Colliding black holes with perturbation theory

- Setting up the initial data and evolving: the Misner example.
- Other families of data: Brill-Lindquist, Bowen-York, Baker-Puzio-Krivan-Price.
- Results of evolutions.

We need to provide initial data to the Einstein equations. By this we mean a three dimensional metric on a three surface and its time derivative.

The initial value problem in general relativity is not free. For people unfamiliar with this, it is good to examine as an example the case of Maxwell fields. Suppose one wishes to evolve in time an electromagnetic field. As initial data one could set up the electric and magnetic fields and their time derivatives. Unfortunately not any vector field will work as initial data. In vacuum the electric field has to satisfy div E=0 at any time, in particular at the initial time. Therefore the initial data has to satisfy this (linear) equation.

In general relativity the corresponding equations are “G00” and “G0i”. These equations only involve the metric and its first time derivatives and therefore constrain the initial data.

The initial value problem for general relativity is non-linear. This is physically understandable. One cannot superpose two non-trivial solutions of general relativity without taking into account "the mutual attraction". This is ubiquitous in the two black hole problem.

What one does normally do for binary situations, like two black holes that will collide? The typical attitude has been to cast the equations in such a way that the equations that govern some of the variables are linear. One then obtains a solution for those equations simply superposing known solutions for individual black holes. One then proceeds to solve the remaining equations in full non-linearity.

It is clear that this procedure may yield appropriate initial data only in certain circumstances. Since one does not have control on the non-linear equations one solves, one in the end is left with "whatever initial data the method provides"! There will never be a way out of this!

The resulting solutions usually resemble two black holes, but there usually is added spurious gravitational radiation. The amount of spurious radiation is "what we need to add in order for some of the variables of the problem to simply superpose linearly". It is artificial.

For most solutions the amount of spurious radiation introduced decreases if the black holes are far away (the farther the black holes the more natural is to linearly superpose variables).

So in principle a strategy would be to set up initial data with the black holes far away, evolve them for a while allowing the system to "flush itself" of spurious radiation, and then one ends up with two black holes plunging into a collision under more or less realistic circumstances.

The main problem with this strategy is that it is very costly (currently it is actually impossible) to evolve binary black holes long enough to "flush out the radiation".

In spite of these drawbacks, we will discuss here one of the most popular methods of constructing binary black hole families of initial data, the conformally flat solutions due to York, Bowen, Misner and other collaborators.

J. York in “Sources of gravitational radiation”, L. Smarr, ed, Cambridge 1979

J. Bowen, J. York, Phys. Rev. D21, 2047 (1980)

C. Misner, Phys. Rev. 118, 1110 (1960).

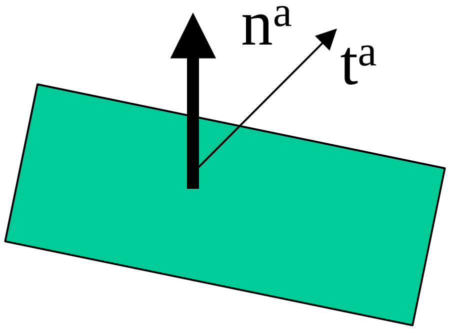
D. Brill, R. Lindquist, Phys. Rev. 131, 471 (1963).

Initial data: the conformal approach

We start by considering the usual decomposition of space-time into 3+1 dimensional space and time. The space-time metric is written as,

$$ds^2 = -(N^2 - N^a N_a)dt^2 + 2N_a dx^a dt + g_{ab} dx^a dx^b$$

Where N is the lapse, N^a is the shift vector and g_{ab} is a three dimensional spatial (positive-definite) metric.



$$t^a = N n^a + N^a$$

The extrinsic curvature is defined as

$$K_{ab} = "\dot{g}_{ab}" = -\frac{1}{2} L_{\vec{n}} g_{ab} = \frac{1}{2N}(\dot{g}_{ab} + N_{a,b} + N_{b,a})$$

and plays the role of "time derivative" of the metric, where n is a vector normal to the foliations.

The "G_{00}" and "G_{0i}" components of the Einstein equations, solely depend on the spatial metric and extrinsic curvature,

$$\nabla_a \left(K^{ab} - g^{ab} K \right) = 0$$
$$^3R - K_{ab} K^{ab} + K^2 = 0$$

The first set of three equations is usually called the "vector constraint" or "momentum constraint". The last equation is the "(super)Hamiltonian constraint" or "scalar constraint". K is the trace of the extrinsic curvature and ^3R is the Ricci curvature of the three metric.

These equations are non-linear, elliptic (no time derivatives) and they hold at every spatial slice. They constrain the initial data of general relativity, which is given by the three metric and the extrinsic curvature, which still carries a certain amount of freedom.

We will not engage in a full discussion of these equations here but jump immediately to a technique for solving them.

The technique (or if you wish a series of choices) consists in assuming that the spatial metric is conformally flat and the trace of the extrinsic curvature vanishes,

$$g_{ab} = f^4 d_{ab}, \qquad K = 0$$

also setting,

$$K_{ab} = \hat{K}_{ab} f^{-2}$$

We get for the constraints,

$$\hat{\nabla}^a K_{ab} = 0,$$
$$\hat{\nabla}^2 f = -\frac{1}{8}\frac{\hat{K}_{ab}\hat{K}^{ab}}{f^7}$$

Where the hats denote flat space operators (gradients and Laplacian).

We are therefore left with a set of flat space equations to solve. The first set of equations is linear, therefore it is reasonably easy to solve and we can superpose solutions. (J. York, JMP14,456 (1973))

York has found solutions for these equations that have ADM momentum and angular momentum at spatial infinity.

$$\hat{K}_{ab} = \frac{3}{2r^2}\left[P_a n_b + P_b n_a - \left(d_{ab} - n_a n_b\right)P^c n_c\right] + \frac{3}{r^3}\left[e_{dac} S^c n^d n_b + e_{dbc} S^c n^d n_a\right]$$

Where P and S are the (three)-momentum and angular momentum of the space-time, and n is a unit normal pointing out from the origin.

In terms of these variables, the ADM momentum and angular momentum of the space-time at spatial infinity are given by,

$$P_a = \frac{1}{8p}\oint_{\infty} K_{ab} d^2 S^b,$$

$$S_a = \frac{1}{8p} e_{abc} \oint_{\infty} \left(x^b K^{cd} - x^c K^{bd}\right) d^2 S_d$$

One can now superpose two of these solutions and solve the Hamiltonian constraint to find initial data that correspond to two black holes.

But how do we know these solutions have anything to do with black holes? Actually, solutions with one hole of this sort were studied by York and Cook and indeed they found apparent horizons. So the space-time has momentum P and an apparent horizon. Later studies have treated these space-times as a perturbation of a single hole and found that they indeed correspond to a black hole with additional radiation (small for small values of the momentum).

R. Gleiser, C. Nicasio, R. Price, JP, PRD57, 3401 (1998).

In fact, we do not even have the complete initial data, we have not tackled the Hamiltonian constraint. This equation can be tackled either numerically or approximately. Before doing that we need to discuss the issue of boundary conditions.

For a single hole the spatial surface we are trying to construct is OK outside the hole. What about inside the horizon? Well, we do not even know where the horizon is unless we evolve the space-time. Therefore, in which domain should we integrate these elliptic equations? Again, we face a choice.

