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Generalized Solutions for Massless Free Fields and Consequent Generalized Conservation Laws

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Generalized solutions to the equations for a massless free field with arbitrary spin are written down. It is shown that they lead immediately to generalizations of all the usual conservation laws.

INTRODUCTION

NEW conservation laws for the electromagnetic field have been discovered recently by Lipkin.[1] Morgan[2] has shown Lipkin's conserved third-rank tensor is a special case of a generalized energy-momentum tensor density. The procedure adopted by Morgan was to construct a generalized tensor by analogy with the usual expression for the conventional $T_{\mu\nu}$. He then showed that the generalized tensor was divergence-free. We will show instead that a possibly more basic concept is the consideration of generalized solutions to the usual field equations. This immediately leads to generalized conservation laws. We will then demonstrate how the conserved quantities of Lipkin and Morgan may be simply obtained as particular cases of our general formalism. Furthermore, in addition to the conserved energy-momentum tensor and current displayed by Morgan, we will show that other generalized conserved quantities may be obtained because our procedure admits a generalization of all the usual conserved quantities.

GENERALIZED SOLUTIONS AND GENERALIZED CONSERVATION LAWS

The equation of a massless free field may be written down in the formalism of Pauli–Fierz[3] or Hammer–Good.[4] For our purpose it is more convenient to use the method of the latter authors who show that the wave equation may be written in the form[5]

$$(S_k/S)\frac{\partial\psi(x)}{\partial x_k}+\frac{\partial\psi(x)}{\partial t}=0, \tag{1}$$

where S is the angular momentum matrix for arbitrary spin $S=\frac{1}{2}, 1, \frac{3}{2}\cdots$ and ψ is a $(2S+1)$ component spinor. As an auxiliary condition, the only solutions retained are those which satisfy the Bargmann–Wigner[6] criteria of having their spins parallel or antiparallel to the momentum. Good,[7] for example, explicitly shows that Maxwell's equations may be written in this form.

Consider any general operator V so selected that $V\psi(x)$ is a generalized solution of Eq. (1). Thus, we can write

$$(S_k/S)\frac{\partial\psi'(x)}{\partial x_k}+\frac{\partial\psi'(x)}{\partial t}=0 \tag{2}$$

and

$$(S_k/S)\frac{\partial\psi''(x)}{\partial x_k}+\frac{\partial\psi''(x)}{\partial t}=0, \tag{3}$$

where

$$\psi'=V'\psi \quad\text{and}\quad \psi''=V''\psi, \tag{4}$$

and where V' and V'' are so selected that ψ' and ψ'' satisfy Eq. (1).

Good[7] has shown how all the usual conservation laws can be derived quite simply from Eq. (1) without using Lagrangian's or Noether's Theorem. He considers infinitesimal transformations of coordinates[8]

$$x_\mu\rightarrow x_\mu'''=x_\mu+\delta x_\mu,$$

and the corresponding transformations of the wavefunction

$$\psi(x)\rightarrow\psi'''(x''').$$

He then introduces a general operator O defined by

$$\psi'''(x)=O\psi(x).$$

This enables him to deduce the conservation laws

[1] D. M. Lipkin, J. Math. Phys. 5, 696 (1964).
[2] T. A. Morgan, J. Math. Phys. 5, 1659 (1964).
[3] M. Fierz, Helv. Phys. Acta 12, 3 (1939); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A173, 211 (1939).
[4] C. L. Hammer and R. H. Good, Jr., Phys. Rev. 108, 882 (1957).
[5] In our units $\hbar=c=1$. Furthermore, Greek indices run from 1–4 and Latin indices from 1–3.
[6] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. US 34, 211 (1948).
[7] R. H. Good, Jr., Phys. Rev. 105, 1914 (1957).
[8] In order to avoid confusion with our already defined single-prime quantities, we use triple primes here instead of the single primes used by Good.

$$\frac{\partial}{\partial x_k}(\psi^+(S_k/S)O\psi)+\frac{\partial}{\partial t}(\psi^+O\psi)=0. \quad (5)$$

Corresponding to the various coordinate transformations of displacement, rotation, etc., Good obtains appropriate expressions for O. The various operators O when substituted into Eq. (5) give rise to the usual conservation laws of energy–momentum, angular momentum, and so on.

