Lectures on loop quantum gravity

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1) Why quantize gravity?
2) General relativity
3) Hamiltonian treatment of constraint systems
4) Totally constrained systems and the issue of time.

5) Quantization of constrained systems
6) Canonical analysis of general relativity
7) Canonical analysis in terms of Ashtekar variables
8) Loop representation for general relativity

9) Spin networks and quantum geometry
10) The issue of the dynamics.
11) Applications: loop quantum cosmology, black hole entropy, and potentially observable effects.
12) Conclusions

1) Why quantize gravity?

Quantum mechanics and general relativity have given us a profound understanding of the physical world, including scales ranging from the atomic to the cosmological.

Quantum mechanics describes nuclear and atomic physics, condensed matter, semiconductors, superconductors, lasers, superfluids and led to important technological developments, for instance, in modern electronics.

General relativity leads to relativistic astrophysics, cosmology and the GPS technology.

These two theories have nevertheless destroyed the coherent vision of the world given by classical mechanics and non-relativistic theories.

General relativity is local, deterministic and continuum, whereas quantum mechanics is probabilistic, non-local and discrete. In spite of their empirical success, GR and QM offer a schizophrenic understanding of the physical world.

General relativity has taught us that space-time is a dynamical entity just like any physical object. Quantum mechanics has taught us that physical objects are composed of quanta and have states that can be superposition of different behaviors.

These observations lead us to expect that at high energies and small scales the universe should behave as composed of quanta of space-time. How is one to describe such objects?

With the exception of classical mechanics, all current theories of physics are *incomplete and contain inconsistencies.* They are all valid to describe phenomena at certain scales and in certain regimes but they display inconsistencies when applied outside their range of validity.

Electromagnetism: The energy and the mass of a point charge are infinite. The self-interaction of a charge with its own field is ill-defined, yielding “runaway” solutions. The treatment of the charged point particle is clearly incomplete.

The quantum description eliminates some infinities, for instance avoiding the collapse of the electrons into the nucleus.

But even in Quantum Field Theories

a)Divergent vacuum energy $<0|H|0>=\infty$.
b)Distributional field operators

$$< 0 | \phi^2(x) | 0 >= \int \frac{d^3k}{(2\pi)^3 2\omega k} \to \infty$$

c) Ill defined interactive theories $L = L_0 + \lambda\phi^4$

We only have rigorous theories in dimensions less than four or highly symmetric theories as N=4 supersymmetry.

d) Physical quantities as scattering cross sections are infinite when all radiative corrections are taken into account,

$$< 0 | T\big(\phi(x_1)\ldots\phi(x_n)\big)| 0 >= \sum \lambda^n G_n(x_1,\ldots,x_n)$$

The divergences in G may be reabsorbed redefining the constants and the fields λ,m,ϕ, so G results well defined. The series, however, for many physically Interesting cases are divergent.

Renormalization may be considered as a short-cut which allow us to compute physical quantities without worrying about what is going on at extremely short distances.

We are ignoring any possible space-time microstructure.

One also has infinities in general relativity.

A generic space-time containing matter will develop singularities in its evolution (Hawking and Penrose singularity theorems).

At a singularity (big bang, black holes) the curvature diverges and matter acquires pathological behaviors. More generally, a space-time is singular if it contains at least one incomplete geodesic.

The geometric description of space-time breaks down at the singularities and only quantum considerations could solve these pathologies.

Summarizing: all known theories of modern physics are partial. Inconsistencies appear when we attempt to apply them beyond their realm of validity.

Only quantum gravity could be complete. It will be relevant at scales when inconsistencies and infinities arise (big bang, black holes singularities, ultra high energy, black hole evaporation).

The problem of unifying quantum mechanics and general relativity is quite complex. Both theories are radically different.

Quantum mechanics in its most developed form, quantum field theory, uses a background space-time in which the notion of particles makes sense. This preferred structure is incompatible with general relativity where space-time is dynamical.