One of the choices is to make the spatial surface look like a throat connecting two asymptotically flat universes. In such case one chooses to make the whole spatial slice symmetric under inversion. The idea is that one asymptotically flat universe will represent the exterior of the black hole and the other one the interior. This corresponds to the traditional picture of the "wormhole".

In practice it means that one pick a radius a, and requires that the solution be symmetric under inversion through the sphere at radius a. When one has more than one throat, one has to require that inversion through all throats be implemented. This can be done by the method of images, but it might take summing an infinite summation.

This was studied in great detail by G. Cook and others, Phys.Rev.D47:1471,(1993) This inversion type of solution led to the first binary black hole solution to the initial value problem, found by Misner. He considered "momentarily stationary" black holes at a given separation. That meant K_{ab}=0. The Hamiltonian constraint then simmply becomes,

$$\nabla^2 f = 0$$

And one wishes to find solutions which respect the inversion symmetry through two throats. Misner accomplished this by considering the solution of the Laplace equation in bi-spherical coordinates. He starts from an ordinary, flat "donut" (torus),

$$ds_D{}^2 = dm^2 + (dq^2 + \sin^2 q\, df^2) \qquad -p < m < p$$

And notices that if one could "break up" the donut at π=0 and identify the points of the breakup one would end up with a wormhole.

To achieve this, Misner notices that the metric of flat space in bi-spherical coordinates looks like a conformal factor times a donut,

$$ds_F{}^2 = (\cosh m - \cos q)^{-2} ds_D{}^2$$

And since flat space obviously solves the Hamiltonian constraint, if one could make the above solution periodic in π, one would have the desired result. This can be achieved by superposition (method of images),

$$f = \sum_{n=-\infty}^{\infty} \left[\cosh(m + 2nm_0) - \cos q\right]^{-1/2}$$

This solution is periodic with period $2\mu_{0.}$ This parameter governs the ratio of radius to separation of the holes. For a fixed radius it measures the separation of the holes.

If one compares with the asymptotic ($\mu^2+\theta^2$)->infinity form of the Schwarzschild metric in bispherical coordinates,

$$ds_S{}^2 = \left[1 + m\left(m_0{}^2 + q^2\right)^{1/2}\right]\left[\frac{4}{\left(m^2 + q^2\right)}\right] ds_D{}^2$$

One finds that the Misner initial data has an ADM mass of,

$$M = 4\sum_{-\infty}^{\infty} (\sinh n m_0)^{-1}$$

And the separation of the throats can be found by computing the integral along the throat of the shortest closed loop through the wormhole

$$L = \int_{-\infty}^{\infty} f^2(m, q = p)\, dm$$

This integral can indeed be evaluated in closed form! Its expression is given in terms of elliptical integrals (see Misner) and is a function of μ_0.

Brill and Lindquist considered the solution of the same problem as Misner, but without requiring symmetry through the throats. Their solution for the conformal factor is remarkably simple,

$$f = \frac{M_1/2}{r - r_1} + \frac{M_2/2}{r - r_2}$$

It is essentially given by the Newtonian potential of each hole!

What can one do for solutions with momentum? One can generalize both the Misner and Brill-Lindquist solutions. Cook found the generalization of the Misner solutions and Brandt and Bruegman the Brill-Lindquist generalization to the case of momentum.

An interesting aspect is that for "slowly moving" holes one can find an approximate solution. Since the extrinsic curvature is linear in the momentum of the holes (either linear or angular), if the black holes are slowly moving, the value of K_{ab} will be small.

Therefore the right hand side of the Hamiltonian constraint will be small too, and at a zeroth level of approximation one can ignore it. What we are saying is that if one takes a conformal factor that solves the Hamiltonian constraint with zero momentum, and takes the extrinsic curvature of Bowen-York, one has an approximate solution of the initial value problem. This procedure can be iterated, now solving for the order P^2 correction to the conformal factor, and so on.

Since in perturbation theory one is interested in approximate solutions, it turns out that using approximate initial data is good enough for the situations of interest.

There is one caveat: if one wishes to compute the ADM mass of the spacetime, it is given by the integral,

$$M = \frac{1}{4p} \oint_{\infty} \nabla_a f \, d^2 S^a$$

Because the ADM mass depends solely on the conformal factor, and this in turn satisfies a highly non-linear equation, the ADM mass is poorly approximated by polynomial approximations. (For instance, the zeroth order solution we considered had no contribution to the ADM mass due to the momentum).

Ignoring this last point, the approximate solutions really work very well. Because the extrinsic curvature is linear in the momentum, it grows very rapidly when one increases the momentum of the collision. The conformal factor, due to the nonlinearity of the Hamiltonian, remains bounded. Therefore, as soon as the momentum grows, the extrinsic curvature (for which we have an exact solution!) completely dominates the initial data!

J. Baker et. al. PRD55, 829 (1997)

We are now ready to consider the evolution of the initial data we have discussed using the “close limit”. That is, we assume we are considering black holes that are close to each other and therefore the situation should not differ too much from a single, distorted black hole.

For historical reasons, we will consider the Misner case (symmetrized time-symmetric initial data). Here the extrinsic curvature is zero and the metric,

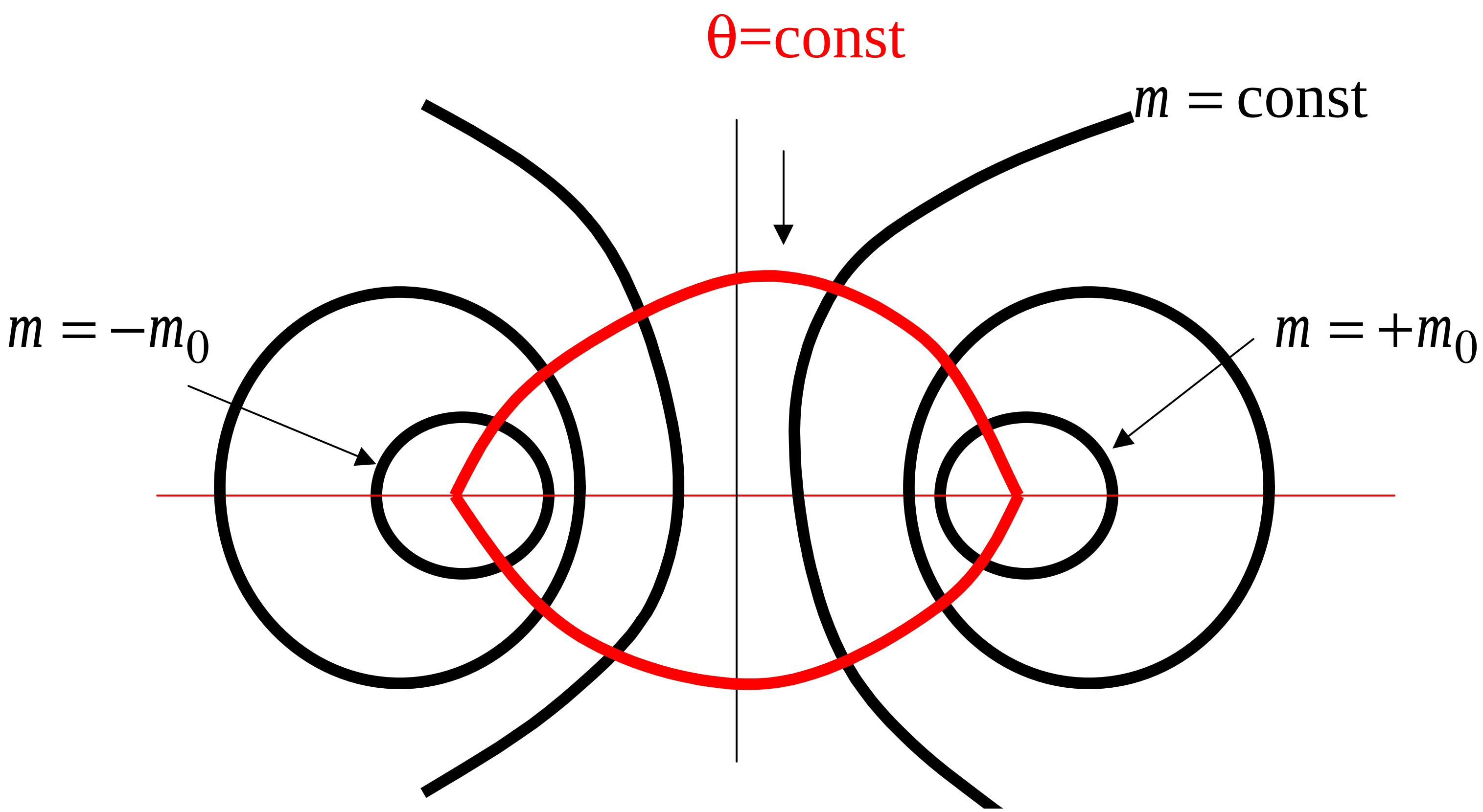
$$ds^2{}_{\text{Misner}} = a^2 f^4 [d\mu^2 + d\eta^2 + \sin^2 \eta \, d\varphi^2]$$

