Colliding black holes using perturbation theory

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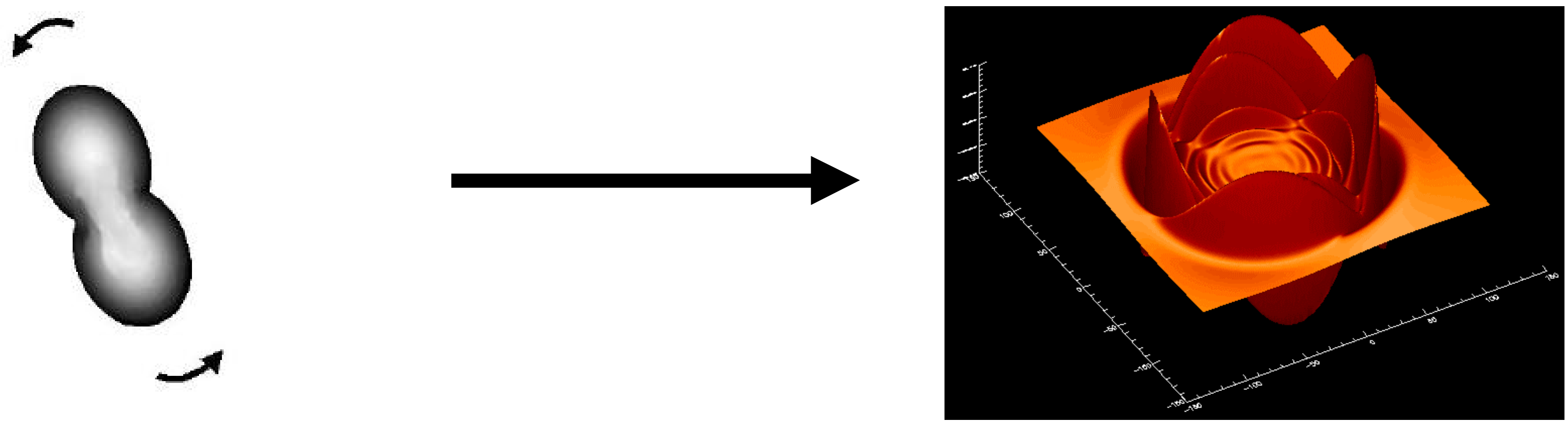
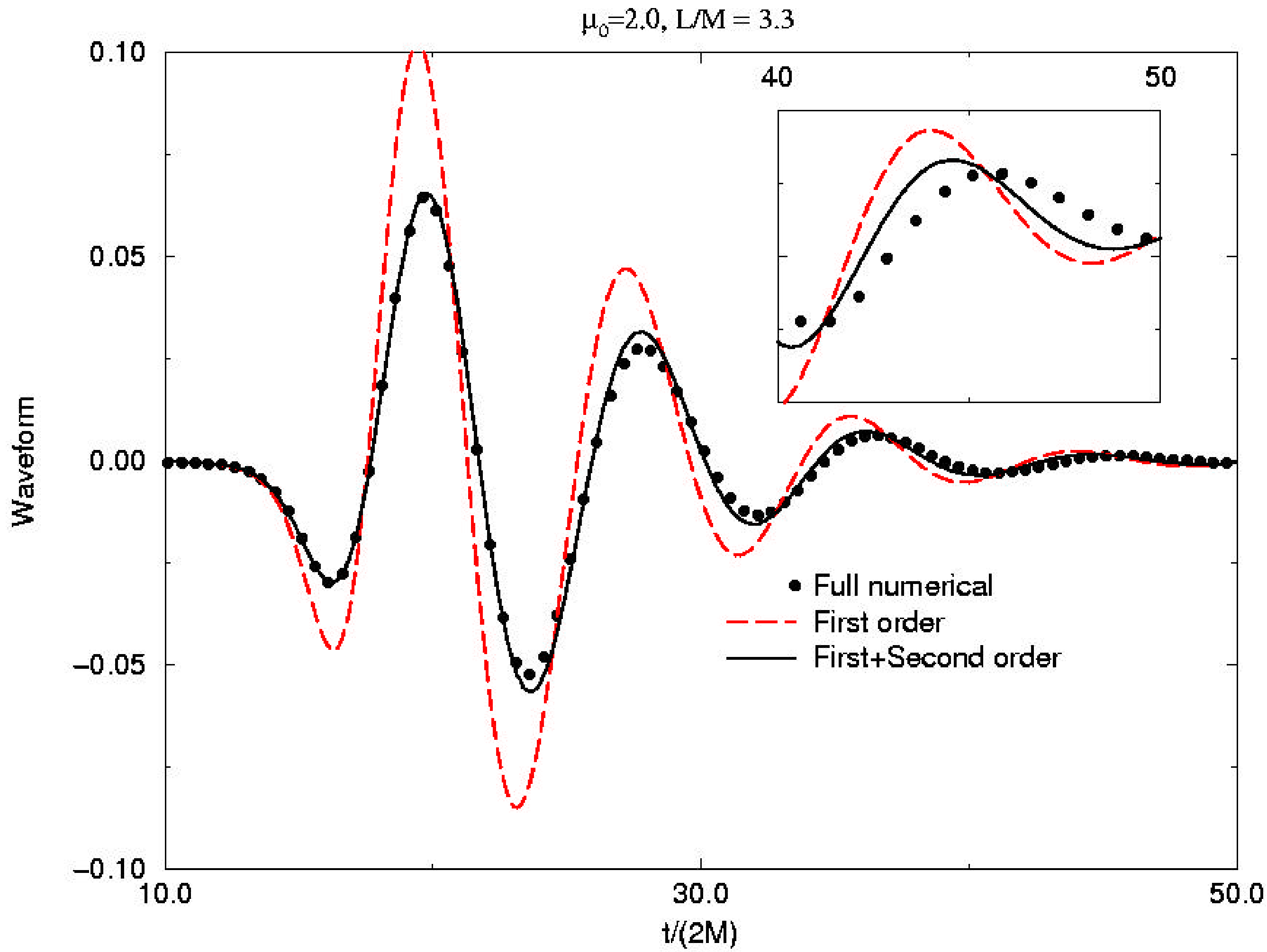
1. First lecture: black hole perturbation theory.

