ACCELERATION-DEPENDENT LAGRANGIANS AND EQUATIONS OF MOTION

B.M. BARKER
Department of Physics and Astronomy, The University of Alabama, University, AL 35486, USA

and

R.F. O'CONNELL
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA

Received 17 May 1980

Acceleration terms in a lagrangian are sometimes eliminated by substituting into the lagrangian the equations of motion which were obtained from the lagrangian. We show that, in general this is an incorrect procedure. In addition, we present a new correct procedure, which we call the *method of the double zero.*

In various branches of physics, one sometimes encounters acceleration-dependent lagrangians. For example, the two-body lagrangian in classical electrodynamics (with $e_1/m_1 = e_2/m_2$ to postpone dipole radiation from the c^{-3} to the c^{-5} order), contains acceleration-dependent terms [1,2], if one works to order c^{-4} (i.e. order c^{-2} beyond the familiar Darwin lagrangian [3]). We have recently given a new and detailed derivation of this result [4] which agrees with the result presented by Landau and Lifshitz [2] and which has corrected some misprints in the original derivation [1].

Next, the authors of refs. [1,2] next used the lowest-order equations of motion in the lagrangian to eliminate the acceleration dependent terms, which are of order c^{-4}. It is our purpose here to show that, in general, this is an incorrect procedure since using the equations of motion in the lagrangian changes its functional form and, hence, leads to different and, thus, incorrect equations of motion.

We shall, first of all, prove our contention in a general way andt hen verify its correctness in the case of a particular example.

It can easily by seen that if one uses the lowest-order equations of motion in a lagrangian $\mathcal{L}$ with acceleration terms of order c^{-4}, to obtain the lagrangian $\mathcal{L}^*$ without acceleration terms, then

$$\mathcal{L}^* \equiv \mathcal{L} + Z, \tag{1}$$

and Z is of order c^{-4}. Also $Z = 0$ upon use of the lowest-order equations of motion. However, upon use of the lowest-order equations of motion $\partial Z/\partial \boldsymbol{r}_i \neq 0$ and $\partial Z/\partial \boldsymbol{a}_i \neq 0$ [though $\partial Z/\partial \boldsymbol{v}_i = 0$ since the lowest-order equations of motion do not contain $\boldsymbol{v}_i$]. The equations of motion for the lagrangian $\mathcal{L}^*$ are

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \mathcal{L}}{\partial \boldsymbol{a}_i}\right) + \left(\frac{\partial Z}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial Z}{\partial \boldsymbol{v}_i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial Z}{\partial \boldsymbol{a}_i}\right) = 0, \tag{2}$$

which, in general, are clearly *not* the same (to order c^{-4}) as those for the lagrangian $\mathcal{L}$. To illustrate our point, let us choose the following very simple one-body lagrangian

$$\mathcal{L} = \tfrac{1}{2}\mu v^2 - e_1 e_2/r + (e_1 e_2/c^2)\boldsymbol{a}\cdot\boldsymbol{n}_0, \tag{3}$$

where $\boldsymbol{n}_0$ is an arbitrary constant unit vector, and μ is the reduced mass. The last term in eq. (3), which is the higher order term, was chosen purely for mathematical simplicity, rather than physical reality. Using eq. (3) in the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}}=\frac{\partial \mathcal{L}}{\partial \boldsymbol{r}}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \mathcal{L}}{\partial \boldsymbol{a}} \tag{4}$$

gives us

$$\mu \boldsymbol{a}=e_1e_2\boldsymbol{r}/r^3. \tag{5}$$

It should be noted that the higher-order term in eq. (3) did not contribute to eq. (5). The last term in $\mathcal{L}$ can be written as $\mathrm{d}[(e_1e_2/c^2)\boldsymbol{v}\cdot\boldsymbol{n}_0]/\mathrm{d}t$, a total time derivative, and, thus, can be dropped to give us the equivalent lagrangian

$$\mathcal{L}'=\tfrac{1}{2}\mu v^2-e_1e_2/r, \tag{6}$$

which gives us the same equations of motion, eq. (5).

We shall now use the equations of motion, eq. (5), in the higher order term of eq. (3) to eliminate the acceleration term. The result is

$$\mathcal{L}^*=\tfrac{1}{2}\mu v^2-\frac{e_1e_2}{r}+\frac{e_1^2e_2^2}{\mu c^2}\frac{\boldsymbol{r}\cdot\boldsymbol{n}_0}{r^3}, \tag{7}$$

which gives us the equations of motion

$$\mu\boldsymbol{a}=\frac{e_1e_2\boldsymbol{r}}{r^3}+\frac{e_1^2e_2^2}{\mu c^2}\left(\frac{\boldsymbol{n}_0}{r^3}-\frac{3(\boldsymbol{r}\cdot\boldsymbol{n}_0)\boldsymbol{r}}{r^5}\right), \tag{8}$$

which is clearly *not* in agreement with eq. (5). Thus we see that the lagrangian $\mathcal{L}^*$ is *not* equivalent to the lagrangian $\mathcal{L}$.

It is the *functional* form of $\mathcal{L}$ which is crucial in leading to the correct equations of motion. Substitution into $\mathcal{L}$ changes its functional form, and thus, upon variation, changes the equations of motion. *We conclude that it is not correct to use the equations of motion in the lagrangian.*

Next, we turn to a new correct procedure for eliminating acceleration terms, which we call *the method of the double zero.* Consider the lagrangian $\bar{\mathcal{L}}$ defined as

$$\bar{\mathcal{L}}\equiv\mathcal{L}+Z_1Z_2, \tag{9}$$

where Z_1Z_2 is of order c^{-4} and both $Z_1=0$ and $Z_2=0$ upon use of the lowest-order equations of motion. The equations of motion for the lagrangian $\bar{\mathcal{L}}$ are

$$\begin{aligned}&\left[\frac{\partial\mathcal{L}}{\partial r_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial v_i}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\mathcal{L}}{\partial a_i}\right]\\&+\left[Z_1\frac{\partial Z_2}{\partial r_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(Z_1\frac{\partial Z_2}{\partial v_i}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(Z_1\frac{\partial Z_2}{\partial a_i}\right)\right]\\&+\left[Z_2\frac{\partial Z_1}{\partial r_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(Z_2\frac{\partial Z_1}{\partial v_i}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(Z_2\frac{\partial Z_1}{\partial a_i}\right)\right]=0,\end{aligned} \tag{10}$$

which *are* clearly the same (to order c^{-4}) as those for the lagrangian $\mathcal{L}$. We conclude that adding a *double-zero* term to a lagrangian does not change the equations of motion (to the order under consideration).

The question of what form to choose for Z_1 and Z_2 will depend on the nature of the specific acceleration-dependent term in the lagrangian which is under consideration. After adding the double zero terms to the lagrangian a total time derivative must also be added in order to completely eliminate the acceleration terms. In particular, we have successfully used this technique [4] to eliminate the acceleration-depen dent terms in the electromagnetic lagrangian, referred to above.

References

[1] V.N. Golubenkov and Ia. A. Smorodinskii, Sov. Phys. JETP 4 (1957) 55.
[2] L.D. Landau and E.M. Lifshitz, The classical theory of fields, 4th revised English Ed. (Pergamon, New York, 1975) see Sec. 75, pp. 209–210.
[3] C.G. Darwin, Philos. Mag. 39 (1920) 537.
[4] B.M. Barker and R.F. O'Connell, to be published.