Canonical quantum gravity

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- Lecture 1: Introduction: The early beginnings (1984-1992)
- **Lecture 2**: Formal developments (1992-4)
- Lecture 3: Physics (1994-present)

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Today's lecture

- Spin networks: how to generate independent Wilson loops
- Well defined operators: areas and volumes.
- Functional integration.
- Entropy of black holes.

In yesterday’s lecture we reviewed how a new canonical formulation of general relativity appeared to offer attractive possibilities, in particular

- The phase space was identical to that of an SO(3) Yang-Mills theory.
- One could solve the Gauss law using Wilson loops.
- One can introduce a representation (the loop representation) where the diffeomorphism constraint can be naturally handled through knot invariants.
- Promising results appeared when analyzing formal versions of the quantum Hamiltonian constraint.

We however found several aspects that need sharpening:

The calculations involving the Hamiltonian were only formal, unregulated ones. We need more experience regulating operators in this formalism.

The Wilson loops were an over-complete basis of functions and that meant that wavefunctions in the loop representations were bound by complicated identities.

The variables that made the Hamiltonian constraint simple were complex variables requiring us to enforce additional reality conditions to make sure we were obtaining real general relativity.

We will see how developments that happened early in the 90s helped significantly with these issues.

Spin networks Rovelli and Smolin 1994 gr-qc/9411005

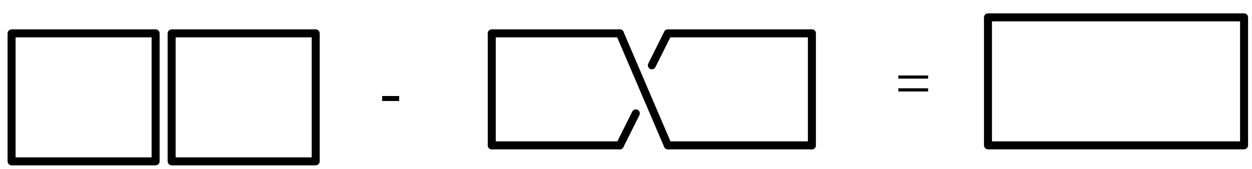
But also Penrose in the 60's, Witten 1991, Kauffman and Lins 90's

We wish to do away with the identities that bound the wavefunctions in the loop representation. If we recall, we had,

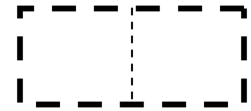
$$W_\gamma[A]W_\eta[A] = W_{\gamma\circ\eta}[A] + W_{\gamma\circ\eta^{-1}}[A]$$

Or, graphically,

Which implies,



Let us put this a different way: suppose one has a lattice, and on this lattice we ask “which Wilson loops can I set up which are independent?”.



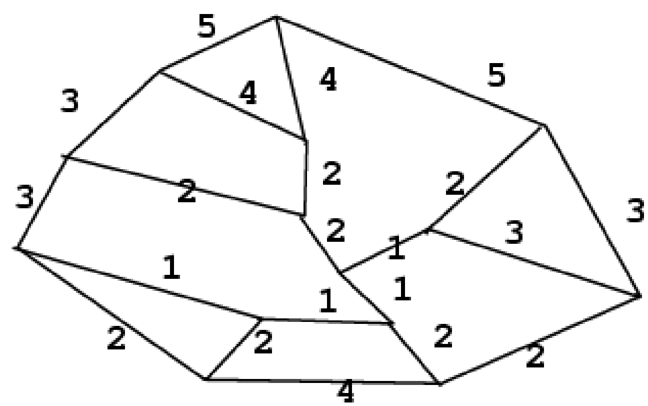
Ignoring multiple windings, the possibilities are

And are not independent. From the previous slide we learnt that one could choose  and the symmetrized combination of the other two.

The moral is: in the center link if we take “no loop” and “symmetrized” loops, we exhaust all independent possibilities.

The result you’ll have to half-believe me is that this construction is general. That is, given any graph, you can construct independent Wilson loops by choosing these two possibilities for each line.

How could this be? The idea is that in considering Wilson loops we had unnecessarily straitjacketed into considering the fundamental representation of SO(3). In general one can construct a “generalized holonomy”. To do this first consider a graph embedded in 3d with intersections of any order.



Now, along each line we consider a holonomy of an SO(3) connection in the j-th representation. We can generate a gauge invariant object by contracting the holonomies at the vertices using invariant tensors for the group.

