r^2(\boldsymbol{a}_1\cdot\boldsymbol{a}_2) + (\boldsymbol{a}_1\cdot\boldsymbol{r})(\boldsymbol{a}_2\cdot\boldsymbol{r})]$$

Using [64] and [65] in [66] we obtain the Ostrogradsky Hamiltonian

$$[67]\quad \mathscr{H}_{\mathrm{OS}} = \mathbf{\Pi}_1\cdot\boldsymbol{v}_1 + \mathbf{\Pi}_2\cdot\boldsymbol{v}_2 + m_1c^2 + m_2c^2 - \tfrac{1}{2}m_1v_1{}^2 - \tfrac{1}{2}m_2v_2{}^2 - \tfrac{1}{8}m_1v_1{}^4/c^2 - \tfrac{1}{8}m_2v_2{}^4/c^2 - \tfrac{1}{16}m_1v_1{}^6/c^4 - \tfrac{1}{16}m_2v_2{}^6/c^4 + \frac{e_1e_2}{r}\left[1 - \frac{\boldsymbol{v}_1\cdot\boldsymbol{v}_2}{2c^2} - \frac{(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})}{2c^2r^2}\right] + \left(\frac{e_1e_2}{8rc^4}\right)[-2(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)^2 + \tfrac{3}{2}v_1{}^2v_2{}^2 - \tfrac{5}{2}v_1{}^2(\boldsymbol{v}_2\cdot\boldsymbol{r})^2/r^2 - \tfrac{5}{2}v_2{}^2(\boldsymbol{v}_1\cdot\boldsymbol{r})^2/r^2 + \tfrac{4}{3}(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)/r^2 + \tfrac{37}{6}(\boldsymbol{v}_1\cdot\boldsymbol{r})^2(\boldsymbol{v}_2\cdot\boldsymbol{r})^2/r^4] - \tfrac{2}{3}(\boldsymbol{v}_1\cdot\boldsymbol{r})(\mathbf{\Phi}_1\cdot\boldsymbol{v}_1)/r^2 + \tfrac{1}{2}v_1{}^2(\mathbf{\Phi}_1\cdot\boldsymbol{r})/r^2 - \tfrac{5}{6}(\boldsymbol{v}_1\cdot\boldsymbol{r})^2(\mathbf{\Phi}_1\cdot\boldsymbol{r})/r^4 + \tfrac{2}{3}(\boldsymbol{v}_2\cdot\boldsymbol{r})(\mathbf{\Phi}_2\cdot\boldsymbol{v}_2)/r^2 - \tfrac{1}{2}v_2{}^2(\mathbf{\Phi}_2\cdot\boldsymbol{r})/r^2 + \tfrac{5}{6}(\boldsymbol{v}_2\cdot\boldsymbol{r})^2(\mathbf{\Phi}_2\cdot\boldsymbol{r})/r^4 + \left(\frac{8rc^4}{e_1e_2}\right)[-\tfrac{1}{3}(\mathbf{\Phi}_1\cdot\mathbf{\Phi}_2)/r^2 - \tfrac{1}{6}(\mathbf{\Phi}_1\cdot\boldsymbol{r})(\mathbf{\Phi}_2\cdot\boldsymbol{r})/r^4]$$

The Ostrogradsky equations of motion are (4, 5)

$$[68]\quad \dot{\boldsymbol{r}}_i = \partial\mathscr{H}_{\mathrm{OS}}/\partial\mathbf{\Pi}_i = \boldsymbol{v}_i, \qquad \dot{\mathbf{\Pi}}_i = -\partial\mathscr{H}_{\mathrm{OS}}/\partial\boldsymbol{r}_i$$

$$[69]\quad \dot{\boldsymbol{v}}_i = \partial\mathscr{H}_{\mathrm{OS}}/\partial\mathbf{\Phi}_i, \qquad \dot{\mathbf{\Phi}}_i = -\mathscr{H}_{\mathrm{OS}}/\partial\boldsymbol{v}_i$$

and it should be noted that the first one of [68] is quite trivial. It should also be noted that giving the initial values of $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, $\mathbf{\Phi}_i$, $\mathbf{\Pi}_i$ ($i = 1, 2$) for the Ostrogradsky equations of motion is equivalent to giving the initial values of $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, $\boldsymbol{a}_i$, $\dot{\boldsymbol{a}}_i$ ($i = 1, 2$) for Lagrange's equations of motion.

Let us next consider the Lagrangian $\mathscr{L}'''$ of [15]. Since this Lagrangian contains only linear acceleration terms, $\mathbf{\Phi}_i$ will *not* depend on $\boldsymbol{a}_1$ or $\boldsymbol{a}_2$ and [60] *cannot* be inverted. *If an Ostrogradsky Hamiltonian is to exist one must be able to obtain* $\boldsymbol{a}_i$ *as a function of* $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$. We, thus, conclude that there is no Ostrogradsky Hamiltonian corresponding to the Lagrangian $\mathscr{L}'''$. There is, however, nothing wrong with the conserved energy and momentum of [56] and [57], respectively, corresponding to $\mathscr{L}'''$.

