# Superfluid Fermi gases in one dimension

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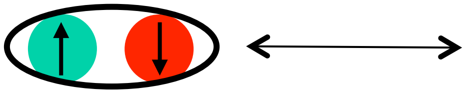
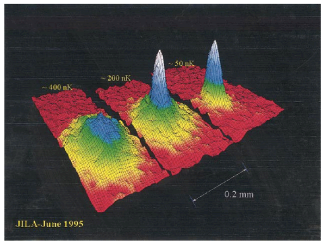
# Outline

- Superfluidity of fermionic atomic gases



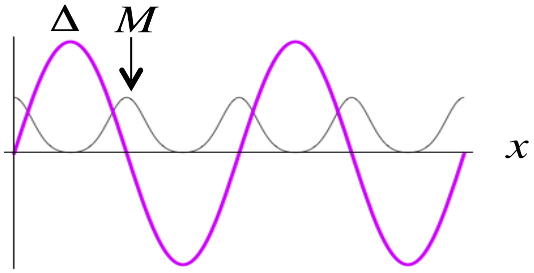
Pairing amplitude $\Delta$

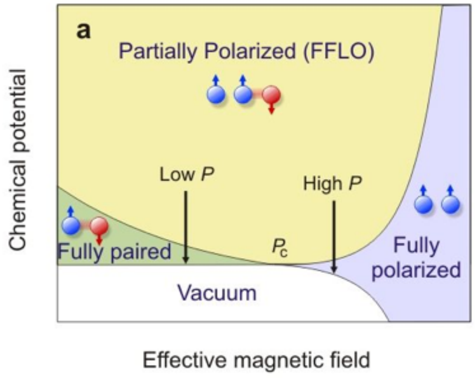
- Recent Work: Apply "population imbalance"
  - – Polarized: More ↑ than ↓
    - ➢ Magnetization: $M = (N_\uparrow - N_\downarrow)/V$
    - ➢ "Zeeman" magnetic field: $h = \mu_\uparrow - \mu_\downarrow$
  - – FFLO phase
- 1D gas: Possibly broad range of stability for FFLO!
- Goal: Simple variational wavefunction for 1D FFLO
  - ➢ "Imbalanced pairing" in a trapped gas
  - ➢ Previous work: Used "Local density approximation"
  - ➢ Our wavefunction has deficiencies!
    - – But perhaps we can fix it…



Anderson et al Science 95

Cartoon picture:





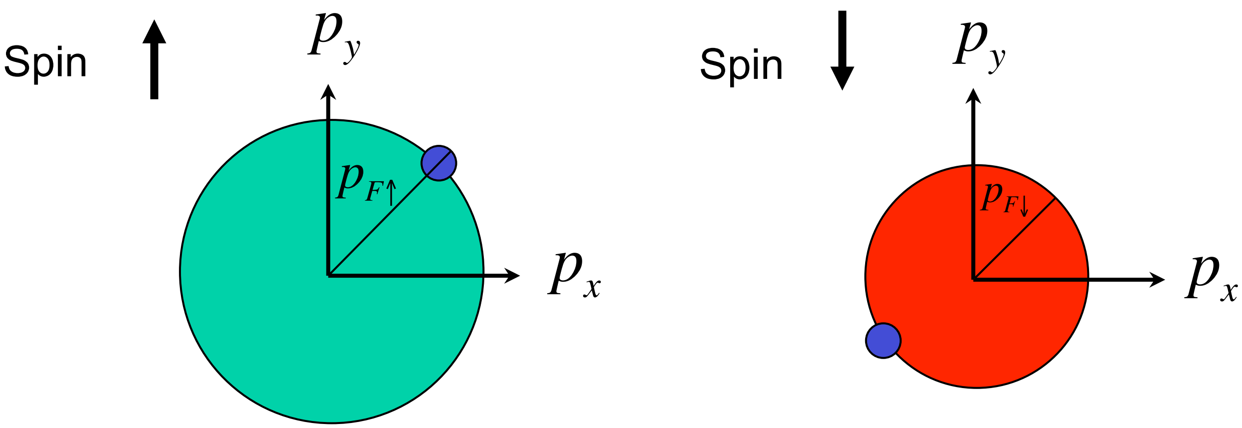
Liao et al Nature 2010

Next: What is the FFLO?

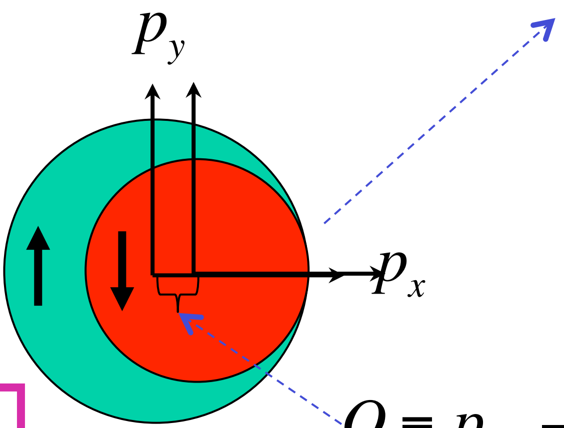
# FFLO state

Fulde&Ferrell PR 1964;
Larkin&Ovchinnikov JETP 1965

- Excess spin ↑ : Larger Fermi surface $p_{F\uparrow} > p_{F\downarrow}$

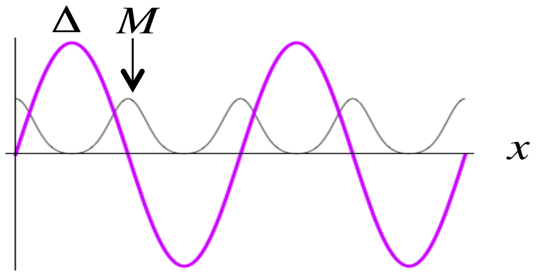


- Pairing of low-energy states near Fermi surface $Q = p_{F\uparrow} - p_{F\downarrow}$
  - ➢ Cooper pairs have finite momentum!
- "Shifted" Fermi seas



Finite-momentum pairing

$$\Delta(x) \propto \cos[Qx]$$



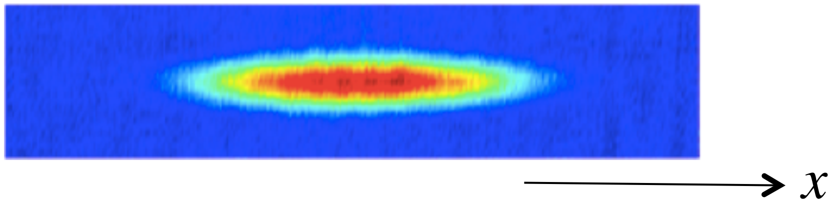
$$Q \equiv p_{F\uparrow} - p_{F\downarrow}$$

What would this look like in a 1D harmonic trap?

Next: 1D atomic gases

# Model Hamiltonian

Partridge et al PRL 2006



- Trapped fermions in 1D

Tight confinement creating cigar-shaped cloud $\Omega_y = \Omega_z >> \Omega_x$

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \sum_{\sigma} \Psi_{\sigma}^{\dagger}(x)[-\frac{1}{2m}\frac{d^2}{dx^2} - \mu_\sigma + \underbrace{\frac{1}{2}m\Omega^2 x^2}_{\text{Axial trap}}]\Psi_{\sigma}(x) + \lambda \int_{-\infty}^{\infty} dx\, \Psi_{\uparrow}^{\dagger}(x)\Psi_{\downarrow}^{\dagger}(x)\Psi_{\downarrow}(x)\Psi_{\uparrow}(x).$$

Spin states $\sigma = \uparrow, \downarrow$

Chemical potentials

Short-range attraction

- No trap: Gaudin-Yang model

Review: Guan et al RMP 2013

  - Bethe ansatz solution: Infinite 1D gas

Densities: $\begin{cases} n_\uparrow(\mu, h) \\ n_\downarrow(\mu, h) \end{cases}$

Average chemical potential & Chemical potential difference $\begin{cases} \mu = (\mu_\uparrow + \mu_\downarrow)/2 \\ h = \mu_\uparrow - \mu_\downarrow \end{cases}$