2. Black hole collisions: initial data and test cases.

3. Boosted black holes and inspiralling collisions.



Motivation: the final stages of black hole collisions are well approximated by a single distorted black hole that "rings down" into equilibrium.



Plan of first lecture:

- Perturbation theory, gauges.
- The Zerilli and Regge-Wheeler equations.
- The integration of these equations.
- Perturbations of rotating black holes.

Plan for today: to describe the framework for doing black hole perturbation theory. We will start with perturbations of Schwarzschild black holes and then consider the Kerr case.

There are many frameworks for doing perturbation theory, I will refer to the two most popular ones: the first one is based on direct examination of the Einstein equations and leads to the so called Zerilli-Regge-Wheeler equations. The second formalism is based on the Newman-Penrose formulation of GR and leads to the Teukolsky equation (for rotating holes) or the Bardeen-Press equation for the non-rotating case.

Both frameworks have pros and cons. The ZRW quantities are first order in the time derivatives of the metric. The Teukolsky function is second order. The latter however, has a better physical motivation: it is a component of the Weyl spinor in a given null tetrad; moreover it can be generalized to rotating holes.

Perturbations, the idea:

Give yourselves a one-parameter family of metrics, that are "nearby" a given exact metric $g^{(0)}$

$$g_{mn} = g_{mn}^{(0)} + e\, g_{mn}^{(1)} + e^2 g_{mn}^{(2)} + \ldots$$

Substitute in the Einstein equations $R_{mn} = 0$

And keep terms order by order in epsilon.

Generically, this sounds pretty much like the way one proceeds any time one studies a problem in perturbation theory in physics.

However, in general relativity there is an added complication: the issue of gauge. That is, how do I know that I am perturbing "the geometry" as opposed to "the metric" (that is, making just a coordinate transformation)?

The issue of gauge:

If one is to confuse coordinate transformations with small perturbations the coordinate transformations must be “small”, $x'^{m} = x^{m} + e\, x^{m}$

It is well known (exercise) that up to first order, an infinitesimal coordinate transformation translates itself, acting on a tensor, into a Lie derivative,

$$g'_{mn}{}^{(1)} = g_{mn}{}^{(1)} + L_x\, g_{mn}{}^{(0)} = g_{mn}{}^{(1)} + x_{(m;n)}$$

Looks like E&M gauge transformation

Notation: $g_{mn}{}^{(1)}$ is usually called h_{mn}

For higher orders in epsilon, the situation is less geometrical and more involved, see Bruni et. al. CQG 14, 2585 (1997); gr-qc/9609040.

Gauge invariance:

To deal with this issue there are two approaches:

a) Construct gauge invariant quantities. Moncrief, Ann. Phys. 88, 323 (1974)

b) Worked in a fixed gauge.

Remarkably under-appreciated is the fact that these two approaches are essentially the same.

As long as one works in a (uniquely fixed) gauge, the quantities one is dealing with are gauge invariant, in the sense that one can translate them into any gauge one wants.

This is kind of obvious, but somehow it leads to interminable arguments...

The Regge-Wheeler notation and the Regge-Wheeler gauge:

(T. Regge, J.A. Wheeler, Phys. Rev. 108, 1063 (1957))

We are interested in considering perturbations in which the background spacetime is the Schwarzschild metric. It therefore makes sense to expand the perturbations in spherical (tensor) harmonics.

Under rotations in the theta-phi sphere, h_{tt} , h_{tr} , h_{rr} behave like scalars. $(h_{tq}; h_{tf})$ and $(h_{rq}; h_{rf})$ behave as vectors.

$$\begin{pmatrix} h_{qq} & h_{qf} \\ h_{fq} & h_{ff} \end{pmatrix}$$ is a tensor.

We can then proceed to decompose. For the scalars it is as usual, they are characterized by “quantum numbers” L, m, and are given by a function of (r,t) times a spherical harmonic $Y_{LM.}$ The parity of these objects is $(-)^L$.

There are two kinds of vectors, of different parity. One kind is simply given by the gradient of a Y_{LM} and has parity $(-)^L$.
The other is the "dual vector" (contraction with the Levi-Civita symbol in two dimensions, and has parity $(-)^{L+1}$ (it's a pseudo-vector).