The resulting object is a generalization of the Wilson loop.

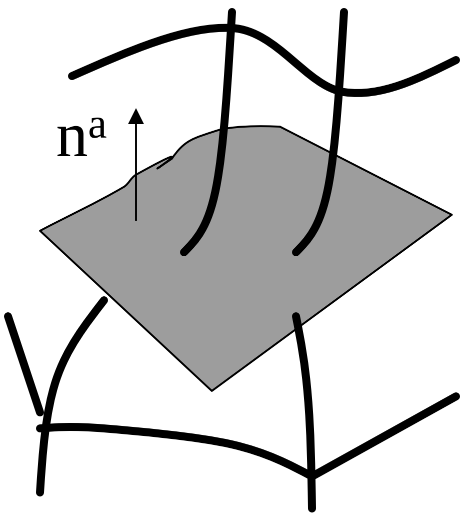
Considering higher order representation is tantamount to the "symmetrization" of the lines we discussed in the simple example.

Why are these objects independent? |Basically because the Mandelstam identities come from identities of the group matrices in a given representation. They appear as a result of considering holonomies constrained in one representation. Therefore when one composed them one had to ensure that the resulting quantity staid in the representation. This is not the case for the objects we have, which involve lines in all possible representations.

Are spin networks different from Wilson loops? They are not. They are linear combinations of Wilson loops. They can simply be seen as an efficient graphical device for keeping track of which combinations are independent. They are also very natural to work with.

Constructing well defined operators: volumes and areas

Rovelli and Smolin 94; Ashtekar, Lewandowski et al 95



Given a surface, we want to compute its area quantum mechanically.

$$A = \int_S d^2\sigma \sqrt{\tilde{E}^{ai}\tilde{E}^{bi}\, n_a n_b}$$

To promote this quantity to an operator, we need to handle the product of triads and also the square root. To do this, we start by partitioning the surface in small elements of area and notice that since the triads are functional derivatives quantum mechanically, one only gets contributions from the small elements of area pierced by a line of the spin net.

So we have $$\hat{A} = \lim_{L\to 0} \sum_I \sqrt{\hat{A}_I^2},$$

We need the action of the triad on a spin network state, which is very similar to on a loop state,

$$\hat{\tilde{E}}_i^a(x)|s>= \int_{e_J} dy^a \delta^3(x-y) X_i^J |s_x>$$

Where X^j_i is a generator of SO(3) in the J level representation and s_x is a spin network that is opened at the point x and the generator X is inserted in that place. Since these quantities are distributional, we need to regularize. We will regularize by: a) smearing the E's along the small surface and point-splitting the product. The result is,

$$\hat{A}_I^2|s>= \int d^2\sigma \int d^2\tau \int_{e_I} dy^a \int_{e_I} dz^b \delta^3(\sigma-y)\delta^3(\tau-z) X_k^I X_k^I |s_{x,y}>$$

Notice that we have six one dimensional integrals and two three dimensional Dirac deltas. All these “cancel each other” and we are left with a simple expression given by the square of the SO(3) generator in the J-th spin representation. From angular momentum theory, we know that the value of such square is j(j+1), so the end result for the area operator is,

$$\hat{A}|s>= \sum_L \sqrt{j_L(j_L+1)}\,\ell_P^2|s>$$

So we see that the area has a well defined action, and although we used background structures to regularize, the final result is topological and background independent. The spectrum of the operator is discrete, and admits a simple interpretation in which the spin of the lines of a spin network can be viewed as “quanta of area”.

Ashtekar and Lewandowski have done a complete analysis that includes the possibility of lines being parallel to the surface and vertices being on the surface and produced the complete spectrum of the operator. gr-qc/9602046,9711031

The prediction of the spectrum is very specific and has immediate consequences. One naively would have assumed that the spectrum of the area would go as,

$$A \sim n\ell_P^2$$

However, it doesn't . With the prediction of

$$\hat{A}|s>=\sum_L \sqrt{j_L(j_L+1)}\,\ell_P^2|s>$$

The spacing of the eigenvalues diminishes rapidly for large values of the area. We will see that this has consequences at the time of computing the entropy of black holes.