IX. Conclusion

Using the method of the double zero we have removed both the quadratic and the linear acceleration terms from our starting Lagrangian $\mathscr{L}''$ of [5] to obtain our final Lagrangian $\mathscr{L}''''$ of [31]. Lagrange's equations for $\mathscr{L}''$ can be written as

$$[70]\quad m_i\boldsymbol{a}_i = f_i(\boldsymbol{r}_l, \boldsymbol{v}_l, \boldsymbol{a}_l, \dot{\boldsymbol{a}}_l, \ddot{\boldsymbol{a}}_l; \qquad l = 1, 2)$$

which means that in general the initial values of $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, $\boldsymbol{a}_i$, $\boldsymbol{a}_i$ ($i = 1, 2$) must be specified. However, to obtain the physically acceptable solutions what is done is to use [70] in itself to eliminate the higher-order acceleration terms and their time derivatives. We then get

$$[71]\quad m_i\boldsymbol{a}_i = g_i(\boldsymbol{r}_l, \boldsymbol{v}_l; \qquad l = 1, 2)$$

and only the initial values of $\boldsymbol{r}_i$, $\boldsymbol{v}_i$ ($i = 1, 2$) need be specified. Lagrange's equations for $\mathscr{L}''''$ give us

$$[72]\quad m_i\boldsymbol{a}_i = h_i(\boldsymbol{r}_l, \boldsymbol{v}_l, \boldsymbol{a}_l; \qquad l = 1, 2)$$

and when [72] is used in itself to eliminate the higher-order acceleration terms, [71] is obtained.

From our Hamiltonian of [39] corresponding to $\mathscr{L}''''$ we have obtained both the Hamiltonian and Lagrangian in center-of-mass coordinates. The Ostrogradsky Hamiltonian corresponding to $\mathscr{L}''$ has also been given and the Ostrogradsky Hamiltonian corresponding to $\mathscr{L}'''$ has been shown *not* to exist.

The Ostrogradsky Hamiltonian can also be written as[3]

$$[73]\quad \mathscr{H}_{\mathrm{OS}} = \mathscr{H}_{\mathrm{OS}}(\boldsymbol{r}_{\mathrm{CM}}, \boldsymbol{r}, \mathbf{\Pi}_{\mathrm{CM}}, \mathbf{\Pi}; \boldsymbol{v}_{\mathrm{CM}}, \boldsymbol{v}, \mathbf{\Phi}_{\mathrm{CM}}, \mathbf{\Phi})$$

Since the coordinate $\boldsymbol{r}_{\mathrm{CM}}$ is cyclic, $\mathbf{\Pi}_{\mathrm{CM}}$ is conserved and may be set equal to zero (to all orders). The center-of-mass Ostrogradsky Hamiltonian is, thus,

$$[74]\quad \mathscr{H}_{\mathrm{OS}} = \mathscr{H}_{\mathrm{OS}}(\boldsymbol{r}, \mathbf{\Pi}; \boldsymbol{v}_{\mathrm{CM}}, \boldsymbol{v}, \mathbf{\Phi}_{\mathrm{CM}}, \mathbf{\Phi})$$

where $\mathbf{\Pi}_{\mathrm{CM}}$ has been set equal to zero. However, since all CM variables have not been eliminated, [74] is only slightly simpler than [73]. On the other hand, the double-zero method allows one to obtain a center-of-mass Hamiltonian (and Lagrangian) where all CM variables have been eliminated.

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[3]Note that $\boldsymbol{P}_{\mathrm{CM}} = \partial\mathscr{L}/\partial\boldsymbol{v}_{\mathrm{CM}}$, $\boldsymbol{P} = \partial\mathscr{L}/\partial\boldsymbol{v}$, $\mathbf{\Phi}_{\mathrm{CM}} = \partial\mathscr{L}/\partial\boldsymbol{a}_{\mathrm{CM}}$, $\mathbf{\Phi} = \partial\mathscr{L}/\partial\boldsymbol{a}$ and $\mathbf{\Pi}_{\mathrm{CM}} = \boldsymbol{P}_{\mathrm{CM}} - \dot{\mathbf{\Phi}}_{\mathrm{CM}}$, $\mathbf{\Pi} = \boldsymbol{P} - \dot{\mathbf{\Phi}}$. Also equations similar to [42] and [43] hold for the $\mathbf{\Pi}$'s and the $\mathbf{\Phi}$'s. The Ostrogradsky equations of motion for the new variables are similar to [68] and [69].