The properties of continuity and differentiability of space-time are essential in general relativity. But in quantum mechanics a quantized space-time is possibly discrete.

We lack experimental evidence of phenomena that are dominated by quantum gravity effects, a theory that becomes relevant in regimes highly difficult to access.

A lot of physicists, motivated by the last observation, have been led to ignore quantum gravity. But ignoring a problem does not make it go away.

We can state that quantum gravity effects are going to be very small, but we do not know how to prove that they actually are (“How do you know the effects of a theory you do not know are small” A. Salam).

The search for consistency:

Searching for consistency in physics has been the source of great discoveries.

Maxwell theory + classical mechanics -> Special Relativity

Special Relativity+ quantum mechanics -> antiparticles, quantum field theory

Special Relativity+ Newtonian gravity -> General Relativity

In all cases progress resulted from taking seriously both points of view and constructing a better synthesis.

Two main approaches:

Canonical quantization and path integral quantization of general relativity-> Loop quantum gravity.

Unification of gravity with other interactions -> string theory.

The existence of more than one approach reflects the state of the art. We still do not have a theory that is completely satisfactory.

General relativity:

Riemannian geometry, a brief review.

Einstein noticed that non inertial systems of reference are locally equivalent to systems in a gravitational field and therefore a theory of gravity will be generally covariant.

General relativity is a theory of gravity but instead of describing the latter as a force, it describes it as a deformation of space-time.

The geometrical properties in a given coordinate system are given by the metric tensor:

$$ds^2 = g_{ab}(x) dx^a dx^b$$

Let us recall the properties of a Riemannian geometry in a metric manifold without torsion.

The covariant derivative defines a mapping from (k,l) tensors to (k,l+1) tensors,

$$\nabla_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$$

The covariant derivative, as a map, has the following properties:

$$\text{Linearity:}\quad \nabla_c\left(\alpha \mathrm{T}_1 + \beta \mathrm{T}_2\right) = \alpha \nabla_c T_1 + \beta \nabla_c T_2$$

$$\text{Leibniz:}\quad \nabla_c\left(\mathrm{T}_1 \mathrm{T}_2\right) = \left(\nabla_c T_1\right) T_2 + T_1\left(\nabla_c T_2\right)$$

It commutes with the contraction:

$$\nabla_c T^{\dots d \dots}{}_{\dots d \dots} = \nabla_c\left(T^{\dots d \dots}{}_{\dots d \dots}\right)$$

On scalars it reduces to the partial derivative: $\nabla_a f = \partial_a f$

and due to vanishing torsion: $\left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) f = 0$

The previous conditions do not define uniquely the covariant derivative.

Any $\nabla_a t^b + \Gamma^b{}_{ac} t^c$ with Γ symmetric satisfies the conditions. Under coordinate transformations, Γ transforms in such a way that the covariant derivative of a vector is a (1,1) tensor.

In Riemannian geometries one chooses “metric compatibility”,

$$\nabla_a g_{bc} = 0$$

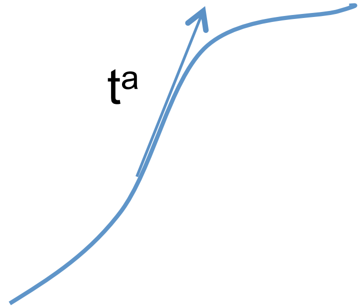
Which is convenient because contractions commute with derivatives. A metric compatible derivative with no torsion has a uniquely defined form,

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right)$$

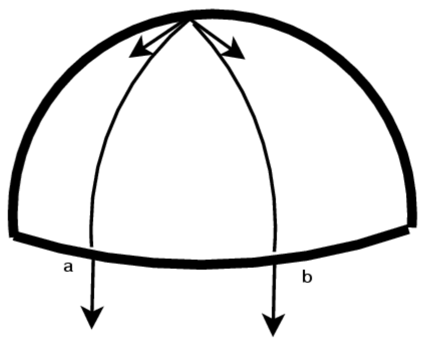
known as the “Christoffel symbols”.