Volume operator:

$$V = \int d^3x \sqrt{\mathrm{detg}} = \int d^3x \sqrt{\tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k \epsilon^{ijk} \epsilon_{abc}}$$

I will omit all the details since the calculation goes very much as for the area operator: one first breaks the integral into a sum over little cubic regions. In each of these regions one smears the E operators with two dimensional integrals. Because of the epsilons the quantity is only non-vanishing at a place where there is a vertex of the spin network. One is left with three two dimensional integrals, three one dimensional integrals and three three-dimensional Dirac deltas so the result is finite.

The group factor is a bit more difficult to compute than before, but corresponds to three traced generators contracted with epsilon. It can be computed. The end result is that the volume is finite, has a discrete spectrum, and the non-vanishing contributions come from **four-valent** intersections or higher. The eigenvalue depends on the value of the valences that enter the intersections.

So a picture of quantum geometry emerges in which the lines of flux are associated with “quanta of area” and the intersections of lines with quanta of volume.

A measure for integration:

A key ingredient for discussing quantum physics is to have at hand an inner product to compute expectation values. It is not easy to develop functional measures in infinite dimensional non-linear spaces like the space of connections modulo gauge transformations. Measures have been introduced for connections in cases like 1+1 Yang-Mills, or Chern-Simons theories, but in these cases one is dealing with finite dimensional spaces.

As part of the development of the techniques for dealing with quantum gravity, mathematically rigorous measures were introduced in these kinds of spaces, in some cases for the first time ever.

This is work due to Ashtekar, Lewandowski, Marolf, Mourao, Thiemann. Due to its heavy mathematical nature, we will only give a brief sketch here to highlight the main concepts.

(Easy to follow presentation: Ashtekar, Marolf, Mourao, Lanczos Proceedings, also in gr-qc

One wishes to compute: $(\psi_1, \psi_2) = \int_{\mathcal{A}/\mathcal{G}} d\mu([A]) \ \overline{\psi_1([A])}\psi_2([A])$

To motivate the functional space we will consider let us start with a simpler example, that of a scalar field *f* satisfying the Klein-Gordon equation.

A configuration space *C* for such a theory would be given by the set of all smooth field configurations with appropriate falloff conditions at infinity, for instance C^2 functions. One therefore expects to have wavefunctions ***Y(f),*** and wishes to compute,

$$(\Psi_1, \Psi_2) = \int_C d_\mu \phi \bar{\Psi}_1(\phi)\Psi_2(\phi)$$

And we therefore need a suitable measure and integration theory. To construct this, let us consider the set of test (or smearing) functions on R^3, that is, functions that fall off such that the integral,

$$F_f(\phi) =< f, \phi >= \int_{\mathbb{R}^d} d^d x f(x)\phi(x) \ .$$

The functions f are called “Schwarz space” and define the simplest linear functionals on C.

A set of functions on C one can introduce are the “cylindrical” functions. Consider a finite dimensional subspace of the Schwarz space V_n, with a basis $(e_1,\ldots,e_n)$. We can define the projections,

$$\pi_{e_1,\ldots,e_n}(\phi) = \{< e_1, \phi >, \ldots, < e_n, \phi >\}$$

A cylindrical function on C is a function that depends on C only through these projections,

$$f(\phi) = F(< e_1, \phi >, \ldots, < e_n, \phi >)$$

For any function $F:R^n \to C$

This representation is not unique. In particular any function cylindrical with respect to V_n is cylindrical with respecto to any V_m that contains V_n.

A cylindrical measure is a measure that allows to integrate cylindrical functions. Any measure in R^n would allow us to integrate cylindrical functions, but the tricky part is that there has to be consistency of these measures for different choices of V_n's.

$$\int_{\mathcal{C}} d\mu(\phi) f(\phi) = \int_{\mathrm{I\!R}^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n)$$

Suppose one has V_n and V_m which have non-vanishing intersection, and with m>n, and,

$$V_n^*(e_1, \ldots, e_n) \subset \tilde{V}_m^*(\tilde{e}_1, \ldots, \tilde{e}_m) \quad \text{with} \quad e_i = \sum_{j=1}^{m} L_{ij} \tilde{e}_j \ ; \quad i = 1, \ldots, n$$