- Density in the trap: LDA

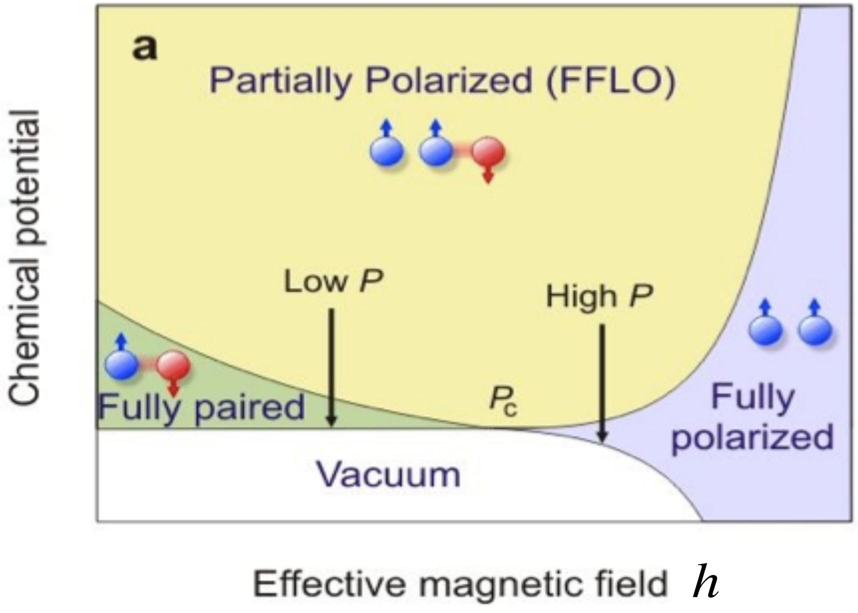
Local density approximation

$$n_\sigma(x) = n_\sigma\left(\mu - \frac{1}{2}m\Omega^2 x^2, h\right)$$

Next: How good is it?

# One-dimensional case: Wide FFLO regime

Bethe Ansatz:
Orso PRL 2007
Hu et al PRL 2007

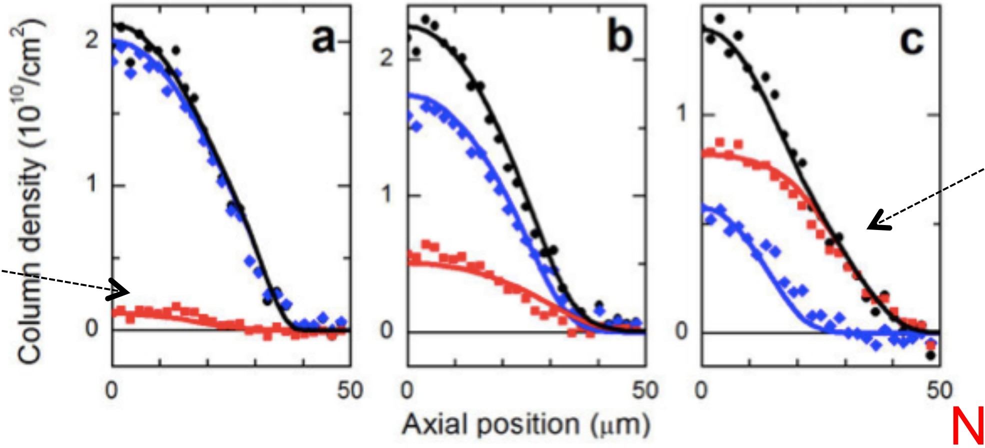


Experiment:
Liao et al Nature 2010

$^6$Li in “tube” geometry

- Density profiles consistent with phase diagram
  - LDA: Trap acts like a spatially-varying chemical potential

Red curves:
Magnetization



Low P: Magnetized in center, fully paired on edge

High P: Magnetized in center, fully polarized on edge

But how to observe the “FFLO”?

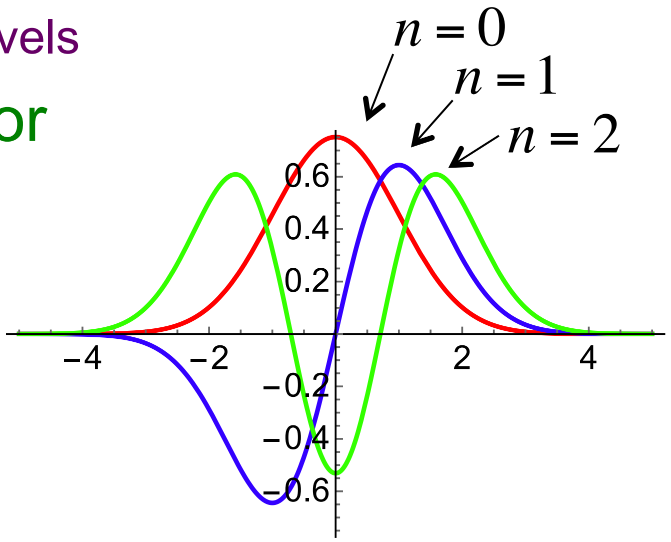
Next: Alternate method

# Our strategy: Exact single particle states

- Hamiltonian for interacting fermions in a harmonic trap

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \sum_{\sigma} \Psi_{\sigma}^{\dagger}(x)[-\frac{1}{2m}\frac{d^2}{dx^2} - \mu_{\sigma} + \frac{1}{2}m\Omega^2 x^2]\Psi_{\sigma}(x) + \lambda \int_{-\infty}^{\infty} dx\, \Psi_{\uparrow}^{\dagger}(x)\Psi_{\downarrow}^{\dagger}(x)\Psi_{\downarrow}(x)\Psi_{\uparrow}(x).$$

  - ➢ No interactions: Fermions in oscillator levels



- Single particle states: Harmonic oscillator

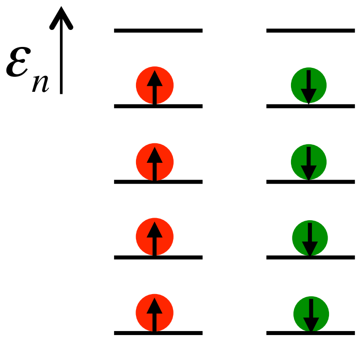
Harmonic oscillator wavefunction

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\frac{1}{\pi^{1/4}}\mathrm{e}^{-x^2/2}H_n(x),$$

Field operator at position x

$$\Psi_{\sigma}(x) = \sum_n \psi_n(x) a_{n\sigma},$$

Annihilation operator for one oscillator state



- Fermi gas of oscillator states

Single particle energy: $\varepsilon_n = \hbar\Omega\left(n + \frac{1}{2}\right)$

Next: Resulting Hamiltonian

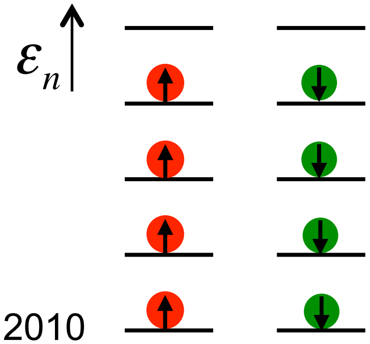
# Hamiltonian in oscillator basis

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \sum_{\sigma} \Psi_{\sigma}^{\dagger}(x)[-\frac{1}{2m}\frac{d^2}{dx^2} - \mu_{\sigma} + \frac{1}{2}m\Omega^2 x^2]\Psi_{\sigma}(x) + \lambda \int_{-\infty}^{\infty} dx\, \Psi_{\uparrow}^{\dagger}(x)\Psi_{\downarrow}^{\dagger}(x)\Psi_{\downarrow}(x)\Psi_{\uparrow}(x).$$

$$\Downarrow$$

$$\mathcal{H}/(\hbar\Omega) \;=\; \underbrace{\sum_{n,\sigma}(\hat{\varepsilon}_n - \hat{\mu}_{\sigma})a^{\dagger}_{n\sigma}a_{n\sigma}}_{\text{Single-particle energy}} + \hat{\lambda}\sum_{n_i} \underbrace{\tilde{\lambda}_{n_1,n_2,n_3,n_4}}_{\text{Interaction matrix element}} a^{\dagger}_{n_1\uparrow}a^{\dagger}_{n_2\downarrow}a_{n_3\downarrow}a_{n_4\uparrow}$$