Finally, there are three kinds of tensors. One is given by the double covariant gradient of Y_{LM} and has parity $(-)^{L.}$ Another is a constant times the metric of the sphere, also with parity $(-)^L$. The last is obtained by "dualizing" the first tensor with the Levi-Civita symbol in each index; it has parity $(-)^{L+1}$.

We can therefore group perturbations into two separate groups, depending on their parity behavior with respect to the sphere:
even $(-)^L$ and odd $(-)^{L+1}$ parity perturbations.

The corresponding metric tensors are:

$$h_{m\,n}{}^{\text{odd parity}} = \begin{pmatrix} 0 & 0 & -h_0(t,r)(1/\sin(q)\partial/\partial f)Y_{LM} & h_0(t,r)(\sin(q)\partial/\partial q)Y_{LM} \\ 0 & 0 & -h_1(t,r)(1/\sin(q)\partial/\partial f)Y_{LM} & h_1(t,r)(\sin(q)\partial/\partial q)Y_{LM} \\ \text{sym} & \text{sym} & h_2(t,r)(\partial^2/\sin(q)\partial q\,\partial f - \cos(q)/\sin^2(q)\partial/\partial f)Y_{LM} & \text{sym} \\ \text{sym} & \text{sym} & h_2(t,r)(\partial^2/\sin(q)\partial f\,\partial f + \cos(q)\partial/\partial q - \sin(q)\partial^2/\partial q^2)Y_{LM} & -h_2(t,r)(\sin(q)\partial^2/\partial q\partial f - \cos(q)\partial/\partial f)Y_{LM} \end{pmatrix}$$

$$h_{mn}{}^{\text{even parity}} = \begin{pmatrix} (1-2M/r)H_0(t,r)Y_{LM} & H_1(t,r)Y_{LM} & h_0(t,r)(\partial/\partial q)Y_{LM} & h_0(t,r)(\partial/\partial f)Y_{LM} \\ \text{sym} & (1\text{-}2\text{M/r})^{-1}H_2(t,r)Y_{LM} & h_1(t,r)(\partial/\partial q)Y_{LM} & h_1(t,r)(\partial/\partial f)Y_{LM} \\ \text{sym} & \text{sym} & r^2\left[K(t,r)+G(t,r)(\partial^2/\partial q^2)\right]Y_{LM} & r^2G(t,r)\left[\partial^2/\partial q\,\partial f - \cos(q)/\sin(q)\partial/\partial f\right]Y_{LM} \\ \text{sym} & \text{sym} & \text{sym} & \text{r}^2\begin{bmatrix} K(t,r)\sin^2(q) + \\ G(t,r)\left(\partial^2/\partial f^2 + \sin(q)\cos(q)\partial\ /\partial q\right) \end{bmatrix}Y_{LM} \end{pmatrix}$$

$$\text{Odd functions}: \; h_0, \; h_{1,} \; h_2$$

$$\text{Even functions}: \; H_0, \; H_1 \; , H_2, \; \text{G}, \; \text{K}, h_0, \; h_1$$

The Regge-Wheeler gauge:

Perform a gauge transformation that eliminates the second angular derivatives. The final form of the metrics are,

$$h_{mn}^{\text{odd}} = \begin{bmatrix} 0 & 0 & 0 & h_0(t,r) \\ 0 & 0 & 0 & h_1(t,r) \\ 0 & 0 & 0 & 0 \\ h_0(t,r) & h_1(t,r) & 0 & 0 \end{bmatrix} \times \sin(q)(\partial/\partial q)P_L(\cos(q))$$

$$h_{mn}^{\text{even}} = \begin{bmatrix} H_0(t,r)(1-2M/r) & H_1(t,r) & 0 & 0 \\ H_1(t,r) & H_2(t,r)(1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2K(t,r) & 0 \\ 0 & 0 & 0 & r^2K(t,r)\sin^2(q) \end{bmatrix} \times P_L(\cos(q))$$

Consider the equation we introduced for the gauge transformations,

$$g'^{(1)}_{mn} = g^{(1)}_{mn} + x_{(m;n)}$$

And assume for g and g’ that they have the Regge-Wheeler “form”.
And also assume that g’ is in the Regge-Wheeler gauge.

For instance, for even waves, assume h_0, h_1, H_1 and G are zero.

And also assume an appropriate angular decomposition for the gauge transformation vector,

$$x^0 = M_0(t,r)Y_{LM}, \quad x^1 = M_1(t,r)Y_{LM},$$

$$x^2 = M_2(t,r)(\partial/\partial q)Y_{LM}, \quad x^3 = M_2(t,r)/\sin(q)\ (\partial/\partial f)Y_{LM},$$

Then, remarkably, the quantities M_0, M_1, M_2 are completely determined by the following equations (Gleiser 1996),

$$
\begin{aligned}
{}^{(1)}A_0 &= \left(\frac{1}{2}r^2\frac{\partial\,{}^{(1)}\widetilde{G}}{\partial t} - {}^{(1)}\widetilde{h_0}\right) \\
{}^{(1)}A_1 &= (1-2M/r)\left(-\frac{1}{2}r^2\frac{\partial\,{}^{(1)}\widetilde{G}}{\partial r} + {}^{(1)}\widetilde{h_1}\right) \\
{}^{(1)}A_2 &= \tfrac{1}{2}\,{}^{(1)}\widetilde{G}
\end{aligned}
$$