Instead of considering Eq. (1) as Good did, we will consider Eqs. (2) and (3). This enables us to deduce a generalization of Good's conservation laws. We obtain

$$\frac{\partial}{\partial x_k}(\psi'^+(S_k/S)O\psi'')+\frac{\partial}{\partial t}(\psi'^+O\psi'')=0, \quad (6)$$

which constitute our generalized conservation laws. We notice that the usual conservation laws contain only ψ; whereas our generalized conservation laws contain both ψ' and ψ''.

As an example, we will discuss in detail the generalization of the energy–momentum tensor $T_{\mu\nu}$ for the electro-magnetic field. The generalization consists of replacing E and H by $E'('')$ and $H'('')$. Thus, Maxwell's equations for the free electromagnetic field in vacuum may be written

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}E_k'=-\frac{\partial}{\partial t}H_i', \quad (7a)$$

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}H_k'=\frac{\partial}{\partial t}E_i', \quad (7b)$$

$$(\partial/\partial x_i)E_i'=0, \quad (7c)$$

$$(\partial/\partial x_i)H_i'=0, \quad (7d)$$

and we have a similar set with the primes replaced by double primes. Associated with ψ' is a generalized antisymmetric field tensor $F_{\mu\nu}'$ and a generalized potential A_μ' defined by

$$\psi_k'=i(-\delta_{4\mu}\delta_{k\nu}+\tfrac{1}{2}\epsilon_{k\mu\nu})F_{\mu\nu}', \quad (8a)$$

and

$$F_{\mu\nu}'=\frac{\partial A_\nu'}{\partial x_\mu}-\frac{\partial A_\mu'}{\partial x_\nu}, \quad (8b)$$

and we have similar quantities with the primes replaced by double primes. This immediately enables us to construct[9] a conserved energy–momentum tensor $\tilde{T}_{\mu\nu}$ given by

$$\tilde{T}_{\mu\nu}=-\tfrac{1}{2}(F_{\mu\gamma}'F_{\nu\gamma}''+F_{\mu\gamma}'^{*}F_{\nu\gamma}''^{*}), \quad (9)$$

where $F_{\mu\nu}'^{*}$ is the dual of $F_{\mu\nu}'$. In the particular case of

$$V'=\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n};\quad V''=\partial_{\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_m}, \quad (10)$$

we see that $\tilde{T}_{\mu\nu}$ is identical with Morgan's conserved tensor $T_{\mu\nu\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}$. We wish to emphasize two points. First of all, Morgan's tensor is a particular case of our general $\tilde{T}_{\mu\nu}$, as given by Eq. (9). Secondly, Morgan proceeded by initially postulating a $T_{\mu\nu\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}$ and then showing that it is conserved while our approach emphasizes the basic concept of generalized solutions to the field equations. From this concept a conserved $\tilde{T}_{\mu\nu}$ follows immediately in a similar manner to the derivation of the usual $T_{\mu\nu}$ from the field equations with the usual solutions.

Morgan has shown that Lipkin's third-order tensor is a particular case of $T_{\mu\nu\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}$, but it will be instructive to derive Lipkin's results explicitly by our method. To do this it will be convenient to rewrite Eqs. (7) in vector notation,

$$\nabla\times\mathbf{E}'=-\partial\mathbf{H}'/\partial t, \quad (11a)$$

$$\nabla\times\mathbf{H}'=\partial\mathbf{E}'/\partial t, \quad (11b)$$

$$\nabla\cdot\mathbf{E}'=0, \quad (11c)$$

$$\nabla\cdot\mathbf{H}'=0. \quad (11d)$$

In the usual case we derive a conserved tensor $T_{\mu\nu}$ which, when written in component form, expresses the laws of conservation of energy density and momentum density. These laws are more readily identifiable when the components of $T_{\mu\nu}$ are expressed in terms of field quantities. The generalizations of these equations are easily shown to be

