Energy flow vector of the electromagnetic field

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Poynting's choice for the energy flow vector of the electromagnetic field has certain unattractive physical features. In order to eliminate such features Hines proposed an alternative choice. Here we show that Hines's choice does *not* lead to Larmor's result for the rate of radiation by an accelerated non-relativistic charge.

Le choix du vecteur de Poynting pour représenter le courant d'énergie dans un champ électromagnétique a certaines conséquences indésirables du point de vue physique. Pour remédier à ces inconvénients, Hines a proposé un autre choix. Nous montrons ici que ce choix de Hines ne conduit *pas* à la formule de Larmor pour le rayonnement d'une charge accélérée non relativiste.

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The conservation of energy and momentum for electromagnetic fields (E and B) and particles may be written as (1)

[1] $\quad \partial W_P/\partial t + \nabla \cdot S_P = -E \cdot J$

where J is the current density,

[2] $\quad W_P = (1/8\pi)(E \cdot E + B \cdot B)$

is the energy density, and (taking $c = 1$)

[3] $\quad S_P = (1/4\pi)(E \times B)$

represents the energy flow. The vector S_P is called the Poynting vector and, since only its divergence appears in the conservation law, it is arbitrary to the extent that the curl of any vector field may be added to it.

However, Poynting's vector has some undesirable counter intuitive properties in the following situations (1): (*a*) when a constant current flows through a wire, (*b*) when non-parallel static electric and magnetic fields exist (as when a charge is near a bar magnet), and (*c*) for a slowly charging capacitor. Intuition tells us that we should expect for (*a*) the energy to flow along the wire, (*b*) no energy flow, and (*c*) the energy to come from the current charging the capacitor. In each case Poynting's vector gives a radically different answer, as shown by various authors (for example, see ref. 1). The conservation law, [1], may be derived by starting with the relation

[4] $\quad \nabla \cdot \left[\dfrac{1}{4\pi} E \times B\right] = \dfrac{1}{4\pi}(B \cdot \nabla \times E - E \cdot \nabla \times B)$

Then using Maxwell's equations, [4] becomes

[5] $\quad \nabla \cdot \left[\dfrac{1}{4\pi} E \times B\right]$

$\qquad = -\dfrac{\partial}{\partial t}\left[\dfrac{1}{8\pi}(E \cdot E + B \cdot B)\right] - E \cdot J$

and making the identifications in [2] and [3] one obtains [1].

An alternative to Poynting's result (eq. [1]) was obtained by MacDonald (2) who started with the relation

[6] $\quad \nabla \cdot (A \times B) = B \cdot B - A \cdot \nabla \times B$

where A is the vector potential and $B = \nabla \times A$. Adding $(8\pi)^{-1}$ times the partial time derivative of [6] to [5], gives, upon rearranging terms,

[7] $\quad \partial W_M/\partial t + \nabla \cdot S_M = -E \cdot J$

where, taking $C \equiv (\nabla \times B)$,

[8] $\quad \dfrac{\partial W_M}{\partial t} = \dfrac{1}{4\pi}\left[E \cdot C + \dfrac{1}{2}\dfrac{\partial}{\partial t} A \cdot C\right]$

is an energy density and

[9] $\quad S_M = \dfrac{1}{4\pi}[E \times B + \dfrac{1}{2}\dfrac{\partial}{\partial t}(A \times B)]$

represents energy flow.

The energy flow vector defined by [9] also shares the undesirable properties of Poynting's vector.

Hines (3) proposed a modification of MacDonald's results which removes the undesirable properties contained in [3] and [9]. Substituting

$E \equiv -\nabla \phi - \partial A/\partial t$

where ϕ is the scalar potential, in [9] we obtain

[10] $\quad S_M = \dfrac{1}{4\pi}\left[\phi \nabla \times B - \nabla \times (\phi B)\right.$

$\qquad\qquad \left. + \dfrac{1}{2}\left(A \times \dfrac{\partial B}{\partial t} - \dfrac{\partial A}{\partial t} \times B\right)\right]$

Using Maxwell's equations and noting that the term $\nabla \times (\phi B)$ does not contribute to the conservation law, Hines obtained

[11] $W_H = W_M$

as the energy density, and

[12] $S_H = \dfrac{1}{4\pi}\left[\dfrac{1}{2}\left(A \times \dfrac{\partial B}{\partial t} - \dfrac{\partial A}{\partial t} \times B\right)\right.$

$\left. + \phi\left(\dfrac{\partial E}{\partial t} + 4\pi J\right)\right]$

as the energy flow vector.