Notation: A=M

And the components of the metric in the Regge-Wheeler gauge can be written in terms of the components of a metric in any gauge as,

$$
\begin{aligned}
{}^{(1)}K^{RW} &= {}^{(1)}\widetilde{K} + (r-2M)\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\,{}^{(1)}\widetilde{h}_1\right) \\
{}^{(1)}H_2^{RW} &= {}^{(1)}\widetilde{H}_2 + (2r-3M)\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\,{}^{(1)}\widetilde{h}_1\right) + r(r-2M)\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\,{}^{(1)}\widetilde{h}_1\right)_{,r} \\
{}^{(1)}H_1^{RW} &= {}^{(1)}\widetilde{H}_1 + r^2\,{}^{(1)}\widetilde{G}_{,tr} - {}^{(1)}\widetilde{h}_{1,t} - \frac{2M}{r(r-2M)}\,{}^{(1)}\widetilde{h}_0 + {}^{(1)}\widetilde{h}_{0,r} + \frac{r(r-3M)}{r-2M}\,{}^{(1)}\widetilde{G}_{,t} \\
{}^{(1)}H_0^{RW} &= {}^{(1)}\widetilde{H}_0 - M\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\,{}^{(1)}\widetilde{h}_1\right) + \frac{2r}{r-2M}\,{}^{(1)}\widetilde{h}_{0,t} + \frac{r^3}{(r-2M)}\,{}^{(1)}\widetilde{G}_{,tt}\ .
\end{aligned}
$$

Details can be found in R. Gleiser, C. Nicasio, R. Price, JP gr-qc/9807077

Why does this happen? Because the Regge-Wheeler gauge is unique.

Therefore any quantity computed in such a gauge is in itself a gauge invariant. Explicit proof of this are the formulas we just introduced: they represent the value of the computed quantity in terms of the metric in any gauge!

The reason for this long detour is that in the following I will use calculations in the Regge-Wheeler gauge. Some people might have the impression that these calculations are only useful in a particular gauge and are lacking in generality. THEY ARE NOT! Any result we compute can be expressed straightforwardly in a "manifestly gauge invariant" manner by substituting the Regge-Wheeler gauge quantities in terms of a general gauge using the formulas we just introduced.

The field equations:

I illustrate here with the odd parity case, which is simpler. This was worked out by Regge and Wheeler in the reference cited. The more important (and involved) even parity case was worked out later by F. Zerilli (Phys. Rev. Lett 24, 737 (1970)).

One now proceeds to insert the metrics we just considered into the usual Einstein equations, and we keep only terms linear in epsilon,

$$R_{23}=0: \quad (1-2M/r)^{-1} w\, h_0 + [(1-2M/r)h_1]' = 0$$

$$R_{13}=0: \quad (1-2M/r)^{-1} w\, (h_0' - w\, h_1 - 2h_0/r) + (L-1)(L+2)h_1/r^2 = 0$$

$$R_{03}=0: \quad (w\, h_1 - h_0') + 2w\, h_1/r = (4Mh_0/r - L(L+1)h_0)/[r^2(1-2M/r)]$$

Where we have assumed that the perturbations are harmonic in time with frequency omega. All other Einstein equations vanish. Moreover, the latter is a combination of the first two.

If we now eliminate h_0 between the two first equations, and defining,

$$y_{\text{odd}} = (1 - 2M/r)h_1/r, \qquad \text{and} \quad r_* = r + 2M\ln(r/2M - 1)$$

We get, $\quad d^2 y_{\text{odd}}/dr_*^2 + w_{\text{eff}}^2(r) y_{\text{odd}} = 0$

Or, in the time domain,

$$\boxed{\frac{\partial^2 y_{\text{odd}}}{\partial r_*^2} - \frac{\partial^2 y_{\text{odd}}}{\partial t^2} + V(r) y_{\text{odd}} = 0,}$$

With $\quad V(r) = [-L(L+1)/r^2 + 6M/r^3](1 - 2M/r)$

Which is known as the Regge-Wheeler equation and describes odd parity perturbations of a Schwarzschild black hole.

The even parity equation has generically the same form,

$$\frac{\partial^2 y_{\text{even}}}{\partial \mathrm{r}_*^{\,2}} - \frac{\partial^2 y_{\text{even}}}{\partial \mathrm{t}^2} + V(r)y_{\text{even}} = 0,$$

But the "potential" is different. It is known as the Zerilli equation.

$$V(r) = (1-2M/r)\left(\frac{1}{l^2}\left[\frac{72M^3}{r^5} - \frac{12M}{r^3}(L-1)(L+2)\left(1-\frac{3M}{r}\right)\right] + \frac{(L-1)L(L+1)(L+2)}{r^2 l}\right)$$

where $l = L(L+1) - 2 + 6M/r$ and

$$y_{\text{even}} = \frac{4r(r-2M)}{L(L+1)(r(L-1)(L+2)+6M)}\left[H - rK' - \frac{r-3M}{r-2M}K\right] + \frac{r^2}{(r(L-1)(L+2)+6M)}K$$

In terms of the Regge-Wheeler gauge perturbation quantities.