Where a is a constant, μ,η are bispherical coordinates and f is the conformal factor we discussed before.

The bispherical coordinates are related to ordinary polar coordinates by,

$$R = a\sqrt{\frac{\cosh m + \cos h}{\cosh m - \cos h}}$$

$$q = \tan^{-1}\frac{\sin h}{\sinh m}$$



So we can translate the Misner geometry into more familiar polar coordinates:

$$ds^2{}_{\mathrm{Misner}} = \Phi^4(R, q, m_0)\left(dR^2 + R^2\left[dq^2 + \sin^2 q\; dj^2\right]\right)$$

with the new conformal factor Φ given by,

$$1 + \sum_{n\neq 0} \frac{\sqrt{\cosh m - \cosh h}}{\sqrt{2\sinh^2 nm_0 + \sinh m \sinh 2nm_0 + (\cosh m - 1)\cosh 2nm_0 + 1 - \cosh h}}$$

Which can be rewritten as,

$$\Phi = 1 + \frac{a}{R}\sum_{n\neq 0} \frac{1}{\sqrt{\left(1 + \frac{a^2}{R^2}\right)\sinh^2 nm_0 + \frac{a}{R}\cos q \sinh 2nm_0 + \frac{a^2}{R^2}}}$$

And if you were college students taking an electricity and magnetism class, you would quickly recognize this as the generating function of the Legendre polynomials (!).

Which therefore allows us to write the conformal factor as,

$$\Phi = 1 + 2\sum_{\ell=0,2,4} k_{\ell}(m_0)\left(\frac{M}{R}\right)^{\ell+1} P_{\ell}(\cos q)$$

And the only dependence on the separation μ_0 is given in the coefficient kappa,

$$k_{\ell}(m_0) = \frac{1}{\left(4\sum_{n=1}^{\infty}\frac{1}{\sinh nm_0}\right)^{\ell+1}}\sum_{n=1}^{\infty}\frac{(\coth nm_0)^{\ell}}{\sinh nm_0}$$

If one now evaluates explicitly the l=0 contribution to the sum, one can rewrite the conformal factor as,

$$\Phi = \left(1 + \frac{\mathrm{M}}{2R}\right)F$$

With the function F given by,

$$F = 1 + 2\left(1+\frac{M}{2R}\right)^{-1} \sum_{\ell=2,4,\ldots} k_\ell \left(\frac{M}{R}\right)^{\ell+1} P_\ell(\cos\theta)$$

We now notice that we have the three metric written as a conformal factor that to leading order looks the same as the form of the conformal factor of the Schwarzschild metric in isotropic coordinates!

$$ds^2 = -\frac{\left(1-\frac{M}{2R}\right)^2}{\left(1+\frac{M}{2R}\right)^2} dt^2 + \left(1+\frac{M}{2R}\right)^4 \left(dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2\right)$$

It is therefore natural to identify the coordinate R with the isotropic radius of Schwarzschild,

$$R = \frac{1}{4}\left(\sqrt{r} + \sqrt{r-2M}\right)^2$$

And we can therefore finally (!) make contact with the usual Schwarzschild coordinates. Our spatial metric reads,

$$ds^2 = \Phi(r,\theta)^4 \left(\frac{dr^2}{1-\frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right)$$

But how is this metric a “perturbation” of Schwarzschild? In order to see this we need to study the dependence on μ_0, which characterized the separation of the holes. Since all the dependence came through the kappa coefficients, we just need to know that (this takes a small effort to prove, see Anninos et al. PRD52, 4462 (1995)),

$$\kappa_\ell \approx \frac{\zeta(\ell+1)}{|4\ln\mu_0|^{\ell+1}}, \qquad \frac{L}{M} \approx \frac{\pi^2}{|4\ln\mu_0|} \quad \text{for } \mu_0 \to 0$$

So we could go back now to our notes on the Regge-Wheeler notation and, starting with this metric, we could read off the various coefficients of the perturbation. Fortunately for us, the higher order multipoles are heavily suppressed given the explicit form of kappa. This is really a blessing, since the conformal factor is raised to the fourth power. If suppression did not happen, one would obtain contributions for l=2 coming from products of all the higher order ell's, yielding the problem intractable in practice! The suppression allows us to only consider the terms linear in kappa for any given ell, even if we wish to consider higher order ells (See Anninos et al).

These terms turn out to be all even parity, so reading off the Regge-Wheeler coefficients,