Dimensionless coupling



- Dimensionless coupling:

$$\hat{\lambda} = \frac{\lambda}{\hbar\Omega a} = -\frac{2a}{a_{1D}} \approx -52$$

Experiment:
Liao et al Nature 2010

Oscillator length $a = \sqrt{\frac{\hbar}{m\Omega}}$

Atom scattering length

- Interaction matrix element

Hermite wavefunction

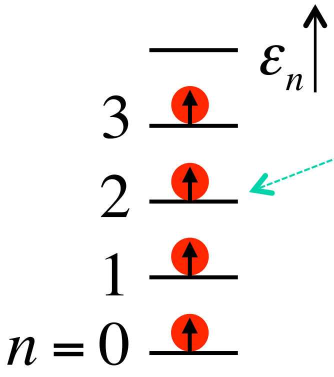
$$\tilde{\lambda}_{n_1,n_2,n_3,n_4} \;\equiv\; \int_{-\infty}^{\infty} dx\, \psi_{n_1}(x)\psi_{n_2}(x)\psi_{n_3}(x)\psi_{n_4}(x).$$

Next: Interaction matrix

# Effective coupling parameter

- Matrix element of oscillator wavefunctions:

$$\tilde{\lambda}_{n_1,n_2,n_3,n_4} \equiv \int_{-\infty}^{\infty} dx\, \psi_{n_1}(x)\psi_{n_2}(x)\psi_{n_3}(x)\psi_{n_4}(x).$$



Indices: Oscillator level

Oscillator wave function

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\frac{1}{\pi^{1/4}} e^{-x^2/2} H_n(x),$$

Gaussian multiplying
Hermite polynomial

- Integral of the product of four harmonic oscillator w.f.'s
  - ➢ Plug in the wave functions …

$$\tilde{\lambda}_{n_1,n_2,n_3,n_4} = \frac{1}{\pi}\frac{1}{\sqrt{2^{n_1+n_2+n_3+n_4} n_1! n_2! n_3! n_4!}} \int_{-\infty}^{\infty} dx\, e^{-2x^2} H_{n_1}(x)H_{n_2}(x)H_{n_3}(x)H_{n_4}(x).$$

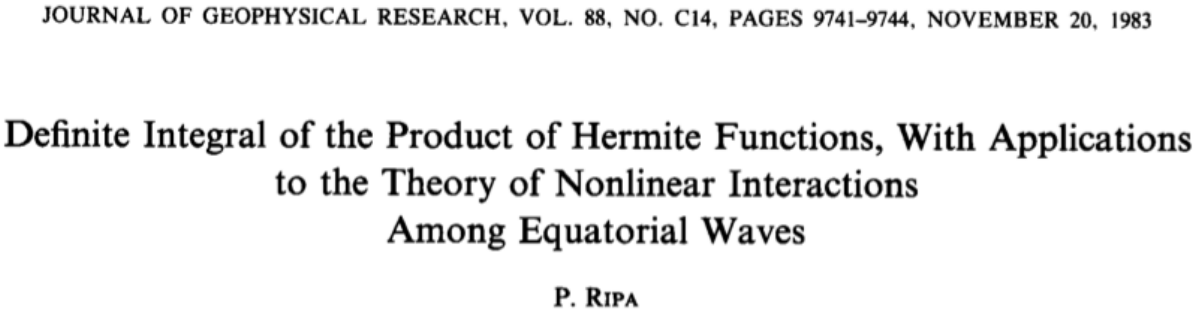
  - ➢ Is this a known integral?

Next: No

# Coupling integral

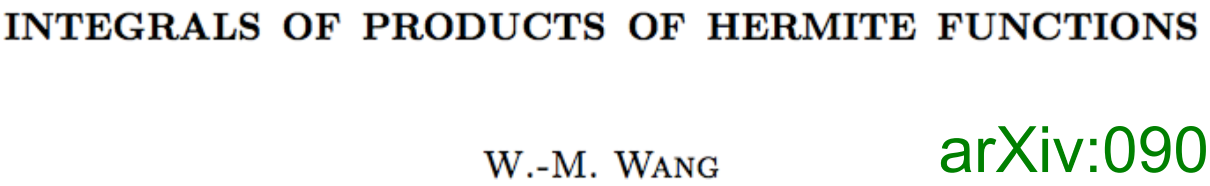
$$\tilde{\lambda}_{n_1,n_2,n_3,n_4} = \frac{1}{\pi}\frac{1}{\sqrt{2^{n_1+n_2+n_3+n_4}n_1!n_2!n_3!n_4!}}\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-2x^2} H_{n_1}(x)H_{n_2}(x)H_{n_3}(x)H_{n_4}(x).$$

- Previous work:


JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 88, NO. C14, PAGES 9741–9744, NOVEMBER 20, 1983

**Definite Integral of the Product of Hermite Functions, With Applications to the Theory of Nonlinear Interactions Among Equatorial Waves**

P. RIPA

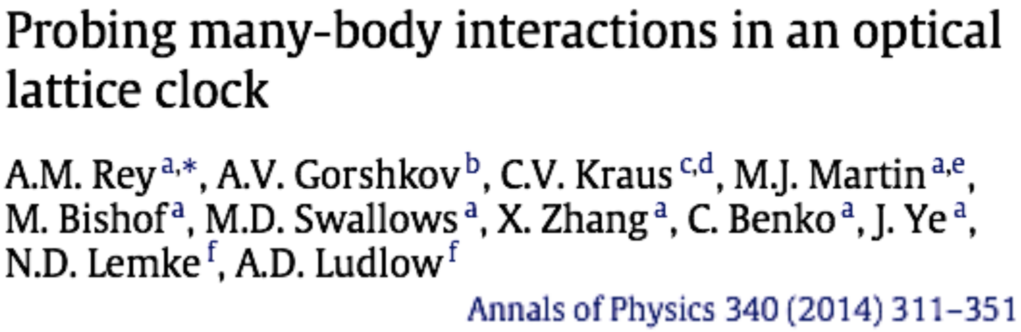

**INTEGRALS OF PRODUCTS OF HERMITE FUNCTIONS**

W.-M. WANG

Asymptotic properties

arXiv:0901.3970

- Also in other AMO contexts


**Probing many-body interactions in an optical lattice clock**

A.M. Rey[a,*], A.V. Gorshkov[b], C.V. Kraus[c,d], M.J. Martin[a,e], M. Bishof[a], M.D. Swallows[a], X. Zhang[a], C. Benko[a], J. Ye[a], N.D. Lemke[f], A.D. Ludlow[f]

Annals of Physics 340 (2014) 311–351

Numerical

Next: Our result

# Analytical result

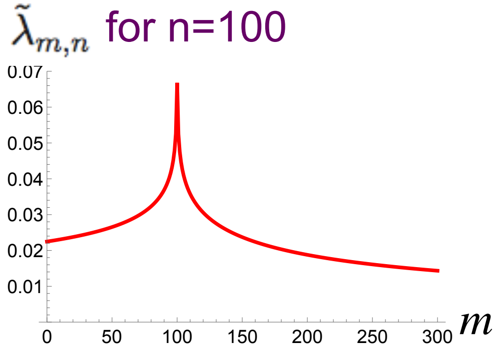
S. Kudla, D.M. Gautreau, DES, Arxiv:1404.4081

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\frac{1}{\pi^{1/4}}e^{-x^2/2}H_n(x),$$

- Simplified case: $n_1 = n_2$ & $n_3 = n_4$ (Or similar combinations…)

$$\tilde{\lambda}_{m,n} = \int_{-\infty}^{\infty} dx\, \psi_n(x)\psi_n(x)\psi_m(x)\psi_m(x),$$

$$= \frac{1}{2^{m+n}}\frac{1}{\pi n!m!}\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-2x^2} H_n^2(x)H_m^2(x).$$



$\tilde{\lambda}_{m,n}$ for n=100

- We find (using Hermite polynomial identities)

$$\tilde{\lambda}_{m,n} = \frac{(-1)^m}{\sqrt{2}m!}\frac{{}_3F_2(\frac{1}{2},\frac{1}{2},-n;1,\frac{1}{2}-m;1)}{\Gamma[\frac{1}{2}-m]}.$$