The Zerilli and the Regge-Wheeler equations each describe one of the two degrees of freedom of linearized gravity propagating in a black hole background. With minor modifications they also describe electromagnetic and scalar fields.

The derivations we followed were in the Regge-Wheeler gauge, but by now we know that the quantities of interest, ψ, are gauge invariant quantities that can be expressed in terms of the metric perturbations in any gauge.

It is worthwhile mentioning that approaches that are "manifestly gauge invariant" to these equations can be constructed. Moncrief for instance, has a beautiful construction from the Hamiltonian formulation V. Moncrief, Ann. Phys. 88, 323 (1974)

Notice that the equations are wave equations in Cartesian 1+1 dimensional spacetime (in spite of the fact that we are in spherical symmetry), and that they are written in terms of the "tortoise" coordinate r_*.

The tortoise coordinate:

$$r_* = r + 2M \ln(r/2M - 1)$$

Arises from absorbing a (1-2M/r) factor,

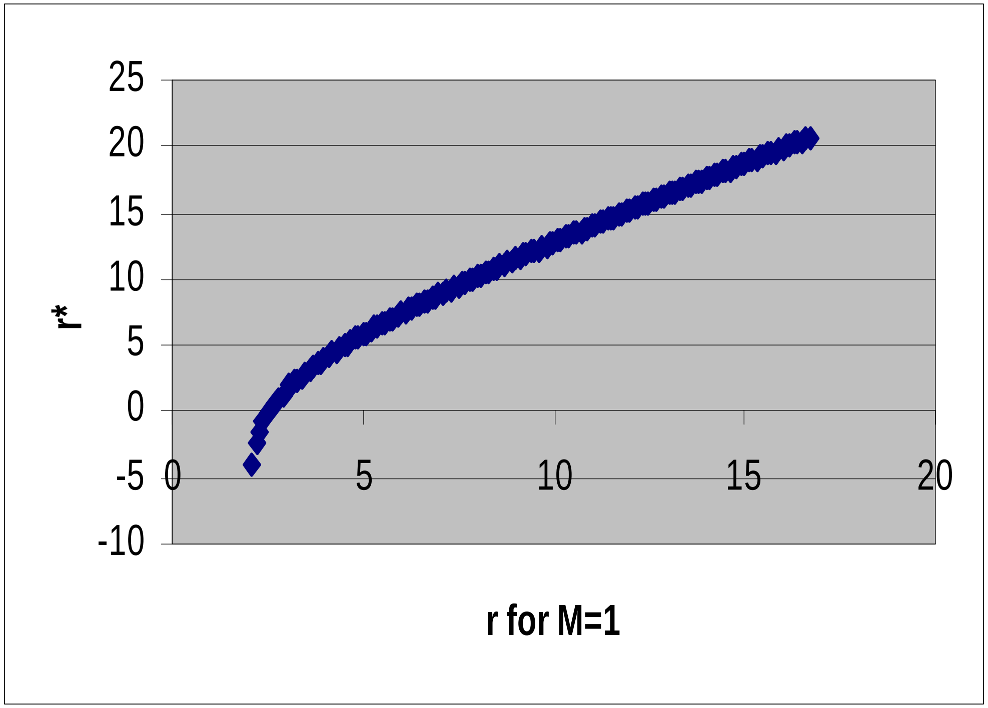
$$dr_* = \sqrt{g_{11}/g_{00}}\,dr = (1 - 2M/r)^{-1} dr$$

And covers the “exterior” of the black hole, since,

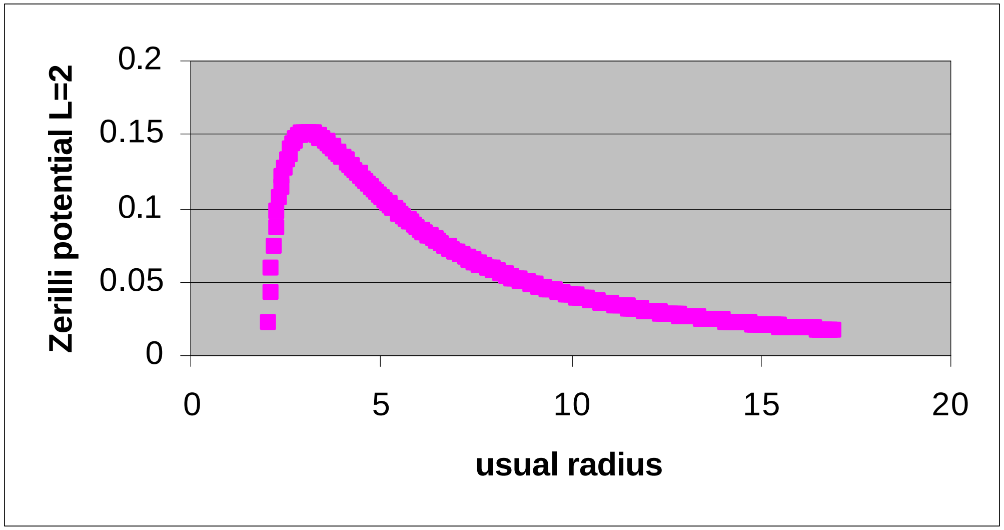
$$r \to \infty \quad \text{for} \quad r_* \to \infty$$