Then for every cylindrical function f with respect to V_n defined by a function F on R^n one can make it cylindrical with respect to V_m via,

$$\begin{aligned} f(\phi) &= F(< e_1, \phi >, \ldots, < e_n, \phi >) = F(< L_{1j} \tilde{e}_j, \phi >, \ldots, < L_{nj} \tilde{e}_j, \phi >) \\ &= \tilde{F}(< \tilde{e}_1, \phi >, \ldots, < \tilde{e}_m, \phi >) \end{aligned}$$

And therefore one has to have that,

$$\int_{\mathbb{R}^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) = \int_{\mathbb{R}^m} \tilde{F}(\tilde{\eta}_1, \ldots, \tilde{\eta}_m) d\mu_{\tilde{e}_1,\ldots,\tilde{e}_m}(\tilde{\eta}_1, \ldots, \tilde{\eta}_m)$$

Any set of measures one finite dimensional spaces satisfying these conditions for any cylindrical function F, defines a cylindrical measure via,

$$\int_{\mathcal{C}} d\mu(\phi) f(\phi) = \int_{\mathbb{R}^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n)$$

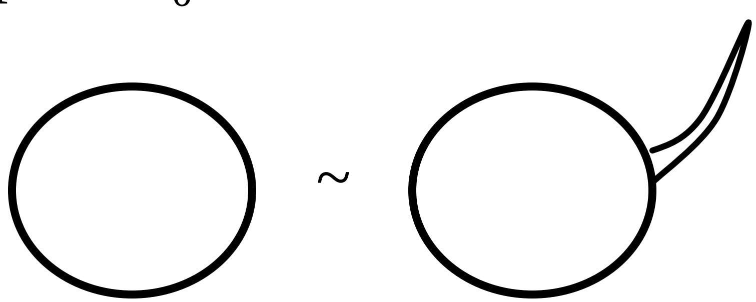
And conversely, a cylindrical measure defines consistent sets of measures in finite dimensional settings.

A particularly simple example of this construction is to consider the normalized Gaussian measures in R^n. The resulting measure on C is the one used in textbooks when quantizing the scalar field. The Fock space is obtained by completion of the sets of cylindrical measures with a certain weight.

The situation is strikingly similar to ordinary quantum mechanics, where the Hilbert space of physical states is obtained by suitable completions of square integrable functions on the configuration space. In field theory the situation is more involved. Not every physical state is a function on just the configuration space, but distributions on the time=constant hypersurface are also generically involved.

How does one generalize this to non-Abelian connections?

We introduce the notion of "hoops" (holonomic loops), that is, loops that yield the same holonomy for any connection. Such quantities form a group (Gambini 1980's) under composition at a given basepoint x_0.



Let us consider a set of independent hoops (β_1,...,β_n) (hint: use spin networks). If we consider the holonomy along each of these loops for a given connection, I get a map from the space of connections modulo gauge transformations to n copies of the gauge group modulo the adjoint action.

$$\begin{aligned} \pi_{\beta_1,\ldots,\beta_n}([A]) &: \quad \mathcal{A}/\mathcal{G} \to G^n/Ad \\ \pi_{\beta_1,\ldots,\beta_n}([A]) &= [H(\beta_1, A), \ldots, H(\beta_n, A)], \end{aligned}$$

We can now define a cylindrical function very much as we did before, in this case considering a function on G^n/Ad,

$$\int_{\mathcal{A}/\mathcal{G}} f([A]) d\mu([A]) = \int_{G^n/Ad} F([g_1, \ldots, g_n]) d\mu_{\beta_1,\ldots,\beta_n}([g_1, \ldots, g_n])$$

A particularly simple choice of measure is to consider the Haar measure on each G/Ad. This choice turns out to be consistent (hard to prove with loops, easier with spin nets).

Since the measure was defined without reference to any background structure it is naturally diffeomorphism invariant!

The construction looks intimidating but the end result is amazingly simple, especially if one casts it in terms of spin nets. It simply states that