Symmetric under exchanging $n,m$

Generalized Hypergeometric function…

- Only need this special case!
  - Due to our choice of variational wavefunction

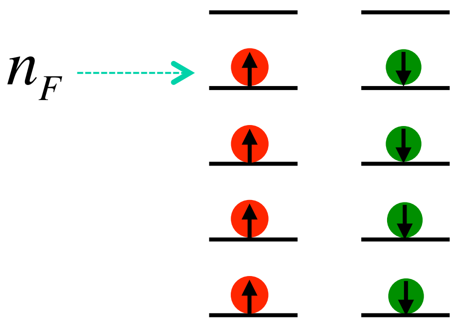
Next: Variational wavefunction

# Balanced gas

- Balanced gas: $N_\uparrow = N_\downarrow$ → $h = 0$

$$\mathcal{H}/(\hbar\Omega) = \sum_{n,\sigma}(\hat{\epsilon}_n - \hat{\mu})a^\dagger_{n\sigma}a_{n\sigma} + \hat{\lambda}\sum_{n_i}\tilde{\lambda}_{n_1,n_2,n_3,n_4}a^\dagger_{n_1\uparrow}a^\dagger_{n_2\downarrow}a_{n_3\downarrow}a_{n_4\uparrow},$$

- Approximate solution: BCS variational wavefunction
  - No interactions: Fermi gas of each species



Fermi level: uppermost filled level

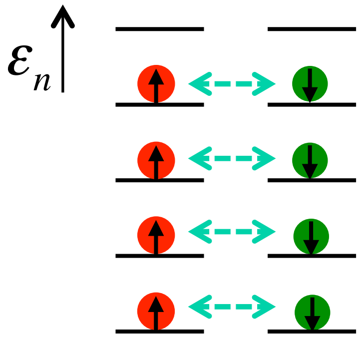
Wavefunction:

Vacuum

$$|\Psi\rangle = \prod_{n=0}^{n_F}(a^\dagger_{n\uparrow}a^\dagger_{n\downarrow})|0\rangle,$$

States up to the Fermi energy are
Filled with probability 1

  - BCS wavefunction: Quantum fluctuations of pairs



BCS pairing

$$|\Psi\rangle = \prod_{n=0}^{\infty}(u_n + v_n a^\dagger_{n\uparrow}a^\dagger_{n\downarrow})|0\rangle,$$

Amplitude to have no
fermions in nth level

Amplitude to have two
fermions in nth level

Next: Variational theory

# Variational theory

- True ground state energy lower than any estimate

$$E_{G,\text{true}} < \underbrace{E_G = \langle\Psi|\mathcal{H}|\Psi\rangle}_{\text{Estimate using our trial wavefunction}}$$

We only need the simplified coupling integral!

- BCS estimate:

$$E_G = \langle\Psi|\mathcal{H}|\Psi\rangle = 2\sum_n \xi_n |v_n|^2 + \hat{\lambda}\sum_{n,m}\tilde{\lambda}_{n,m}\left(u_n^* v_n v_m^* u_m + |v_n|^2|v_m|^2\right),$$

"Pairing" correlations

"Hartree-Fock" interactions

- Minimize $E_G$ subject to constraint $|u_n|^2 + |v_n|^2 = 1,$

Probability to have no fermions in nth level

Probability to have two fermions in nth level

- Solution: Bogoliubov-de Gennes

Normalized chemical potential

Amplitude to have two fermions in nth level: $v_n = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\tilde{\xi}_n}{E_n}},$

$$\tilde{\xi}_n = \underbrace{n + \frac{1}{2}}_{\text{Harm. oscillator energy}} - \hat{\mu} + U_n$$

Hartree-Fock energy

$$E_n = \sqrt{\tilde{\xi}_n^2 + \Delta_n^2}$$

Pairing energy

Next: Self-consistent equations

# Variational solution

- Ground-state values of pairing, Hartree-Fock energies

  ➢ Self consistent equations:

$$\Delta_n = -\hat{\lambda} \sum_{m=0}^{\infty} \tilde{\lambda}_{n,m} \frac{\Delta_m}{2E_m},$$

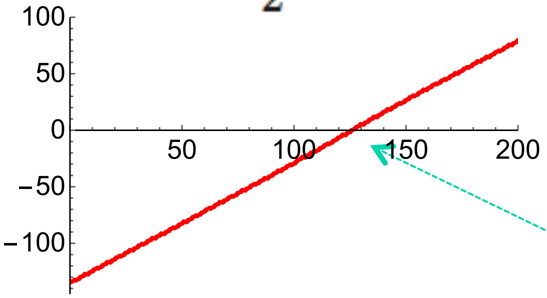
$$U_n = \hat{\lambda} \sum_{m=0}^{\infty} \tilde{\lambda}_{n,m} \frac{1}{2}\left(1 - \frac{\tilde{\xi}_m}{E_m}\right),$$

$$\boxed{\tilde{\xi}_n = n + \frac{1}{2} - \hat{\mu} + U_n}$$

$$\boxed{E_n = \sqrt{\tilde{\xi}_n^2 + \Delta_n^2}}$$

- Numerical solution: Weak coupling $\boxed{\hat{\lambda} = -5,}$ $N = 250$ fermions

$\tilde{\xi}_n = n + \frac{1}{2} - \hat{\mu} + U_n$ = single fermion energy

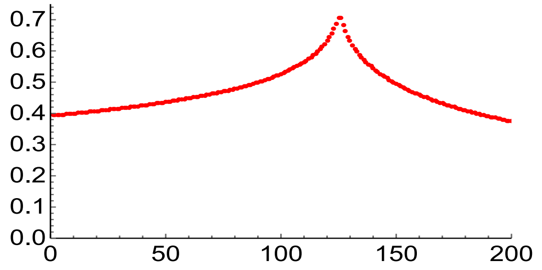


$n$ =oscillator level

Vanishing point defines Fermi level

Pairing should be strongest here

$\Delta_n$ =pairing strength



$n$

Next: Fermion occupation

# Fermion occupation

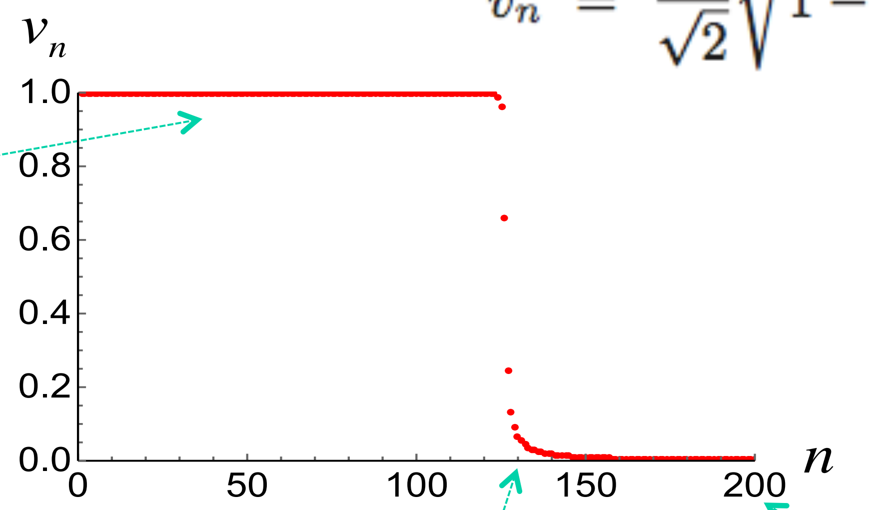
Recall:

$$\tilde{\xi}_n = n + \frac{1}{2} - \hat{\mu} + U_n$$

$$E_n = \sqrt{\tilde{\xi}_n^2 + \Delta_n^2}$$

- Occupation amplitude:

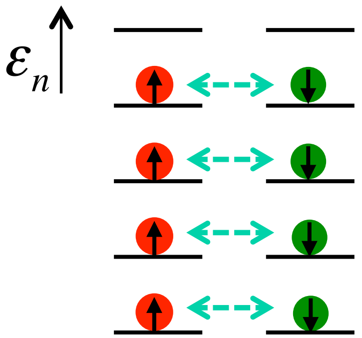
$$v_n = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\tilde{\xi}_n}{E_n}},$$



Levels with low n: almost always occupied

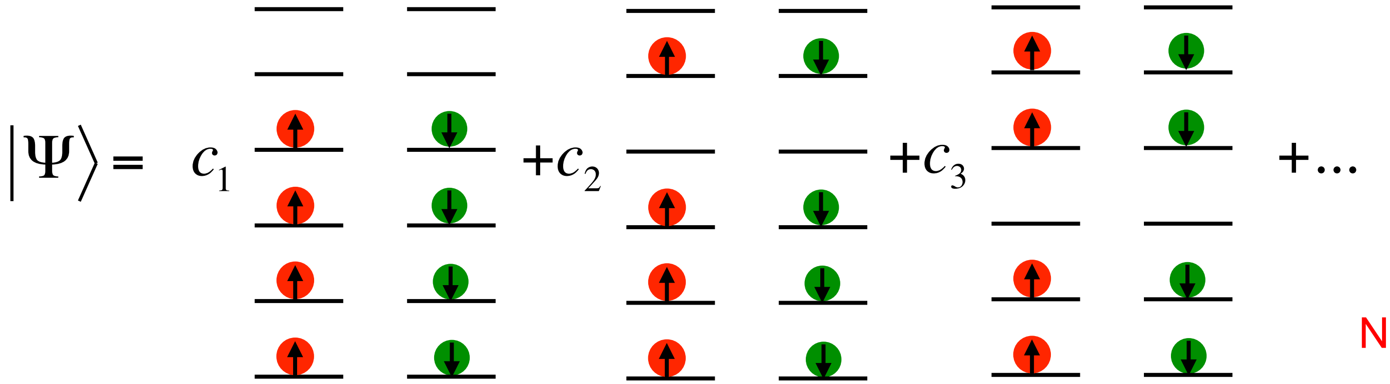
Near Fermi level: Superposition of occupied and empty

Levels with high n: almost always empty



BCS pairing

- Better cartoon picture of wavefunction:



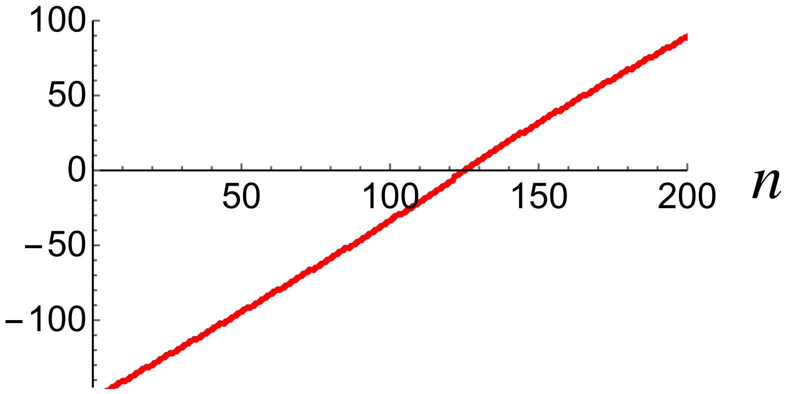
Quantum pair fluctuations near the Fermi level

Next: Stronger coupling

# Stronger coupling: $\boxed{\hat{\lambda} = -15,}$ $N = 250$ fermions

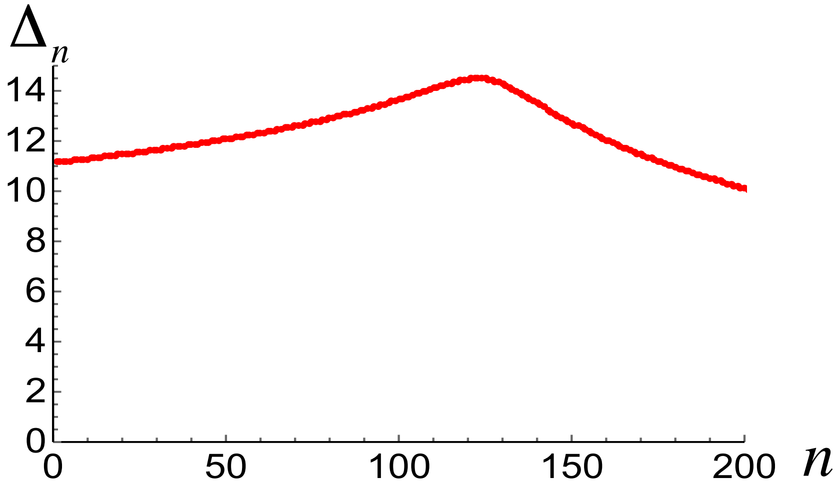
- Single fermion energy

$$\tilde{\xi}_n = n + \frac{1}{2} - \hat{\mu} + U_n$$



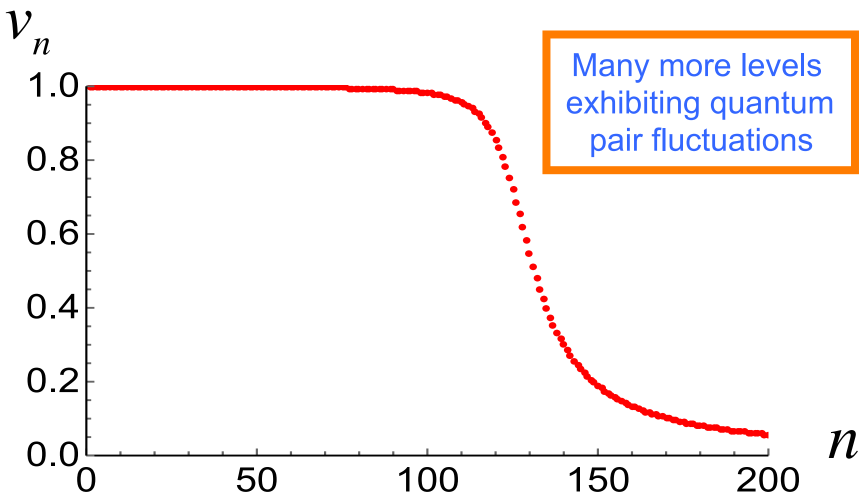
Almost the same as weak coupling!
**Slightly steeper…**

- Pairing energy



Pairing amplitude
**larger**

- Occupation amplitude



Many more levels exhibiting quantum pair fluctuations

Next: what about FFLO

# Recap: 1D trapped fermions

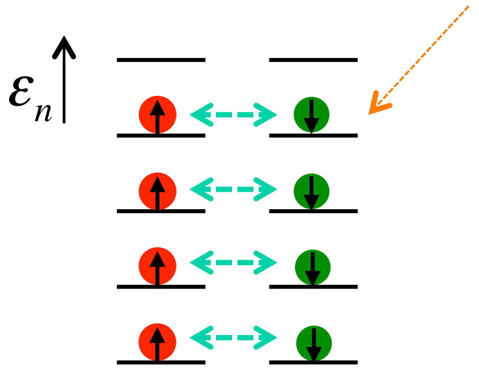
- Experimentally-realized Hamiltonian: 1D trapped fermions

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \sum_{\sigma} \Psi_{\sigma}^{\dagger}(x)[-\frac{1}{2m}\frac{d^2}{dx^2} - \mu_{\sigma} + \frac{1}{2}m\Omega^2 x^2]\Psi_{\sigma}(x) + \lambda \int_{-\infty}^{\infty} dx\, \Psi_{\uparrow}^{\dagger}(x)\Psi_{\downarrow}^{\dagger}(x)\Psi_{\downarrow}(x)\Psi_{\uparrow}(x).$$

- Balanced case: $N_{\uparrow} = N_{\downarrow}$ and $\mu_{\uparrow} = \mu_{\downarrow}$

Our variational wavefunction:
BCS pairing in oscillator basis

$$|\Psi\rangle = \prod_{n=0}^{\infty}(u_n + v_n a_{n\uparrow}^{\dagger} a_{n\downarrow}^{\dagger})|0\rangle,$$