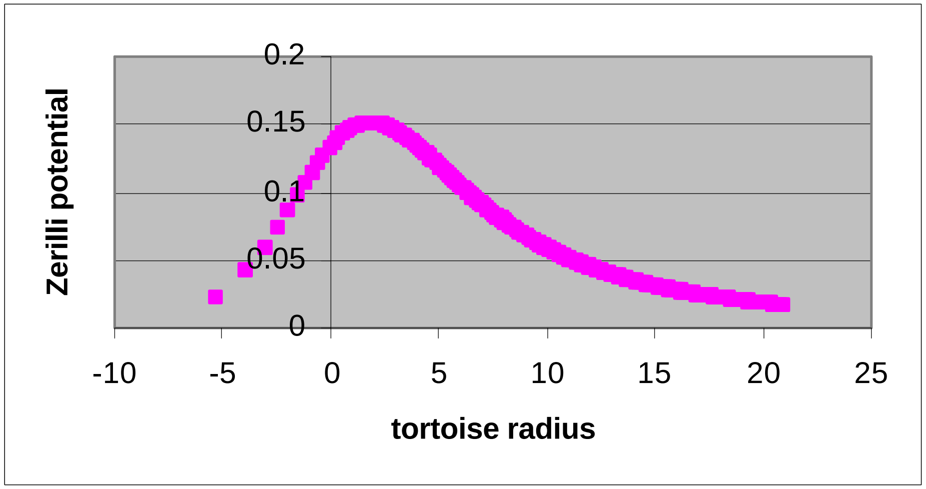
$$r \to 2M \quad \text{for} \quad r_* \to -\infty$$

$$[2M, \infty] \to [-\infty, \infty]$$



The potential:





The potential is an analog of the usual centrifugal plus Newton plus GR corrections potential of the two body problem.

The peak corresponds to the barrier that normally determines the ISCO (innermost stable circular orbit) in the general relativistic two body problem.

The physical meaning of the Zerilli function:

We manipulated the Einstein equations to obtain a second order wave equation for a certain quantity. Why not choose another quantity? You can do it, but the one we chose has interesting properties. To see this we will go to a region where things are under control, far away from the hole. There it is customary to describe things in the radiation gauge (MTW),

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{2M^2}{r^2} O(\frac{1}{r^3})\right)dt^2 - \left(4e_{jkl} S^k \frac{x^l}{r} + O(\frac{1}{r^3})\right)dt dx^j$$
$$+ \left[\left(1 + \frac{2M}{r} + \frac{3M^2}{2r^2}\right)d_{jk} + O_{jk}(\frac{1}{r})\right]dx^j dx^k$$

So in the Regge-Wheeler notation, $h_1 = O(1/r), \quad H_2 = O(1/r^2)$

And the tracelessness condition $h_{\hat{q}\hat{q}} + h_{\hat{f}\hat{f}} = O(1/r^2)$ implies

$K = 3G + O(1/r^2)$ which in turn means that $y_{\text{even}} = 12rG$

And therefore,

$$\boxed{h_{\hat{q}\hat{q}} = \frac{1}{12}\frac{y_{\text{even}}}{\mathrm{r}} Y_{LM}}$$

So the Zerilli function really captures the “essence” of gravitational radiation!

With this setup, it is straightforward to work out formulas for radiated energy and angular momentum, using the Landau-Lifshitz pseudo-tensor,

$$\frac{d\text{Power}}{d\Omega} = \frac{r^2}{16p}\left[\dot{h}_{qf}^{\;2} + \frac{1}{4}\left(\dot{h}_{qq} - \dot{h}_{ff}\right)^2\right]$$

$$\boxed{\text{Power} = \frac{1}{384p}\dot{y}^{\,2}}$$

C.Cunningham, R. Price, V. Moncrief, Ap. J. 230, 870 (1979)

Exercise: the time domain code:

Solving the Zerilli equation: what is going on mathematically?

E. Ching, P. Leung, W.-M. Suen, K. Young Phys. Rev. D52, 2118 (1995) and references therein.

In the numerical experiments we saw that perturbations of a black hole of finite duration in time generate, in addition to an initial transient, a characteristic ring-down followed by a power-law tail. Let us try to understand better analytically how these behaviors come about.

Consider the Green's function solution to the Zerilli equation,

$$f(x,t) = \int dy [G(x,y,t)\dot{f}(y,0) + \dot{G}(x,y,t) f\ (y,0)$$

Where G(x,y,t) is the Green's function for the time-domain Zerilli equation. It can be obtain via Fourier transform from the frequency domain Green's function, which is easier to obtain,

$$\tilde{G}(x,y,t) = \int_0^\infty dt\, G(x,y,t) e^{iwt}$$

One way of obtaining the frequency-domain Green's function is by constructing two independent solutions f(ω,x) and g(ω,x) to the homogeneous equation, one of them satisfying the left boundary condition, the other one the right boundary condition, multiplying and dividing by the Wronskian,

$$\tilde{G}(x,y,w)=\begin{cases}\dfrac{f(w,x)g(w,y)}{W(w)}, x<y\\[2ex]\dfrac{f(w,y)g(w,x)}{W(w)}, y<x\end{cases}$$