$$H_2 = K = 8\sqrt{\frac{4p}{2\ell+2}}k_\ell(m_0)\frac{(M/R)^{\ell+1}}{1+M/2R}$$

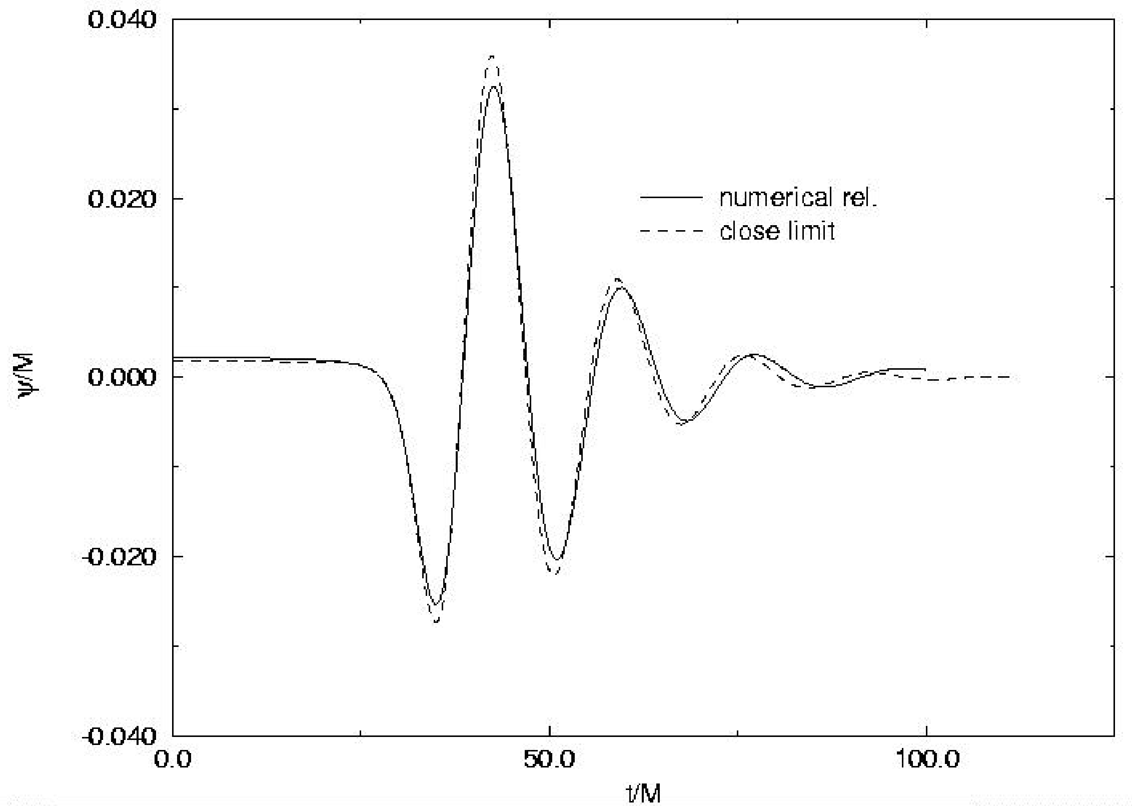
And go ahead and compute the Zerilli function,

$$\Psi = \frac{(\ell+2)(\ell-1)}{(\ell+2)(\ell-1)+6M/r} q$$

$$q = 2r(1-2M/r)\left[K - \sqrt{1-2M/r}\left(\frac{rK}{\sqrt{1-2M/r}}\right)\right] + \ell(\ell+1)rK$$

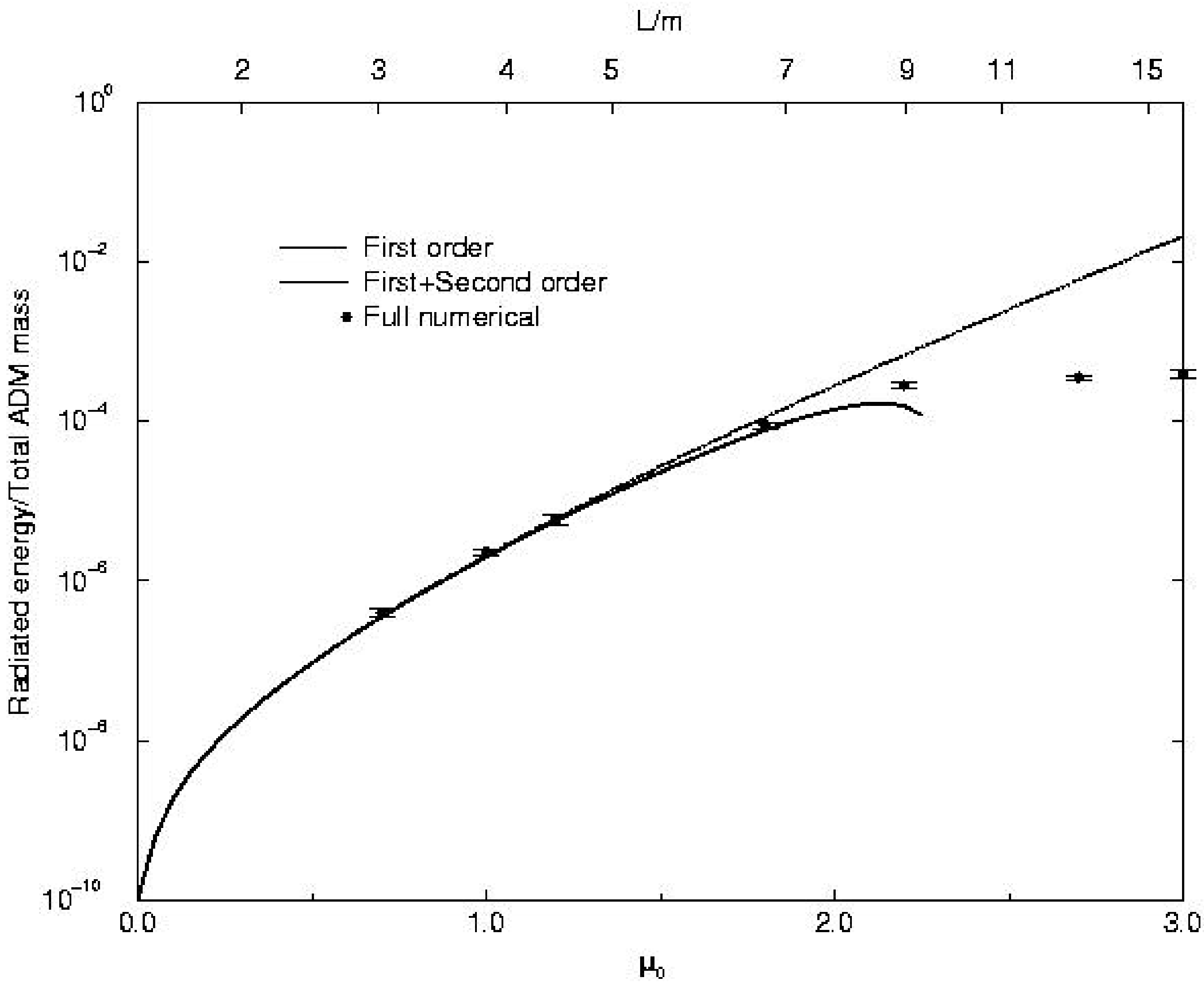
And we are ready to evolve! Notice that all the dependence on the separation of the holes comes linearly as a prefactor of kappa. Therefore one can pull this factor out, make one evolution and rescale the results to obtain the evolution for all values of the separation! The computational economy is astonishing.

If one proceeds along these lines these are the results one gets:



Zerilli function for a separation of μ_0=1.2 as a function of time for a fixed radial position, compared with the full numerical relativity results of Seidel's group.

If one computes the radiated energy as a function of the separation one gets.