Given a curve with tangent vector t^a, one defines the "parallel transport" of a vector v^a as,

$$t^a \nabla_a v^b = 0$$

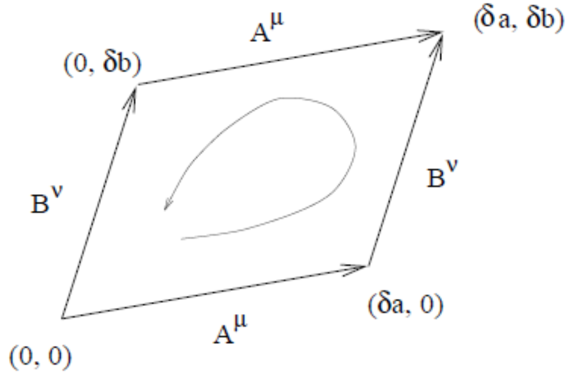


We still do not have a satisfactory definition of curvature. Notice that in curvilinear coordinates Γ^a_{bc} can be non-vanishing and still have a flat manifold. To determine if a manifold is curved one takes a vector and parallel transports it around a closed circuit,



For instance, in the example someone starts with a vector in the north pole, carries it as parallel to itself as possible (and tangent to the Earth) to the equator, then move from *a to b* and then brings it back. The fact that it does not come back parallel to its original orientation is proof the Earth is curved. The angle depends on the area of the circuit traveled and how curved the manifold is.

To make the previous concept precise, we consider an infinitesimal closed circuit. We have that,



$$\delta v^c = R_{abd}{}^c v^a A^b B^d$$

Where $R_{abd}{}^c$ is known as the curvature tensor or Riemann tensor and is defined by,

$$\left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) v_c = R_{abc}{}^d v_d$$

And it satisfies certain algebraic identities,

$$R_{abc}{}^d = -R_{bac}{}^d,$$

$$R_{[abc]}{}^d = 0,$$

$$R_{abcd} = -R_{abdc} = R_{cdab},$$

And the Bianchi identity,

$$\nabla_{[a} R_{bc]d}{}^e = 0.$$

One can define important “traces” of the Riemann tensor as the Ricci tensor,

$$R_{ab} = R_{acb}{}^{c}$$

And the scalar curvature $$R = R_{ab} g^{ab}$$

In terms of these one can define the Einstein tensor:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

For which the Bianchi identity reads: $$\nabla_a G^{ab} = 0$$

The Einstein equations:

They determine the geometry in terms of the energy and stress present in the matter.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$T_{\mu\nu}$ is the energy-momentum tensor. The above equation may be consider the relativistic generaliztion of the Poisson equation of Newton's theory of gravity,

$$\nabla^2 \phi = 4\pi G \rho$$

Both contain second derivatives but the Einstein equations involve both space and time derivatives. This means that change in the matter content do not propagate instantaneously. The energy is automatically conserved

$$\nabla^\mu T_{\mu\nu} = 0$$

The Einstein equations may be extended to include a cosmological constant,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

The cosmological constant is related to the vacuum energy of the fields, $\rho_{vacuum} = \dfrac{\Lambda}{8\pi G}$ in GR the actual energy matters and not the energy up to a constant.

And the vacuum energy results from the sum of the fundamental energy of each of the modes composing the field,

$$\rho_{vacuum} \approx \hbar\, k^4{}_{\max}$$

this contribution diverges, but if we assume that the Planck energy imposes a natural cutoff we would have that

$$\rho_{vacuum} \approx 10^{112}\,\mathrm{erg/cm^3}$$

But cosmological observations indicate that $\rho_{vacuum} \approx 10^{-8}\,\mathrm{erg/cm^3}$

And this constitutes the “cosmological constant problem”, we have a discrepancy of 120 orders of magnitude.

It is still not clear what could be the role of quantum gravity in the solution of this problem

The Einstein-Hilbert action:

The action that leads to the Einstein equations is,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\det(g)}\left(R - 2\Lambda\right)$$

and variations with respect to the metric yield the field equations.