Harmonic oscillator levels

**Arrows: Quantum fluctuations to adjacent levels**

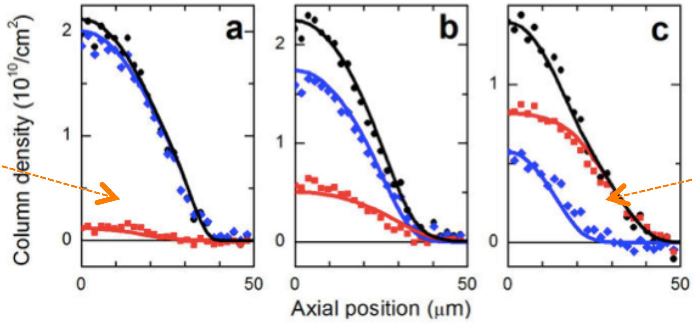
- Next question: Wavefunction for imbalanced case? Magnetization: $M = N_{\uparrow} - N_{\downarrow}$

**Criteria for success:**

- Reduces to noninteracting wavefunction for $\lambda \rightarrow 0$
- Incorporates pairing fluctuations for fermions near Fermi level
- Agrees with existing observations

Red curves: Magnetization

**Locally magnetized** in center, **locally paired on edge!**

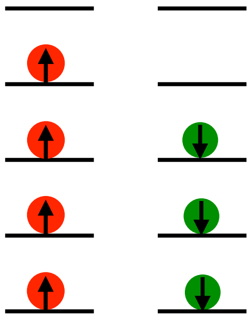


**Partially magnetized** in center, **completely magnetized at edge**

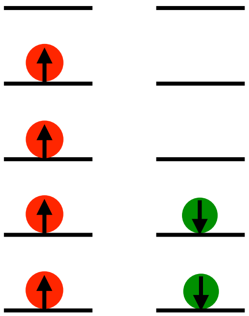
Next: our wavefunction

# 1D FFLO wavefunction

- Noninteracting case: Imbalanced occupation of oscillator levels



$M = 1$

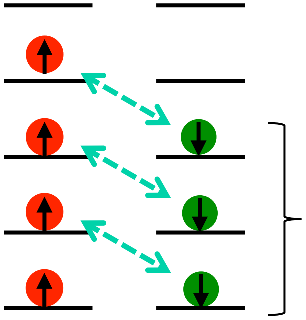


$M = 2$

$$M = N_\uparrow - N_\downarrow$$

$$|\Psi\rangle = \prod_{n=0}^{N_\uparrow} a^\dagger_{n\uparrow} \prod_{n=0}^{N_\downarrow} a^\dagger_{n\downarrow} |0\rangle$$

- We propose: Pair near the Fermi level ⇔ Fermions with lowest excitation energies



Excess unpaired spin-up

Imbalanced pairing

FFLO pairing

$$|\Psi\rangle = \underbrace{\prod_{n=0}^{M-1} a^\dagger_{n\uparrow}}_{\text{Excess unpaired spin-up}} \underbrace{\prod_{n=M}^{\infty} (u_n + v_n a^\dagger_{n\uparrow} a^\dagger_{n-M\downarrow})}_{\text{Imbalanced pairing}} |0\rangle$$

Next: Variational energy

# Variational energy

- Fixed chemical potential & fixed magnetization $\Phi(\hat{\mu}, M) = \langle \mathcal{H} \rangle + hM$

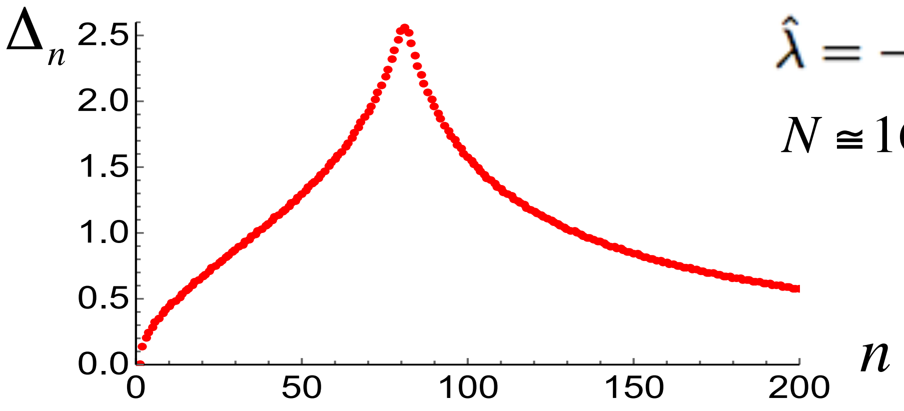
$$\Phi(\hat{\mu}, M) = \frac{1}{2}M^2 - \hat{\mu}M + \sum_{m=M}^{\infty} 2\bar{\xi}_m |v_m|^2 + \hat{\lambda} \sum_{n_1=M}^{\infty} \sum_{n_2=M}^{\infty} \tilde{\lambda}_{n_1,n_2,n_1-M,n_2-M} u^*_{n_1} v_{n_1} v^*_{n_2} u_{n_2}$$

$$+\hat{\lambda} \sum_{n_1=0}^{M-1} \sum_{n_4=0}^{M-1} \sum_{n_2=M}^{\infty} \tilde{\lambda}_{n_1,n_2-M,n_2-M,n_4} |v_{n_2}|^2 + \hat{\lambda} \sum_{n_1=M}^{\infty} \sum_{n_2=M}^{\infty} \tilde{\lambda}_{n_1,n_1,n_2-M,n_2-M} |v_{n_1}|^2 |v_{n_2}|^2.$$

- More complicated variational energy
  - Single-particle energy: $\bar{\xi}_n \equiv \frac{1}{2}(\varepsilon_n + \varepsilon_{n-M} - 2\hat{\mu}).$
  - New coupling functions…

But we can also obtain them analytically

- Similar results for pairing amplitude



$M = 1$

$\hat{\lambda} = -15,$

$N \cong 160$ fermions

Next: Other observables

# Observables

- Total particle density:

Harmonic oscillator wavefunction

$$n(x) \;=\; \underbrace{\sum_{n=0}^{M-1} |\psi_n(x)|^2}_{\text{Excess spins-up}} + \underbrace{\sum_{n=M}^{\infty} |v_n|^2\left(|\psi_n(x)|^2 + |\psi_{n-M}(x)|^2\right)}_{\text{Due to pairs}},$$

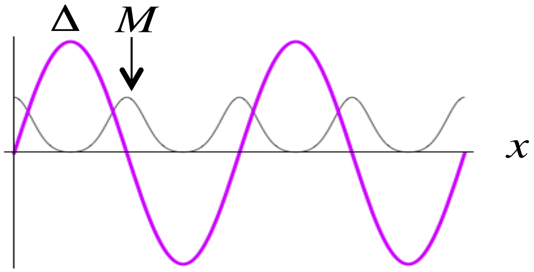
- Local magnetization:

$$M(x) \;=\; \sum_{n=0}^{M-1} |\psi_n(x)|^2 + \sum_{n=M}^{\infty} |v_n|^2\left(|\psi_n(x)|^2 - |\psi_{n-M}(x)|^2\right),$$

- Local pairing amplitude: Should exhibit “FFLO” behavior

$$\Delta(x) = \sum_{n=M}^{\infty} \psi_{n-M}(x)\psi_n(x)u_n v_n$$

FFLO Cartoon picture:

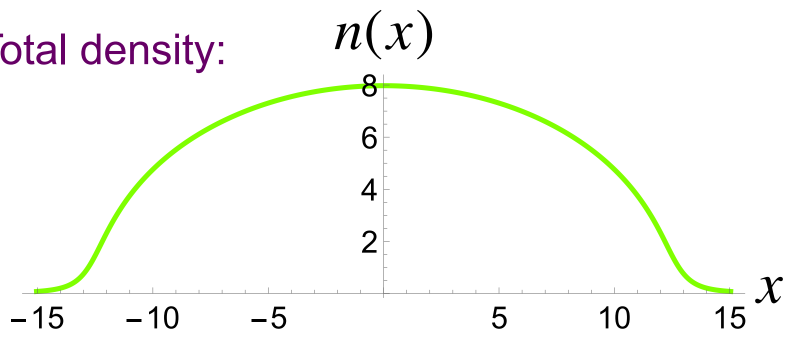


Next: Results

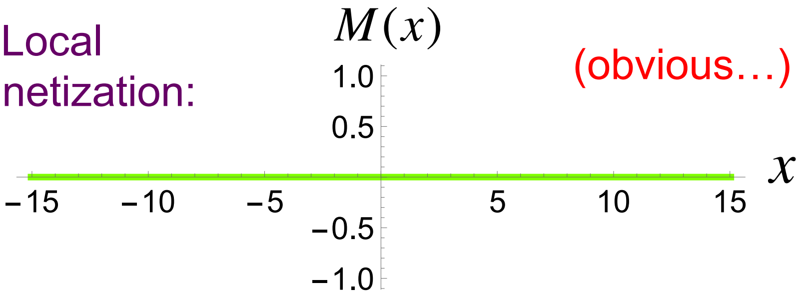
# Results

- Balanced gas: $M = 0$

Total density:

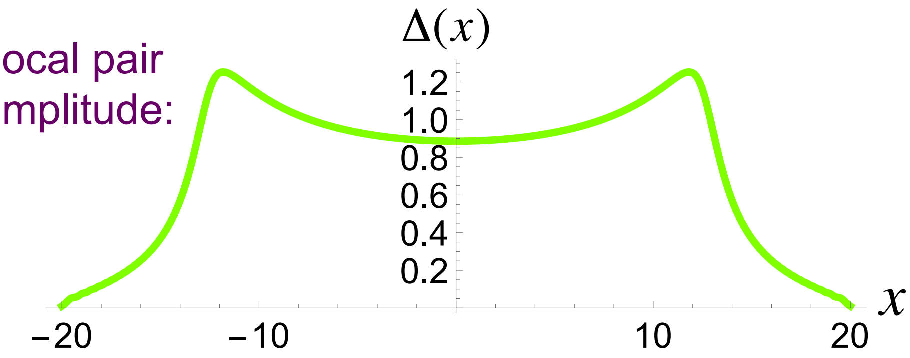


Local magnetization:



(obvious…)

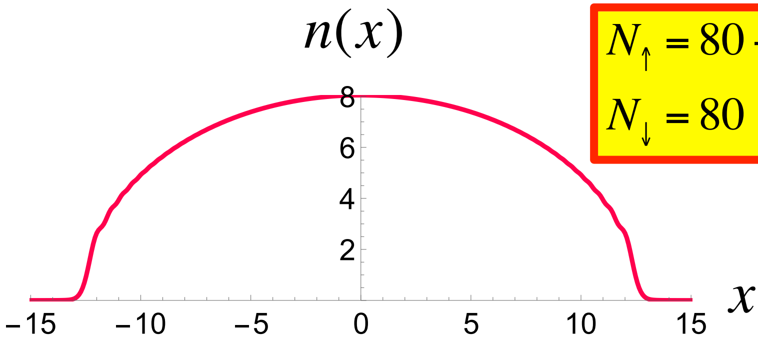
Local pair amplitude:

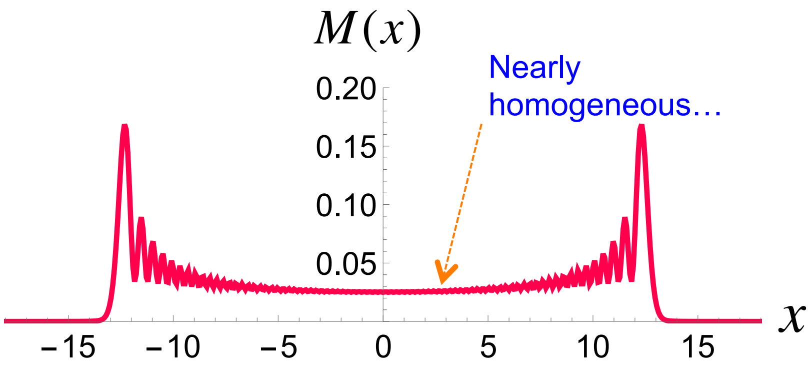


- Imbalanced gas: $M = 1$

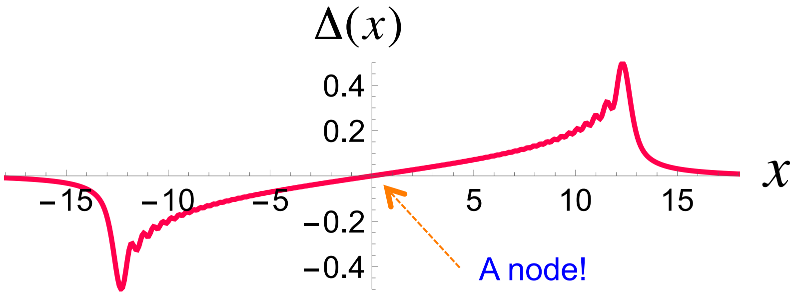
Henceforth:

$$\boxed{\begin{array}{l} N_\uparrow = 80 + M \\ N_\downarrow = 80 \end{array}}$$





Nearly homogeneous…



A node!

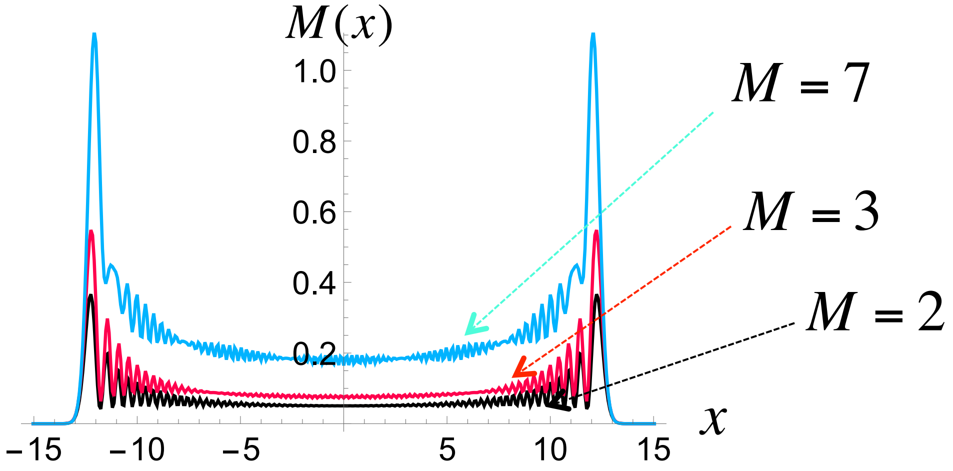
Next: More results

# Results

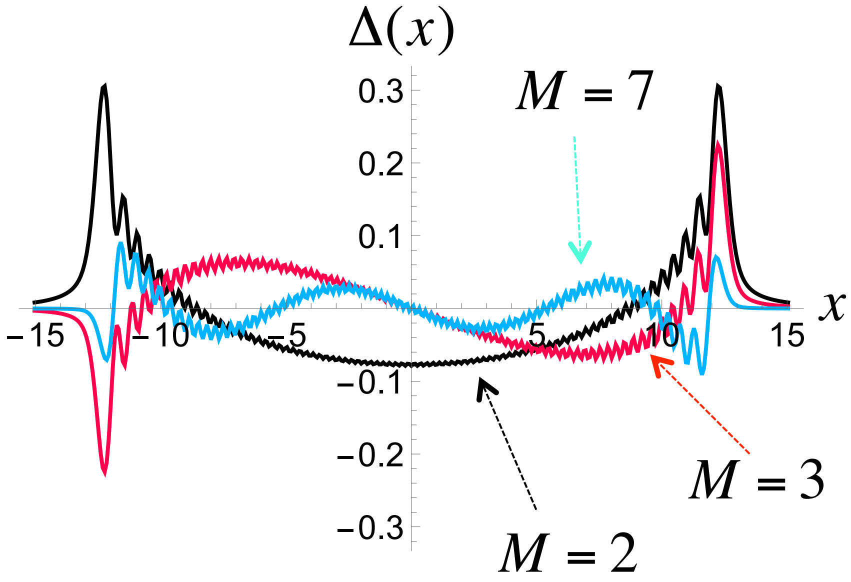
- Interactions constant, increase imbalance $M = 2,3,7$