Where $W(w)=f'g-g'f$

To construct the time-domain Green's function we use the inverse Fourier transform, which for t>0 requires following a contour encircling the lower half of the complex ω plane. Examining the singularity structure of the Green's function in that domain we can see:

a) The Green's function will have poles wherever the Wronskian vanishes. At these points, f and g are linearly dependent, meaning that one is finding a solution that satisfies outgoing boundary conditions at both the horizon and at infinity. Such solutions are the quasinormal modes of the system and have a complex frequency with negative imaginary part.

b) If the potential is of compact support in x, one can impose the outgoing boundary condition immediately outside the domain of the potential. One can then integrate the differential equation for a finite amount of x range to obtain f,g. Therefore these f,g's cannot have singularities. This is also true if the potential decays fast with x. If the potential has a slower than exponential tail, f and g will have singularities in the complex plane. These singularities have the form of a branch cut along the negative imaginary ω axis. When these singularities reach $\omega=0$, they produce the power-law tails.

C) Finally, the "prompt" contribution comes from the circle at $|\omega|$=infinity.

Solving the Zerilli equation: what is going on physically?

In curved space-times Huyghen's principle does not apply: waves do not propagate freely but scatter.

The ringdown can be viewed as waves bouncing around the potential well of the Zerilli potential. Their frequency is therefore determined by the light travel time across the potential, which is proportional to the mass of the black hole.

The tails can be viewed as "accumulation" of waves produced by the back-scattering on the curved spacetime.

Notice that these phenomena happen also for stars. (W-modes in neutron stars).

Second order perturbation theory:

One can repeat all the manipulations we performed keeping quantities up to second order in epsilon.

What about the gauge issues? One can proceed in the same way, based on the following: consider a gauge transformation purely of first order in epsilon. Let us choose it in such a way that the second order quantities are brought into the Regge-Wheeler gauge to first order in epsilon. This can obviously be done, since the second order pieces play no role and we repeat the same calculations as before.

The first order gauge transformation will introduce changes in the quantities at second order.

We can now perform a gauge transformation that is purely of order epsilon^2 in such a way that the second order pieces of the metric are transformed into the Regge-Wheeler gauge. Such a transformation does not affect the first order pieces. Therefore the metric to first and second order is put, via unique formulas, into the Regge-Wheeler gauge.

So we go through the same manipulations as before and end up with an equation that looks exactly like the first order Zerilli equation, but that also contains pieces quadratic in the first order perturbations.

There is more ambiguity in what to call “the second order Zerilli function”. It has a rather unambiguous piece, which depends on the second order metric perturbations exactly in the first way that the first order Zerilli function does on the first order metric perturbations.

But the “second order Zerilli function” has also pieces quadratic in the first order perturbation, that we can choose as we wish.

The bottomline in what is useful is to make the choices in such a way that the physical quantities of interest have a simple and well defined dependence on the chosen Zerilli function.

For instance, a carelessly chosen Zerilli function might diverge for large values of r. This actually happened in the first attempts to finding such a function. Of course, the physical quantities do not diverge, so it just reflects a poor choice of function to work with.

The details of all this are just too long to summarize here. Let it be said that a second order Zerilli formalism, including the relations to the asymptotic energies has been worked out for the L=2 m=0 perturbations.

R. Gleiser, C. Nicasio, R. Price, JP, gr-qc/9807077
G. Davies, gr-qc/9810056

Detail: in the even parity case, when one manipulates the Einstein equations to reach the Zerilli equation, one differentiates and then integrates the equations with respect to t. This is a curious procedure, since in principle it implies that one could add an arbitrary function of r to the Zerilli function. This is true, one simply ignores this freedom.

In the second order case, after differentiating the Einstein equations, one cannot simply integrate back with respect to time because that would mean integrating the pieces quadratic in the first order perturbation. This is a priori feasible, but not in closed form.

One therefore settles by considering a Zerilli equation with third order derivatives, and operating with it. Since by iteration of the Einstein equations at the initial slice one can obtain arbitrarily high time derivatives of the metric, one can in practice evolve.

Perturbations of rotating black holes:

Due to the complexity of the Kerr metric, it becomes impractical to proceed simple-mindedly to massage the Einstein equations to get a perturbative equation. This forces us to think harder about what one is doing. Couldn’t one find a general rewriting of the Einstein equations such that in the case of small perturbations around a spacetime the perturbations immediately become controlled by wave equations?

In fact, a driving force behind the development of current hyperbolic formulations of the Einstein equations for numerical relativity is to achieve such a goal.

For instance, see Anderson, Abrahams, Lea, Phys.Rev.D58:064015,1998

To obtain the perturbation equation for rotating black holes, Teukolsky used the Newman-Penrose formalism.

The Newman-Penrose formalism is a notation to write various quantities and equations that appear in relativity. It starts by considering a (complex) null tetrad $(\vec{l}, \vec{n}, \vec{m}, \vec{\bar{m}})$ such that,

$$\vec{l} \cdot \vec{n} = 1 = -\vec{m} \cdot \vec{\bar{m}}$$

A notation is introduced for the directional derivatives along the tetrad vectors,

$$D = l^m \partial_m, \quad \Delta = n^m \partial_m, \quad d = m^m \partial_m, \quad d^* = \bar{m}^m \partial_m,$$

And $a, b, g, e, k, l, m, n, p, r, s, t$, are a notation for the spin coefficients of the null tetrad.