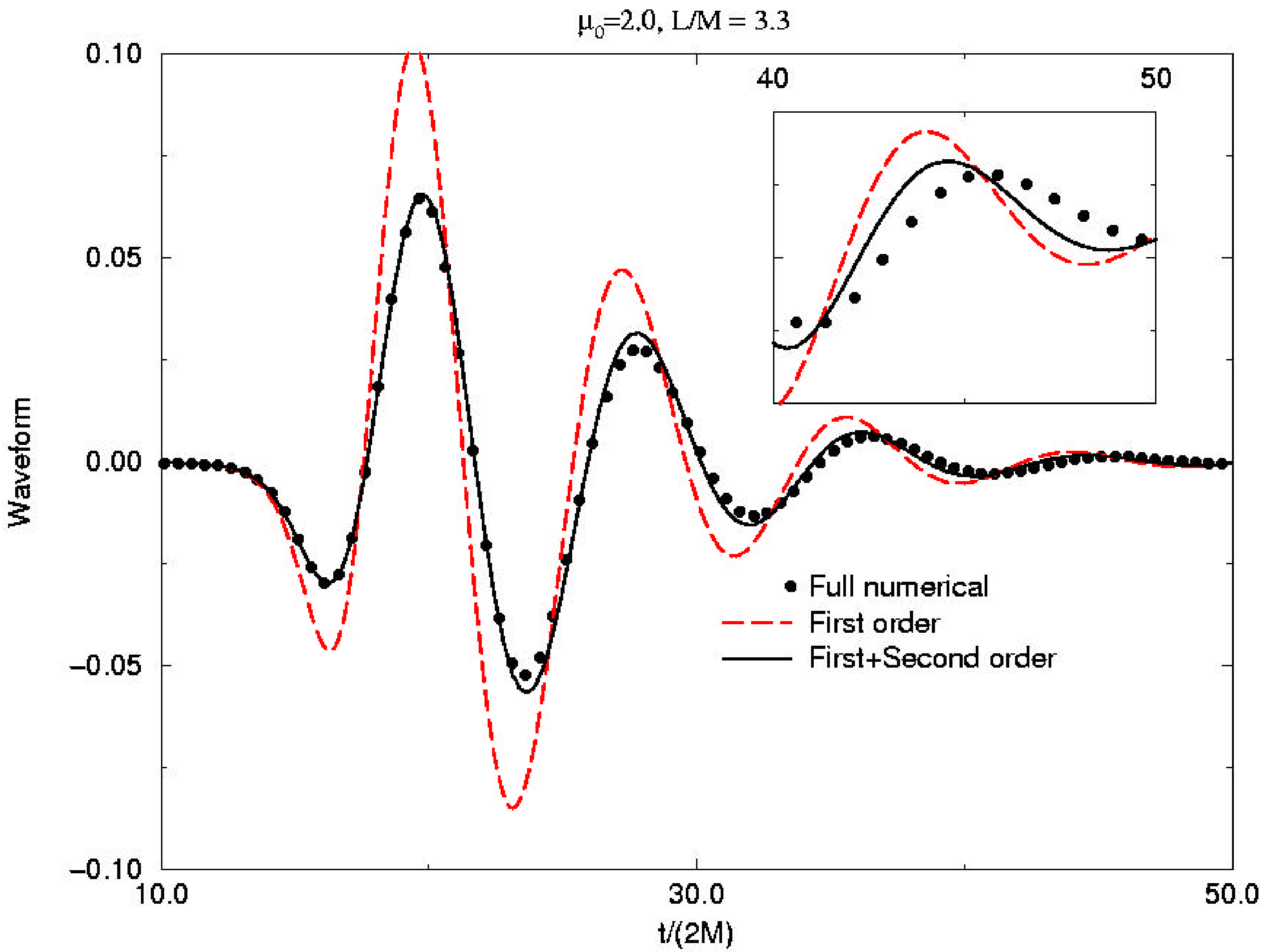


The figure includes second order results. These are technically more complex, but the idea is similar to what we described.

To give an idea of the physics involved (since both measures of the separation are quite artificial when the holes are close), it is worthwhile mentioning that for separations larger than μ_0=1.2 the two holes have separate apparent horizons and that for separations larger than μ_0=1.8 the two holes have separate event horizons. These facts were obtained from the full numerical simulations. In our approach we always treat the problem as if it had a single horizon.

We see that the energy radiated in a (head-on) black hole collision is small, less than 0.1% of the mass. One cannot seem to radiate more by dropping the black holes from farther away. This is due to the fact that the increased kinetic energy also increases the mass, so the fraction of the mass radiated remains constant. This is a feature of these families of initial data, that is also true for boosted and spinning holes: they all “level out” in the amount of radiation produced at values less than 1% of the total mass.

And if one pushes perturbation theory close to the limit of breakdown, but immediately before it, one finds the best place for second order corrections to help:



The Misner initial data were a real “tour de force” from the point of view of calculus, mainly due to the presence of the bispherical coordinates. It illustrates how the issue of “symmetrizing” the initial data complicates calculations. Compared to this, studying the close limit collision of Brill-Lindquist type holes is much simpler. The conformal factor is (assume black holes at +/- z_0 on the z axis),

$$\Psi = 1 + \frac{1}{2}\left(\frac{m}{\sqrt{R^2 \sin^2 q + (R\cos q - z_0)^2}} + \frac{m}{\sqrt{R^2 \sin^2 q + (R\cos q + z_0)^2}} \right)$$

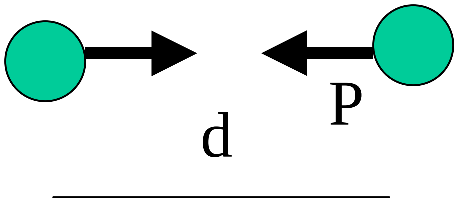
And expanding in spherical harmonics we get,

$$\Psi = 1 + \frac{\mathrm{M}/2R}{1 + \frac{M}{2R}} \sum_{\ell=2,4,\ldots} \left(\frac{z_0}{M}\right)^{\ell} \left(\frac{M}{R}\right)^{\ell}$$

It is instructive to compare the conformal factors of Misner and Brill-Lindquist. One sees that the only real difference is that instead of the kappa factor one has z/M. That means that for a given multipole, the two problems really differ in a single number.

Since the measure of the separation of the black holes is not a well defined physical quantity, for close black holes this issue becomes really more pressing than the difference in the coefficient we see in the Brill-Lindquist and Misner cases! One could, by definition, identify certain value of μ_0 with certain values of z_0 in such a way that the two problems give the same results for radiated energies and waveforms. In a sense, this is good, it would be troublesome that a small difference in the initial data as the symmetrization would cause significant differences in the final physical results.

What about black holes boosted towards each other?