$N_\uparrow = 80 + M$

$N_\downarrow = 80$



➢ Magnetization homogeneous in center, large on edge



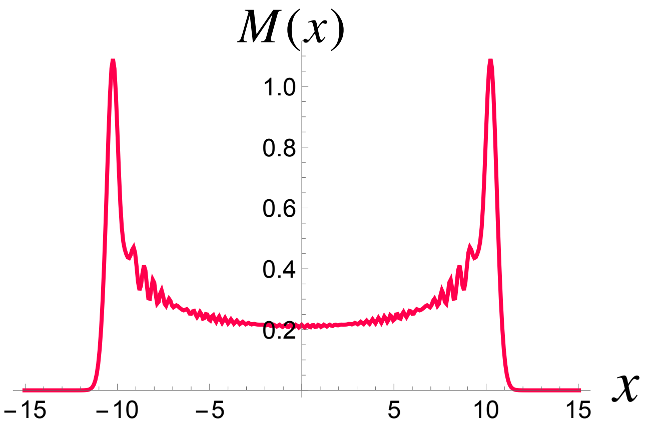
➢ Pairing *suppressed* with increasing imbalance

➢ Number of nodes is equal to $M$

Next: Problem

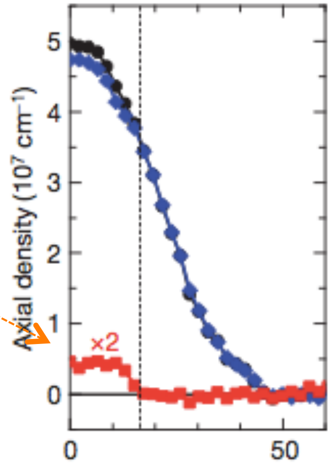
# Our FFLO wavefunction

- Oscillatory “FFLO” pairing correlations
- But: densities do not agree with existing experiments!



Almost the same as noninteracting case!

**Locally magnetized** in center, **locally paired on edge!**



Liao et al
Nature 2010

- Essential problem: Our wavefunction was oversimplified
    - Similar problem occurs in the balanced case! $|\Psi\rangle = \prod_{n=0}^{\infty}(u_n + v_n a^\dagger_{n\uparrow}a^\dagger_{n\downarrow})|0\rangle,$
- Density operator:

$$n(x) = \sum_{\sigma=\uparrow,\downarrow}\sum_{n,m}\underbrace{\psi^*_n(x)\psi_m(x)}_{\text{Can be positive or negative}}a^\dagger_{n\sigma}a_{m\sigma}$$

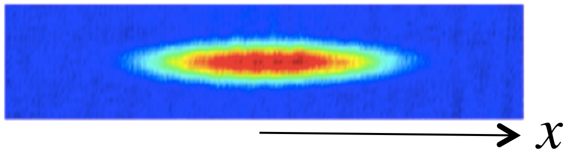
**Wavefunction always occupies pairs!**

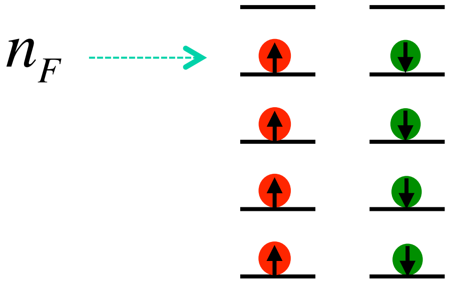
$$n(x) = \sum_{\sigma=\uparrow,\downarrow}\sum_{n}|\psi_n(x)|^2\underbrace{a^\dagger_{n\sigma}a_{n\sigma}}_{\text{“Diagonal” in harmonic oscillator operators}},$$

Next: Consequence of this

# How do interactions affect cloud size?

- Attractive interactions: Decrease cloud size
  - $\lambda = 0$ Fermi gas of each species

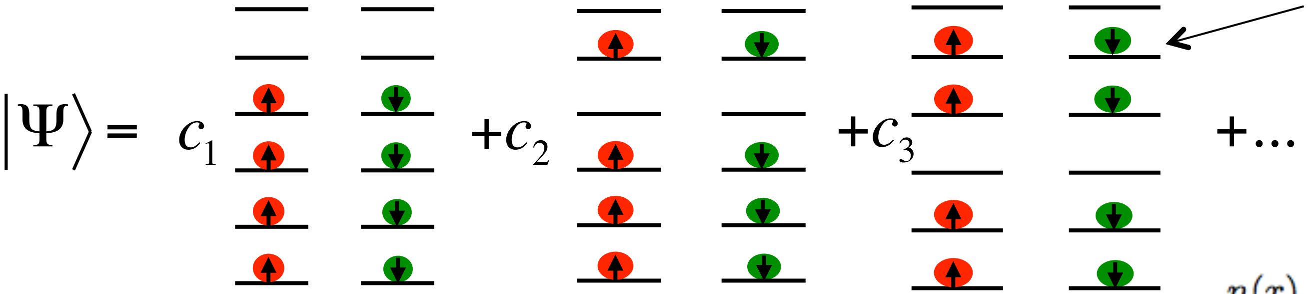




Spatial extent of oscillator wavefunction: $\sim \sqrt{2n}$

Cloud size: $\sim \sqrt{2n_F}$

- Now turn on attraction: We only allow pair fluctuations!



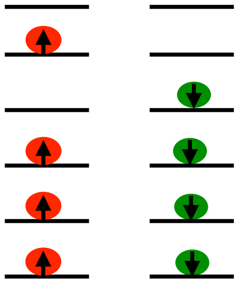
Each fluctuation is into a larger cloud!

BCS w.f. predicts increasing cloud size with attraction!

$$n(x) = \sum_{\sigma=\uparrow,\downarrow} \sum_n |\psi_n(x)|^2 a^\dagger_{n\sigma} a_{n\sigma}$$

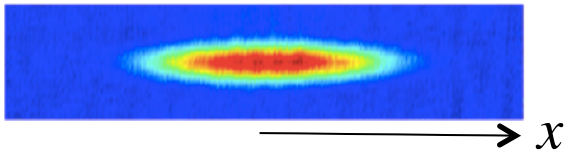
- Correctly obtain “shrinking” cloud?
  - Must include “off-diagonal” contributions to wavefunction

$\langle\Psi|a^\dagger_{n\sigma} a_{m\sigma}|\Psi\rangle \neq 0$ for $n \neq m$

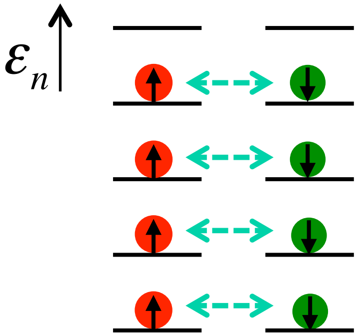


Next: Final remarks

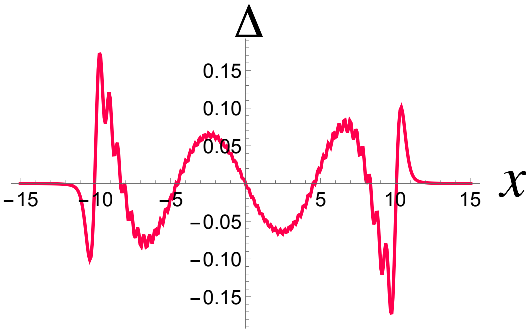
# Concluding remarks



- Trapped 1D superfluid Fermi gases: Simple variational wavefunction
  - Pairing in oscillator basis



- No need for local density approximation
  - Include coupling among 1D tubes & effects of higher transverse bands
- Imbalanced regime:  Prediction for oscillatory pairing correlations
  - Overall densities do not agree!
  - Also a problem in the *balanced* case



- Mean-field theory fails in 1D
  - "Trapped BCS" wavefunction: increasing cloud size with attraction in any dimension
- How to fix our wavefunction?
  - Allow the wavefunctions in our ansatz to have a different oscillator length than the physical system

Additional variational parameter