$$< s_1|s_2 >= \int D_\mu[A] W_{s_1}[A] W_{s_2}[A] = \delta_{s_1, s_2}$$

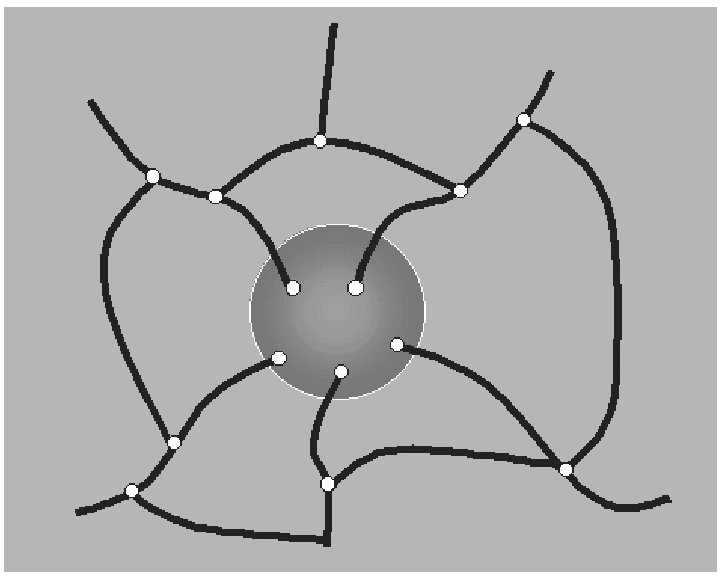
Which means that the inner product of two spin network states vanishes if the two spin network states are different. More precisely, if no representative of the diffeo-equivalence class of spin networks s_1 is present in the class s_2.

That is, not only have we made sense precisely of the infinite dimensional integral present in the inner product, but the result is remarkably simple at the time of doing calculations.

Entropy of black holes and isolated horizons: (brief sketch)

Early ideas: *Smolin (95) Krasnov (96)*, *Rovelli (96)*
More refined treatment: *Ashtekar*, *Baez*, *Krasnov*, *Corichi*, *Lewandowski*, *Beetle*, *Fairhurst*, *Dreyer*, *Krishnan (99-00)*.
(Easy to read presentation: Ashtekar gr-qc/9910101)

Rough idea:



Suppose one has a black hole. The area of the horizon will have an eigenvalue S. There are many rearrangements of spin networks that yield the same eigenvalue. Counting the number of these quantum states gives a measure of the entropy of the black hole.

There are obvious problems with this proposal.

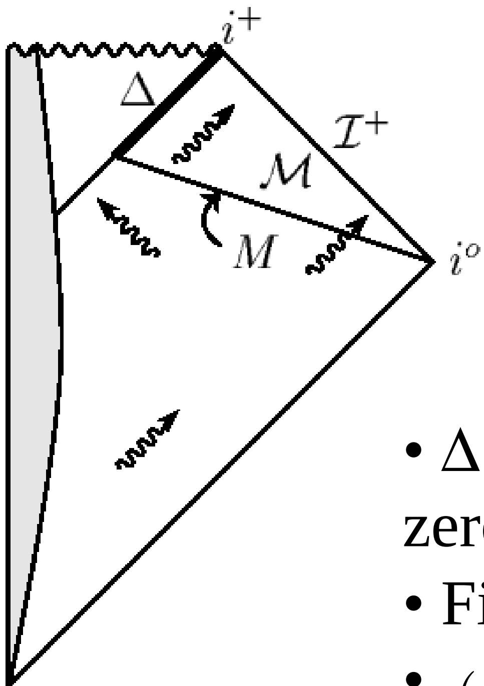
First of all, this reasoning seems to work for any area, not just the horizon of a black hole. This might not be bad, since people speak about a “Bekenstein bound” implying that any area would have a notion of entropy associated with our ignorance of what is inside it, the black hole case maximizing this bound.

The entropy of a black hole should be an observable. Yet, areas in general are not observables. Neither is the area of the horizon (since matter may fall in and the area grow).

How do we actually do the counting?

How does the specific dynamics of general relativity enter the calculation (it should).

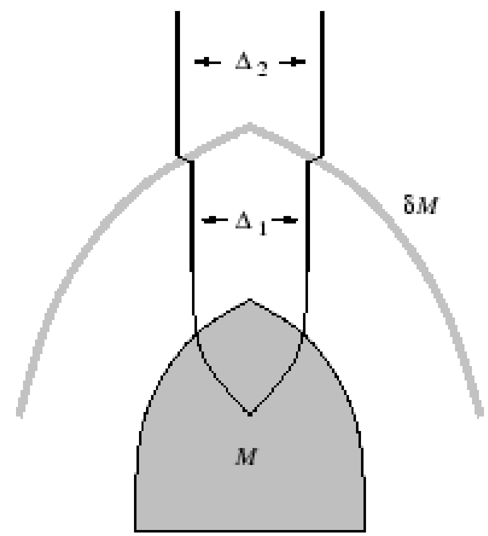
The first task consists on choosing a surface that is reasonably close in its definition to a horizon, but that is an actual observable of the theory. This was accomplished by Ashtekar and collaborators through the notion of "isolated horizon".