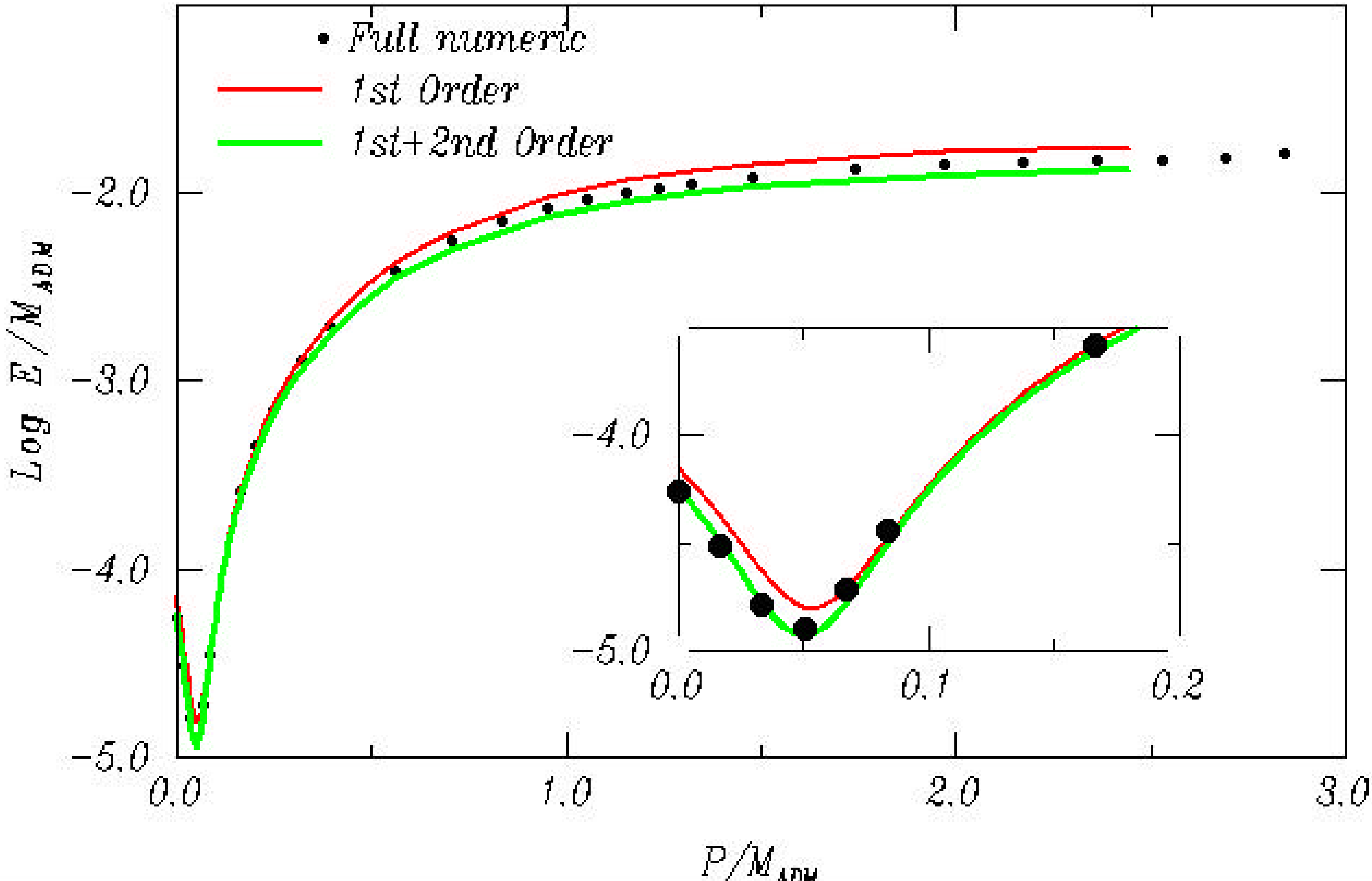


In this case one has a non-vanishing extrinsic curvature. The extrinsic curvature bears the same relation with the time derivative of the Zerilli function as the metric does with the function itself.

If we go with the slow approximation we discussed before, the only thing we need to do different is to include the initial time derivative, since in this approximation the conformal factor (and therefore the Zerilli function) are the same as in the Misner (or Brill-Lindquist) case.

And since the problem is linear, we can break it up into two pieces: in one piece we set the initial time derivative to zero and the initial Zerilli function as in the Misner (or Brill-Lindquist problem) and another piece with non-vanishing time derivative but vanishing initial data for the Zerilli function.

The first problem we already solved. So we solve the second one and add up the results. One gets,



Radiation as a function of momentum for initial separation mu0=1.4

This figure merits several comments:

a) There is this "dip" in the radiated energy. As one starts to smash the black holes harder and harder starting from rest one finds that the energy initially diminishes!

b) We were supposed to be looking at a "slow approximation", yet the results work well even for very large values of P.

c) One cannot exceed 1% of radiation even for very large values of P.

What is going on?

The answer lies in the construction of the initial data. As you remember we used an exact solution for the extrinsic curvature and an approximate (slow) solution for the conformal factor.

The extrinsic curvature grows linearly in P. The conformal factor grows much more slowly with P due to the nonlinearity of the Hamiltonian constraint.

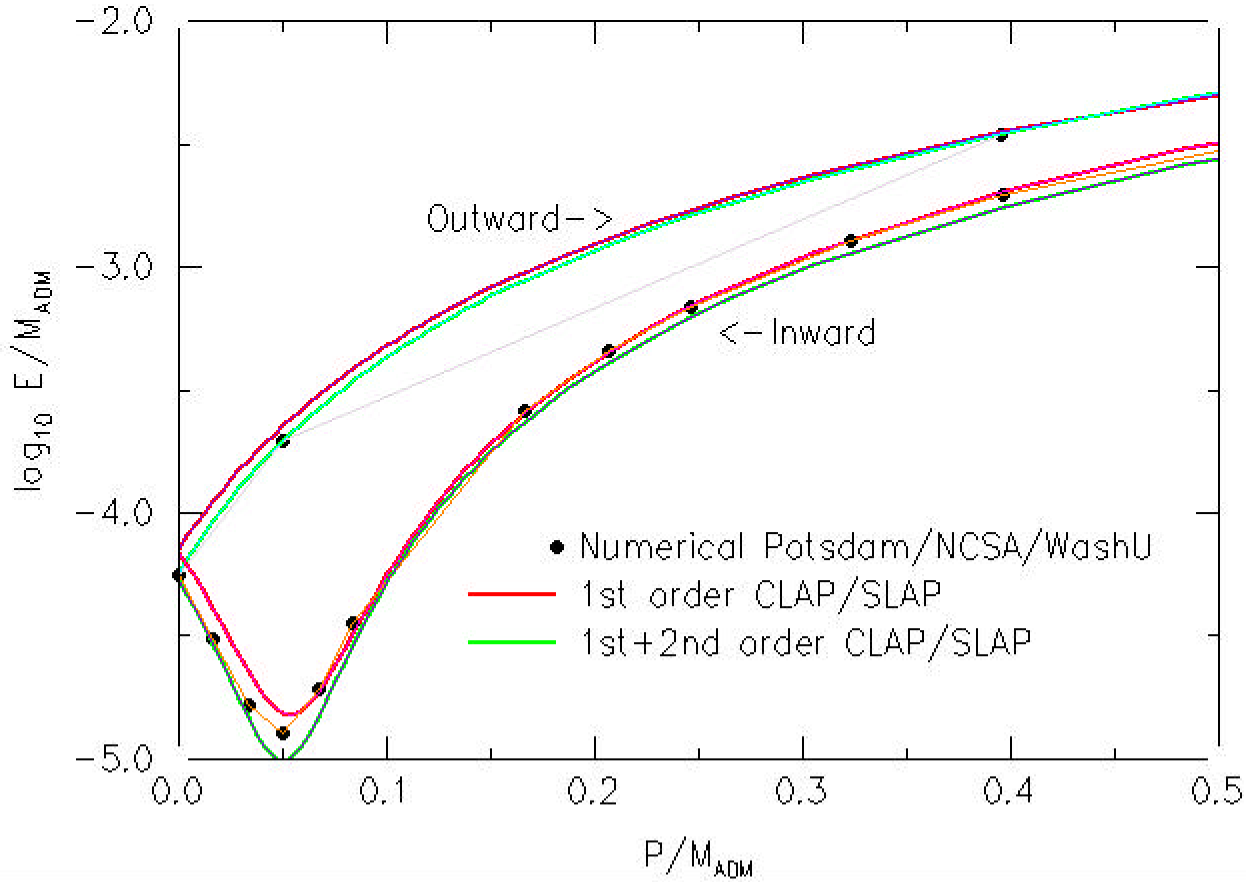
Therefore as one increases the momentum, the data from the extrinsic curvature "overtakes" the data of the conformal factor, and quickly dominates the problem. As the extrinsic curvature is exact, for large values of the momentum, the initial data is quite good. Sure, the slow approximation yields a questionable conformal factor, but this portion of the data is irrelevant for large values of P.

This "overtaking" also leads to the dip, and is called "momentum dominance". It is a simple fact that went overlooked.