It is easy to show that S_P and S_H do not differ by a solenoidal term, i.e., by the curl of a vector field, and that [11] and [12] satisfy the conservation law

[13] $\partial W_H/\partial t + \nabla \cdot S_H = -E \cdot J$

Applying [12] to the previously mentioned examples, we have for (a) that energy flows along the wire at a rate ϕJ; for (b) $S_H \equiv 0$, hence giving no energy flow; and (c) the energy comes into the capacitor along the wires.

Based on the conservation law alone, the choice of Hines is as acceptable as that of Poynting. However, there is a potential way of distinguishing which of the various energy flow vectors in the literature is correct. In fact our purpose here is to demonstrate that a calculation of the power radiated by an accelerating charge enables us to distinguish between the various energy flow vectors (4).

To calculate the total power radiated by an accelerated charge, we use the Liénard–Wiechert retarded potentials (5)

[14] $\begin{aligned}\phi(x, t) &= [e/(1 - \hat{n} \cdot \beta)]_{\text{ret}}\\ A(x, t) &= [e\beta/(1 - \hat{n} \cdot \beta)R]_{\text{ret}}\end{aligned}$

where e is the particle's charge, β is numerically equal to the particle's velocity, $\hat{n}$ is a unit vector from the particle in the direction of the field point x, and R is the distance between the particle and the field point. [ ]$_{\text{ret}}$ means that the quantity in the brackets is to be evaluated at the retarded time, $t_{\text{ret}} = t - R$. Note that [14] is in the Lorentz gauge, i.e.,

[15] $\partial\phi/\partial t + \nabla \cdot A = 0$

For convenience we restrict ourselves to a non-relativistic charge, since such a calculation will be sufficient for our purposes. For an arbitrary energy flow vector S, the power radiated per unit solid angle at rest is given by

[16] $dP/d\Omega = R^2\, S \cdot \hat{n}$

Letting $S = S_P$ in [16] and integrating over all angles gives the well-known Larmor radiation formula

[17] $P = \tfrac{2}{3}e^2\,|\dot{\beta}|^2$

where the dot denotes time derivative.

Substituting [14] into [12] and taking $J = 0$ gives

[18] $4\pi S_H = \dfrac{1}{2}\dfrac{e^2}{R^2}[|\dot{\beta}|^2\hat{n} - (\hat{n} \cdot \dot{\beta})\dot{\beta}]$

$- \dfrac{e^2}{R^2}[(\hat{n} \cdot \ddot{\beta})\hat{n} - \ddot{\beta}]$

Hence

[19] $\begin{aligned}dP_H/d\Omega &= R^2 S_H \cdot \hat{n}\\ &= (1/8\pi)e^2|\dot{\beta}|^2\sin^2\theta\end{aligned}$

where θ is the angle between the acceleration $\dot{\beta}$ and $\hat{n}$.

Integrating [19] over all angles gives

[20] $P_H = \tfrac{1}{3}e^2|\dot{\beta}|^2$

which is one-half of the Larmor result (eq. [17]).

Since S_M and S_H differ by a solenoidal term only we also have that $P_M = P_H$. Since Larmor's result agrees with observations, we conclude that, in contrast to Poynting's choice, the choices of MacDonald and Hines are not acceptable choices for the energy flow vector of the electromagnetic field.

Finally, we would like to make some comments on the more general question of the significance and proper interpretation of Poynting's S vector. As emphasized by Born and Wolf (6), in every physical situation the quantity which is physically significant is not S but the flux of S, i.e., the integral of $S \cdot n$ over a closed surface, where n is the unit outward normal to the surface. This point is also brought out clearly by Pugh and Pugh (7) who take as an example a short length of co-axial cable supporting a dc current. It should also be emphasized that it is only in the case of energy transport by electromagnetic waves that S describes the flow of energy at any point in space and any time (8).

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