It is worthwhile pointing out that in general relativity geometry is the central idea and the theory is covariant in its description of nature. The dynamics is not unique.

Alternative theories:

We mention here a couple of alternatives to general relativity that have been considered in the literature.

The first one are the *scalar tensor theories*, where gravity in addition of being described by a curved geometry is described by a scalar field,

$$S = \frac{1}{16\pi G}\int d^4x\left[f(\lambda)R - \frac{1}{2}h(\lambda)g^{\mu\nu}\partial_\mu\lambda\partial_n\lambda - U(\lambda) + L_M\left(g_{\mu\nu},\psi_M\right)\right]$$

The second one is theories that have higher order terms in the action

$$S = \frac{1}{16\pi G}\int d^4x\left[R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \cdots\right]$$

Hamiltonian treatment of constrained systems:

A theory whose dynamical variables depend on functions that can be chosen arbitrarily is a gauge theory. In such a theory the equations of motion and the initial conditions do not determine the evolution uniquely.

General relativity is a gauge theory since one can perform changes in coordinates as one evolves that yield different metrics starting from the same initial data. The evolution of the space-time metric depends on arbitrary functions. As we will see for each arbitrary function there will exist a constraint on the canonical variables.

Dirac analysis of gauge theories:

$$S = \int_{t_1}^{t_2} L(q^a, \dot{q}^a)dt \qquad\qquad a = 1 \ldots N$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = 0$$

$$\sum_b \left[\frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} \ddot{q}^b + \frac{\partial^2 L}{\partial \dot{q}^a \partial q^b} \dot{q}^b \right] - \frac{\partial L}{\partial q^a} = 0 \qquad H_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}$$

If det(H_{ab}) is non-zero then the acceleration can be determined from the initial data. If it is zero, only certain components can be determined in terms of the others. Similarly, when one determines the canonical momenta,

$$p_a = \frac{\partial L}{\partial \dot{q}^a}$$

If $\det(H_{ab}) \neq 0$ then $\dot{q}^a = \dot{q}^a(q, p)$

If $\det(H_{ab}) = 0$ then $\dot{q}^a = \dot{q}^a(q, p, v^\alpha) \quad \alpha = 1 \ldots M$

And there will be M constraints.

$\phi_\alpha(q, p) = 0$ (primary constraints)

There exists a 2N-M dimensional constraint surface in phase space. One then constructs the Hamiltonian,

$$H(q,p) = \sum_{a=1}^{N} p_a \dot{q}^a - L$$

The canonical equations are derived by considering variations of δq and δp. If the system has constraints such variations are not independent.

Given M arbitrary functions $u^\alpha(q,p)$ one has that,

$$u^\alpha \frac{\partial \phi_\alpha}{\partial q^a} \delta q^a + u^\alpha \frac{\partial \phi_\alpha}{\partial p^a} \delta p^a = 0$$

Then the total Hamiltonian is given by $\quad H_T = H + u^\alpha \phi_\alpha$

$$\dot{q}^a = \frac{\partial H_T}{\partial p_a}, \qquad \dot{p}^a = -\frac{\partial H_T}{\partial q_a}, \qquad \phi_\alpha(q,p) = 0$$

The equations of motion can be derived from the action

$$S = \int_{t_1}^{t_2} \left(p_a \dot{q}^a - H(q,p) - u^\alpha \phi_\alpha(q,p) \right) dt$$

by taking variations with respect to p,q and u.
One has that te time derivative of a physical magnitude is,

$$\dot{F}(q,p) = \{F, H\} + u^\alpha \{F, \phi_\alpha\}$$

One has to satisfy consistency conditions, that ensure that the constraints are preserved in time,

$$\dot{\phi}_\alpha = 0 \;\Rightarrow\; \{\phi_\alpha, H\} + u^\beta \{\phi_\alpha, \phi_\beta\} = 0$$