R. Gleiser, C. Nicasio, R. Price, JP, Phys. Rev. D59, 044024 (1999).

At the “dip” there is a cancellation occurring. Therefore linearized theory does a bad job in following the evolution, since the higher order terms that dominated are not taken into account. Second order perturbation theory does a much better job.

The dip is linear in the momentum, for black holes boosted away from each other it is not there.



How much spurious energy in one of these holes?

The Bowen-York construction is such that if one carries it out for a single boosted hole one does not get a single boosted Schwarzschild spacetime. One gets spurious gravitational radiation and a black hole that “rings down” to Schwarzschild.

Similar comments apply in the case of a single rotating hole. One does not get a Kerr hole, but a hole that rings down to Kerr while emitting gravitational radiation.

As long as the momentum and angular momentum is small, we can treat these problems as single distorted holes and evaluate the resulting radiation.

R. Gleiser, C. Nicasio, R. Price, JP, Phys. Rev. D57:3401-3407,1998
R. Gleiser, G. Khanna, JP, gr-qc/9905067

We have treated both situations as perturbations of Schwarzschild.

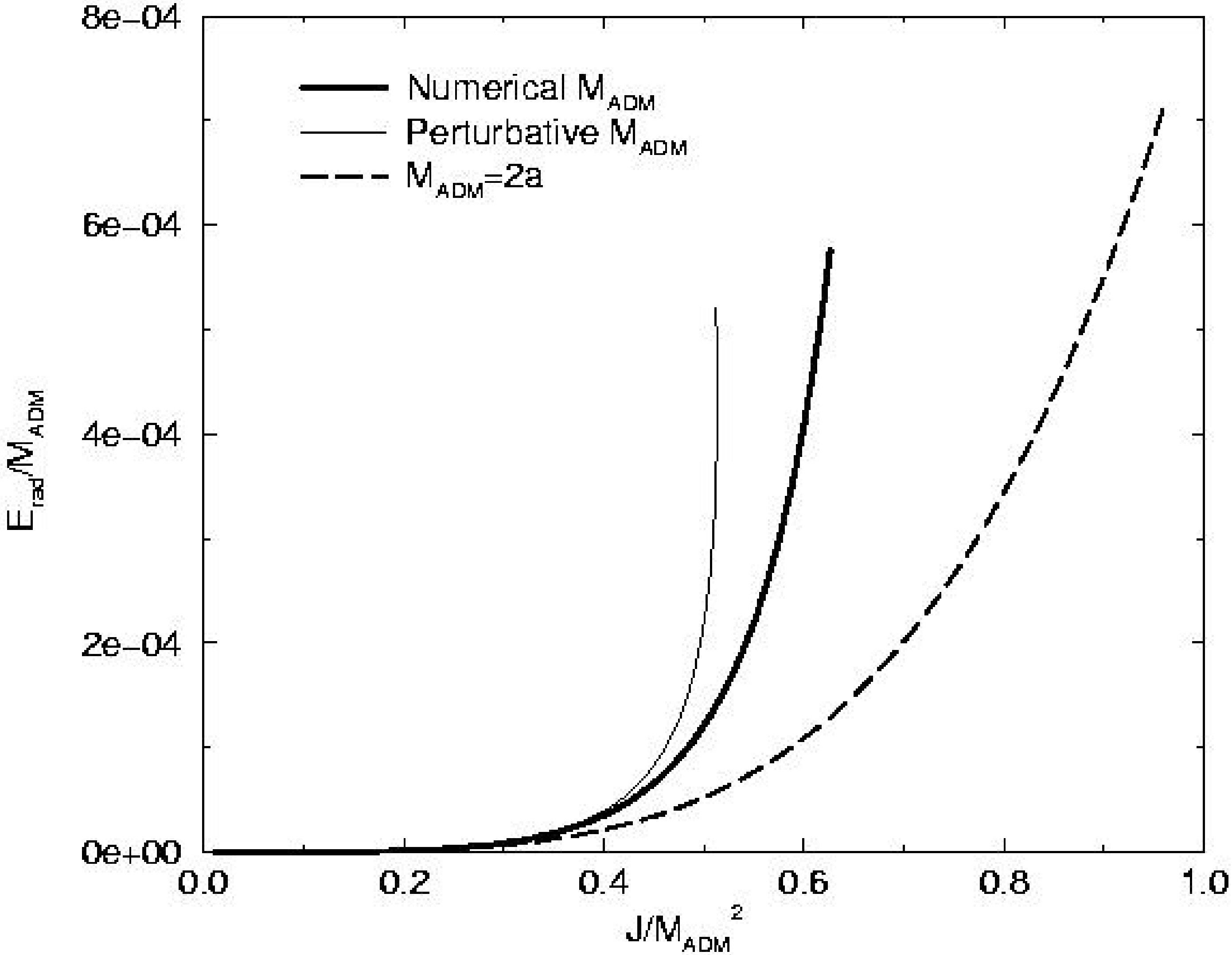
In the spinning case, the first order perturbations correspond to the static rotation that makes Kerr different from Schwarzschild, one needs to go to second order to evaluate the radiation.

In the boosted case, the first order piece corresponds to the boost, it turns out to be pure gauge, and after removal, one needs to evolve the second order pieces.

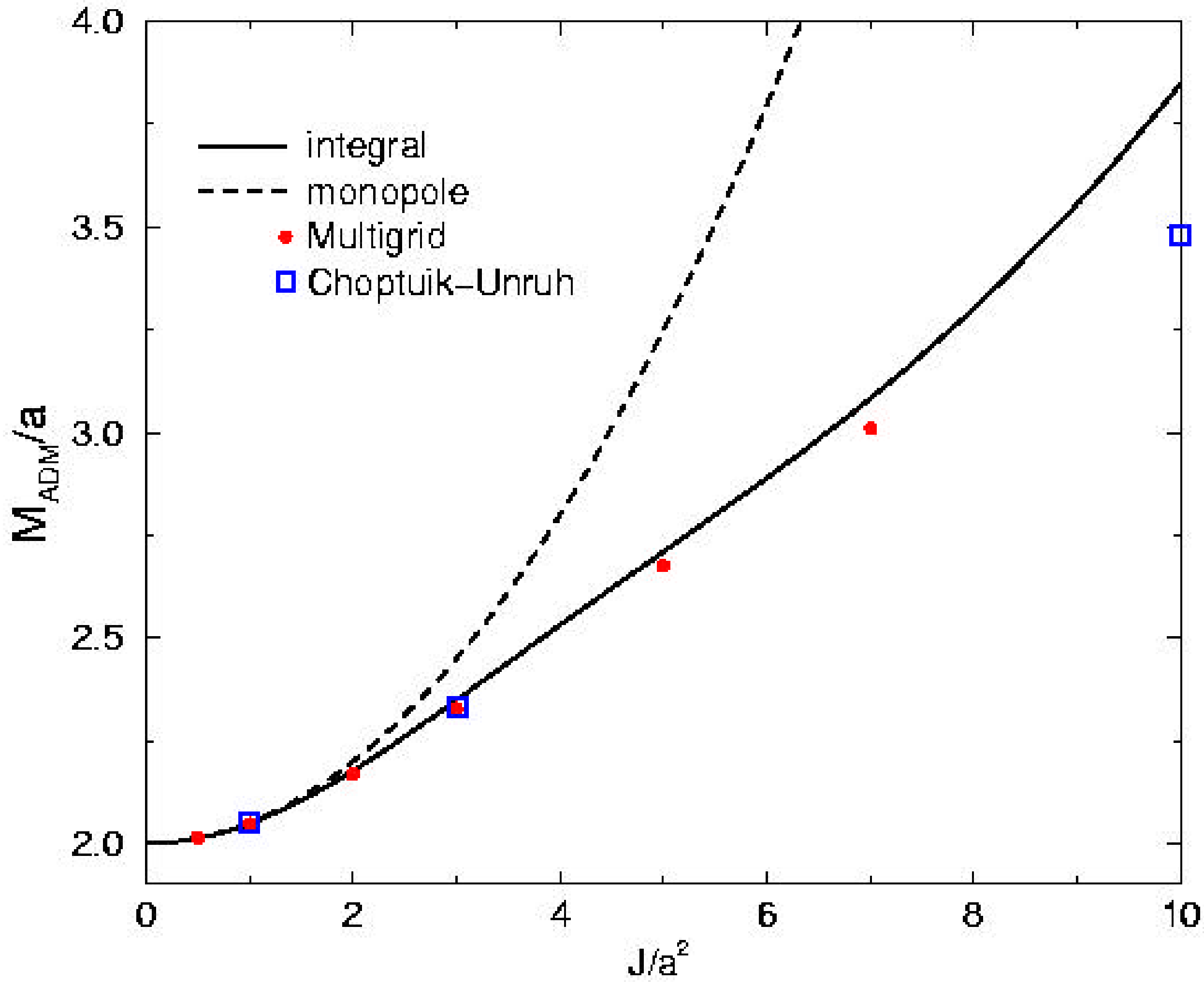
In both cases one finds that each hole carries more energy than a total collision for values of the momentum greater than 0.5.