Finally, the projections of the Weyl tensor,

$$\Psi_0 = -C_{mnrs}\, l^m m^n l^r m^s, \quad \Psi_1 = -C_{mnrs}\, l^m n^n l^r m^s,$$

$$\Psi_2 = -C_{mnrs}\, l^m m^n \bar{m}^r n^s, \quad \Psi_3 = -C_{mnrs}\, l^m n^n \bar{m}^r n^s,$$

$$\Psi_4 = -C_{mnrs}\, n^m \bar{m}^n n^r m^s.$$

One can write the Bianchi identities and the Einstein equations using these and related quantities.
Three of the Bianchi identities, if one substitutes the vacuum Einstein equations in them, read,

$$(d^* - 4a + p)\Psi_0 - (D - 4r - 2e)\Psi_1 - 3k\Psi_2 = 0$$
$$(\Delta - 4g + m)\Psi_0 - (d - 4t - 2b)\Psi_1 - 3s\Psi_2 = 0,$$
$$(D - r - r^* - 3e + e^*)s - (d - t + p^* - a^* - 3b)k - \Psi_0 = 0.$$

Consider now a spacetime given by the Kerr metric plus a small perturbation. One can easily find null vectors that form a Newman-Penrose tetrad. One also can see that for a Kerr spacetime,

$$\Psi_0 = \Psi_1 = s = k = 0 \qquad D\Psi_2 = 3r\Psi_2, \quad d\Psi_2 = 3t\Psi_2$$

And one can combine the above equations into a single equation for Ψ_0 (or Ψ_4; actually, both can be shown to be equivalent, since Ψ_0 becomes Ψ_4 under the interchange of the vectors l and n).

Since these Ψ's are scalars, and vanish for the background under an infinitesimal coordinate transformation they are invariant.

$$\Psi' = \Psi - x^m \partial_m \Psi^{\text{background}}$$

And a similar reasoning leads to the proof that they are invariant under infinitesimal tetrad rotations.

The scalars have remarkably simple connections with physically important quantities. For instance, if one considers outgoing linearized waves of a single frequency, it is reasonably straightforward to notice that,

$$\Psi_4 = -(R_{tqtq} - iR_{tqtf}) = -w^2 (h_{qq} - ih_{qf})/2$$

And using the pseudotensor formulas we introduced before, one gets,

$$\frac{d^2E}{dtd\Omega} = \frac{r^2}{4pw^2} |\Psi_4|^2$$

Details: NP formalism: Chandra's article in "General relativity, an Einstein centenary survey" by Hawking and Israel, Cambridge.
S. Teukolsky, Ap. J. 185, 635 (1973).

What does the Teukolsky equation look like in practice?

The Kerr metric in Boyer-Lindquist coordinates,

$$ds^2 = (1 - 2Mr/\Sigma)dt^2 + (4Mar\sin^2\theta/\Sigma)dt\,d\phi - (\Sigma/\Delta)dr^2 - \Sigma d\theta^2$$
$$- \sin^2\theta\,(r^2 + a^2 + 2Ma^2 r\sin^2\theta/\Sigma)d\phi^2$$

where M is the mass, aM the angular momentum,

$$\Delta = \mathrm{r}^2 - 2Mr + a^2, \Sigma = r^2 + a^2\cos^2\theta$$

It should be noted that this metric is quite more involved than the Schwarzschild metric: it is non-diagonal, rational coefficients with non-trivial dependence on theta, etc.

And the Teukolsky equation:

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2 q\right]\frac{\partial^2\Psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\Psi}{\partial t\partial f} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 q}\right]\frac{\partial^2\Psi}{\partial f^2}$$

$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Psi}{\partial r}\right) - \frac{1}{\sin q}\frac{\partial}{\partial q}\left(\sin q\frac{\partial\Psi}{\partial q}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos q}{\sin^2 q}\right]\frac{\partial\Psi}{\partial f}$$

$$-2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos q\right]\frac{\partial\Psi}{\partial t} + (s^2\cot q - s)\Psi = 0$$

Where if $s = 2, \quad \Psi = \Psi_0$

$s = -2, \Psi = r^{-4}\Psi_4$

$$r = -1/(r - ia\cos q)$$

This equation is considerably more involved to handle than the Zerilli equation. To begin with, it is not separable in angles in the time domain. In the frequency domain it is separable. That is, if you assume,

$$\Psi = e^{-iwt} e^{imf} S(q,w) R(r,w)$$

Then the S's become the spheroidal functions $S_{LM}(-a^2\,\omega, \cos\theta)$.

In other words, if you wish to evolve things in time, you will need a 2+1-dimensional code. This was only recently achieved (W. Krivan, P. Laguna, P. Papadopoulos, Phys. Rev. D54, 4728;D56, 3395 (1997))

When a=0 one does not recover the Zerilli equation. The resulting equation is the Bardeen-Press equation and it contains in its real and imaginary parts the Zerilli and the Regge-Wheeler equation (both parities are handled at the same time).

See Chandra "The mathematical theory of black holes", Oxford.