There exist three possibilities:

a) One gets new constraints. One needs to impose additional conditions, called secondary constraints (which one also needs to check are preserved in time),

$$\phi_\rho(q,p) = 0, \qquad \rho = M+1, \ldots, M+K$$

b) One gets inconsistencies and the theory does not exist.

c) Some of the Lagrange multipliers get fixed. The multipliers must satisfy M+K equations,

$$\{\phi_\xi , H\} + u^\alpha \{\phi_\xi , \phi_\alpha\} = 0$$

If the dynamical system is consistent then $u^\alpha = U^\alpha + V^\alpha$
with U a particular solution of the inhomogeneous equation and V such that

$$V^\alpha \{\phi_\xi , \phi_\alpha\} \approx 0$$

We have introduced the notation $F \approx G \quad \text{if} \quad F - G = c^\mu \phi_\mu$

F is weakly equal to G if they are equal on the constraint surface.

Let us suppose that there are *L* independent solutions for *V* $u^\alpha = U^\alpha + v^l \, V_l^\alpha$

Then: $H_T = H + U^\alpha \phi_\alpha + v^l V_l^\alpha \phi_\alpha$ with v^l *L* independent functions

Functions of the dynamical variables that have vanishing Poisson brackets with all the constraints are called first class. The $\phi_m = V_m^{\alpha}\phi_\alpha$ are primary constraints that are first class.

One can also have secondary constraints that are first class. First class constraints generate gauge transformations (Dirac's conjecture).

Given a function of phase space F(q,p) and assuming one knows $q(t_1)$, $p(t_1)$ that satisfy the constraints one has that,

$$F(t_1 + \Delta t) = F(t_1) + \{F, H\}\Delta t + v^m \Delta t \{F, \phi_m\}$$

And if one chooses to evolve with v' instead of v, one will get an F' such that,

$$\delta F = F' - F = \delta v^m \{F, \phi_m\}$$

And F is gauge dependent. Primary constraints that are first class generate gauge transformations.

Totally constrained systems:

This type of system is very important because general relativity belongs in this class.

In the usual Hamiltonian framework the dynamical variables evolve in time which, although observable, is not a dynamical variable itself.

There exists a more symmetric treatment where one introduces the time as a dynamical variable. *X(t),T(t)* and both space and time are functions of an unobservable parameter *t* that can be redefined freely *t->t'=f(t)* .

As any theory depending on an arbitrary function, it will be treated as a gauge theory. To give an example of such a treatment we consider the parameterized non-relativistic particle.

$$S = \int dT L\left(x, \frac{dx}{dT}\right) = \int dt \dot{T} L\left(x, \frac{dx}{dt}\frac{dt}{dT}\right)$$

$$S = \int dT \frac{1}{2} m\left(\frac{dx}{dT}\right)^2 = \int dt \dot{T} \frac{1}{2} m\left(\frac{\dot{X}}{\dot{T}}\right)^2$$ Configuration variables X(t), T(t).

The canonical momenta are,

$$P_X = \frac{\partial L}{\partial \dot{X}} = m\frac{\dot{X}}{\dot{T}}, \qquad P_T = \frac{\partial L}{\partial \dot{T}} = -m\frac{\dot{X}^2}{\dot{T}^2}$$

And there is a constraint

$$\phi(X,T) = P_T + \frac{P_X{}^2}{2m} = 0$$

And the Hamiltonian vanishes, $$H = P_T\dot{T} + P_X\dot{X} - L = 0$$

The total Hamiltonian is $$H_T = N\left(P_T + \frac{P_X{}^2}{2m}\right)$$

And the equations of motion are: $$\dot{X} = N\frac{P_X}{m}, \qquad \dot{T} = N,$$

$$\dot{\mathrm{P}}_{\mathrm{X}} = 0, \qquad \dot{\mathrm{P}}_{\mathrm{T}} = 0.$$

$H_T(p,q)$ is proportional to a first class constraint and not only generates evolution but simultaneously it generates a gauge transformation. The action is,

$$S = \int_{t_1}^{t_2} \left(P_T \dot{T} + P_X \dot{X} - N \left(P_T + \frac{P_X{}^2}{2m} \right) \right) dt$$

The general form of the action for a totally constrained system is

$$S = \int \left[p_a \dot{q}^a - \mu^\alpha \phi_\alpha(q,p) \right] dt$$

The theory is invariant under "time" reparameterizations and the Hamiltonian is a linear combination of the constraints

$$H_T = \mu^\alpha \phi_\alpha(q,p) = 0$$

We will see that general relativity is a totally constrained system with first class constraints.