$$(\partial/\partial t)\{\mathbf{E}'\cdot\mathbf{E}''+\mathbf{H}'\cdot\mathbf{H}''\}+\nabla\cdot\{(\mathbf{E}'\times\mathbf{H}'')+(\mathbf{E}''\times\mathbf{H}')\}=0, \quad (12)$$

and

$$(\partial/\partial t)\{(\mathbf{E}'\times\mathbf{H}'')+(\mathbf{E}''\times\mathbf{H}')\}_i+\partial/\partial x_j\{(\mathbf{E}'\cdot\mathbf{E}''+\mathbf{H}'\cdot\mathbf{H}'')\,\delta_{ij}-(E_i'E_j''+H_i'H_j'')\}=0. \quad (13)$$

In the process of deriving Eq. (12), if we display components of the fields, we can obtain the result

$$(\partial/\partial t)\{E_i'E_i''+H_i'H_i''\}=\{[E_i'(\nabla\times\mathbf{H}'')_i-H_i'(\nabla\times\mathbf{E}'')_i]+[E_i''(\nabla\times\mathbf{H}')_i-H_i''(\nabla\times\mathbf{E}')_i]\}. \quad (14)$$

We obtain two further equations from Eq. (14) by first replacing E' and H' by $(\nabla\times\mathbf{E}')$ and $(\nabla\times\mathbf{H}')$,

[9] For details of this construction, see for example, J. L. Synge, *Relativity: The Special Theory* (North-Holland Publishing Company, Amsterdam, 1956), p. 323.

respectively, and second by replacing E'' and H'' by $(\nabla \times \mathbf{E}'')$ and $(\nabla \times \mathbf{H}'')$, respectively. Adding these two equations together, and using the wave equation, leads to the conservation law:

$$(\partial/\partial t)\{([(\nabla \times \mathbf{E}')_i E_j'' + (\nabla \times \mathbf{H}')_i H_j''] + [E_i'(\nabla \times \mathbf{E}'')_j + H_i'(\nabla \times \mathbf{H}'')_j]) + (i \leftrightarrow j)\} + (\partial/\partial x_k)\{[(E_i' H_{j,k}'' - H_i' E_{j,k}'') + (E_j'' H_{i,k}' - H_j'' E_{i,k}')] + (i \leftrightarrow j)\} = 0. \tag{15}$$

We now set

$$E' = (\nabla \times \nabla \cdots \nabla \times)_{n \text{ times}} \mathbf{E}, \tag{16a}$$

and

$$H' = (\nabla \times \nabla \cdots \nabla \times)_{n \text{ times}} \mathbf{H}, \tag{16b}$$

and, in a similar manner, define E'' and H'' which are the same as the single primed quantities except that an index m replaces n.

We will now show that Lipkin's results are particular cases of the above. For ease in comparison we use Lipkin's definition of $Z^{\mu\nu\rho}$ [as given by his Eqs. (17)–(22)].

Setting $n = 0$ and $m = 1$ in Eq. (12), we obtain

$$(\partial/\partial t)Z^{000} + (\partial/\partial x_k)Z^{00k} = 0; \tag{17}$$

setting $n = 0$ and $m = 1$ in Eq. (13), we obtain

$$(\partial/\partial t)Z^{0i0} + (\partial/\partial x_k)Z^{0ik} = 0; \tag{18}$$

setting $n = m = 0$ in Eq. (15), and making use of Eq. (17), we obtain

$$(\partial/\partial t)Z^{ij0} + (\partial/\partial x_k)Z^{ijk} = 0; \tag{19}$$

Eqs. (17)–(19) constitute all of Lipkins conserved quantities.

In summary, we have shown that generalized solutions of the usual wave equations are possibly a more fundamental concept than generalized conservation laws and that the latter readily follow once we have established the existence of the former.

Note Added in Proof. In subsequent publications[10,11] we demonstrated the existence of generalized conservation laws for free fields with mass, and examined the physical interpretation of these conservation laws.

[10] R. F. O'Connell and D. R. Tompkins, Nuovo Cimento **38**, 1088 (1965).

[11] R. F. O'Connell and D. R. Tompkins, Nuovo Cimento **39**, 391 (1965).