Which is surprising, since we see no such effect in boosted collisions, this suggests some miraculous cancellation is occurring!

The radiated energy by a single spinning Bowen-York hole. The curves show the importance of using the right conformal factor in the ADM mass calculations.

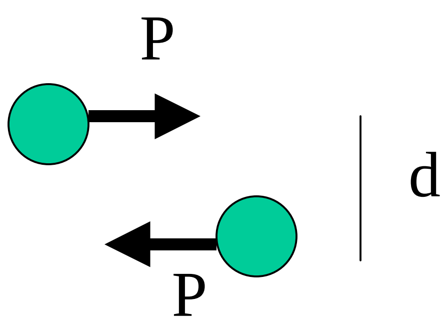


The conformal factor can be found to high accuracy iterating the "slow approximation" we discussed.



Inspiralling non-spinning black holes: Kerr perturbations?

Not so obviously a better choice than Schwarzschild. In any family of initial data for the non-head-on collision of non-spinning holes, the Kerr spacetime only features with a=0



$a=P\,d/M^2$

It leads to a weird perturbation theory: one is working to linear order in a, with an equation where the background metric contains all orders of a! (Inconsistencies with initial data?)

Easy to get confused...

Moreover, there are practical problems:

- Bowen-York initial data is conformally flat. The Kerr spacetime is not known to admit conformally flat spatial slices. It is conformally flat to first order in a.

- Little experience with Teukolksy equation in the time domain. 2+1 dimensional problem. Progress has been made by Krivan, Laguna and Papadopoulos.

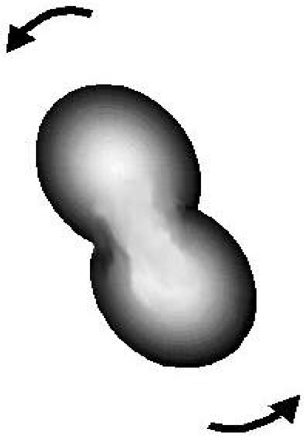
- Also the setup of initial data in the Teukolsky formalism had to be worked out. Campanelli, Krivan, Lousto, Baker, Khanna have done it.

- Radiation of angular momentum: formulas had to be set up in the Zerilli and Teukolsky formalims: Gleiser, Khanna, Campanelli, Lousto just finished it.

Inspiralling black holes:

(G. Khanna, J. Baker, H.-P.Nollert, P. Laguna, R. Gleiser, R.Price)

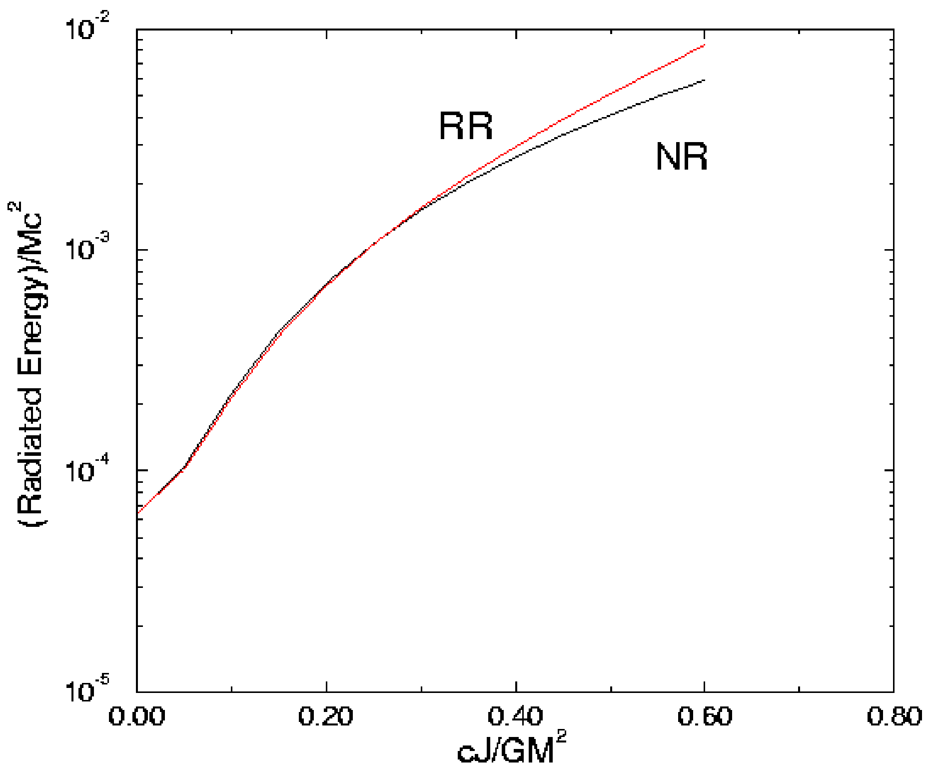
gr-qc/9905081



We treated the problem both as a perturbation of a rotating and a non-rotating black hole. There are several technical complications.

Perturbing a non-rotating black hole works better!

The system radiates somewhat less than 1% of its mass.



The radiated angular momentum puzzle:

Title:
angmom.eps
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with a preview included in it.
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PostScript printer, but not to
other types of printers.

Small quantity, due to interference in phase of different modes.

Both curves obtained with same initial data, evolved with different evolution equations.

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angmom2.eps
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PostScript printer, but not to
other types of printers.

There even is a dip!

Summary:

The close limit approach has been hugely successful, delivering a lot of physics with very low cost. We cannot do the problems of most relevance, but we can still learn a lot.

Future steps: collisions of spinning holes, and a better understanding of Kerr perturbation theory.

A lesson to be followed throughout the binary black hole problem: we need more and novel approximation techniques to compare with numerical results. This is the way physics has always been done, especially with such a big problem.