The concept embodies the natural idea that even in dynamical space-times with radiation, one can have "portions" of horizons that behave as one wishes.

- Δ is a null 3-surface $R \times S^2$. With zero shear and expansion.
- Field equations hold on Δ.
- $(\mathcal{L}_\ell D_a - D_a \mathcal{L}_\ell) V^b = 0$

  With ℓ any normal to Δ



If one studies in detail the Einstein action with this set of boundary conditions, one finds that for the action to be differentiable one needs to add boundary terms. The boundary terms have the form of the integral of a Chern-Simons form built with the

One can then construct a quantum theory with Hilbert space,

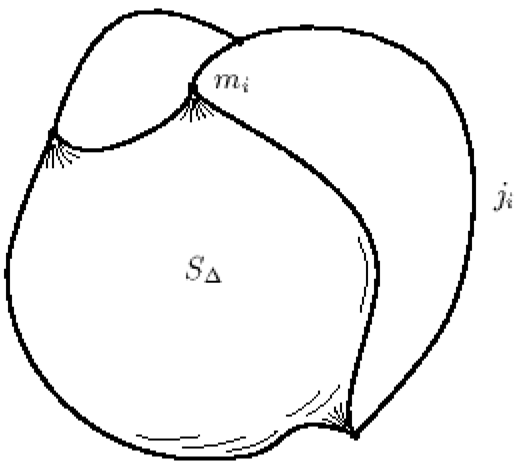
$$\mathcal{H} = \mathcal{H}_{\mathrm{bulk}} \otimes \mathcal{H}_{\mathrm{surface}}$$

These two spaces are not entirely disconnected, it turns out that the “level” k of the Chern-Simons theory is determined by the bulk.

One then wishes to consider a microcanonical ensemble, in terms of the area, $(a_0 - da, a_0 + da)$

The quantum boundary conditions dictate that for a given state in the bulk that punctures P times the surface, the Chern-Simons state has its curvature concentrated at the punctures, one therefore has,

$$\mathcal{H}^{\text{phys}}_{\text{surface}} = \bigoplus_{\mathcal{P}} \mathcal{H}^{\mathcal{P}}_{\text{surface}}$$



Each puncture adds an element of area $8pb\sqrt{j_i(j_i+1)}$ and introduces a deficit angle of value $2pm_i/k$

Where m is in the interval [- j_i ,+j_i] and k is the “level” of the Chern-Simons theory.

The picture of the quantum geometry of the horizon that appears is that it is flat except at punctures where the lines of gravitational flux “pull” the surface up and introduce curvature.

The counting of degrees of freedom on the surface is quite delicate and uses in great detail results of Chern-Simons theory that would be too lengthy to go into detail here. The final result is:

$$\mathcal{N} = \exp\left(\frac{\gamma_o}{\gamma}\,\frac{a_o}{4\ell_{\rm Pl}^2}\right) \qquad \text{where} \qquad \gamma_o = \frac{\ln 2}{\pi\sqrt{3}}$$

It is good that the result (log) is proportional to the area (without the tight boundary conditions one gets results proportional to square root of area, for instance). The result depends on a free parameter, the Immirzi parameter (we called it β in other lectures, in the entropy literature it is usually called γ).

Considering black holes coupled to matter one notices that one gets the correct result for the same value of the Immirzi parameter.

Similarities and differences with the stringy calculation:

Quantum Geometry

- Not worked out in d>4 (perhaps works in d=3)
- Bulk states end on isolated horizon, determine gauge fields on surface.
- Works for non-extremal holes right away.
- Has undetermined parameter
- Works in usual geometric terms

Strings

- Only partially worked out in d=4.
- Strings end on D-brane, determines gauge fields on it.
- Getting away from extremality is hard.
- Result precisely correct.
- Uses features of string theory to turn the calculation into a much simpler system.

Summary:

- Significant problem in major technical issues that plagued the formalism (and related ones) from the beginning.
- Spin networks provide an elegant and powerful calculational tool
- Discreteness of areas and volumes.
- Black hole entropy: very detailed calculation.