A constraint is **second class** if it is not first class. To treat theories with second class constraints one needs to introduce the Dirac brackets $\{\ ,\ \}^*$.

The latter satisfy $\{X,q\}^*=0$, $\{X,p\}^*=0$ with X a second class constraint.

One says that the constraints have been *strongly imposed* because their Dirac brackets with any dynamical variable vanish.

Observables:

Functions of phase space that are gauge invariant are called *observables*,

$$\{F(p,q),\phi_\alpha\} \approx 0$$

Where ϕ_α are the first class constraints.

In a totally constrained system like general relativity the observables are also constants of the motion, since,

$$H_T = \sum \lambda^\alpha \phi_\alpha,$$

$$\{F(p,q),\phi_a\} \approx 0 \quad \Rightarrow \quad \{F(p,q),H_T\} \approx 0$$

This is the root of the *problem of time in canonical quantum gravity*. If the physically relevant quantities are constants of the motion how does one describe evolution?

The issue of time: If the physically relevant quantities in totally constrained systems as general relativity are constants of the motion, how can we describe the evolution?

1) Gauge fixing:

$$\tau = f(q,p), \qquad \tau = q^0$$

2) Evolving observables: Bergmann, DeWitt, Rovelli, Marolf

$$\{Q_i(t),\phi_\alpha\} \approx 0 \qquad Q_i(t,q^a,p_a)\,\big|_{t=q^1} = q_i$$

For instance, for the relativistic particle $\phi = p_0^2 - p^2 - m^2$

Two independent observables:

$$p, X \equiv q - \frac{p}{\sqrt{p^2+m^2}}\,q^0, \qquad Q(t,q^a,p_a) = X + \frac{p}{\sqrt{p^2+m^2}}\,t$$

$$Q(t=q^0,q^a,p_a) = q$$

Notice that one needs to assume that there are variables as q^0 that are physically observable, even though they are not Dirac observables

Quantization of constrained systems

The treatment of second class constraints is the more direct one, although it is not trivial.

The key correspondence rule is that the graded commutator of two quantum operators should be equal to iħ times the operator associated to the Dirac bracket,

$$[\hat{A},\hat{B}]_{\pm} = i\hbar\,\widehat{\{A,B\}}^{*}$$

The procedure may encounter difficulties. One has to find a realization of the algebra of operators. One may admit deviations of order $\hbar^2$,

$$[\hat{A},\hat{B}]_{\pm} = \widehat{i\hbar\{A,B\}}^{*} + O(\hbar^2)$$

In many situations one also wishes to require the operators be self-adjoint. These requirements are generically not easy to satisfy and sometimes can be unsurmountable. Let us now turn to how to treat first class constraints.

a) Reduced phase space quantization.

i) Gauge invariant quantization: one quantizes the observables

$$[\hat{F},\hat{G}]_{\pm} = i\hbar\widehat{\{F,G\}^{*}} + O(\hbar^2)$$

Quite non-trivial, in the case of general relativity it is not known how to proceed.

ii) Gauge fixing

One introduces additional gauge conditions that do not commute with the first class constraints. One ends up with a second class set of constraints.

The main challenge of this approach is how to realize the algebra, constraints that may be non-local in nature and the symmetries are broken.

Example: electromagnetism

$\phi = \partial_i E^i = 0$ Gauge fixing $\partial_i A^i = 0$ E^T, A^T gauge invariants

$$[E_i^T(x), A_j^T(y)] = -i\hbar\left((\delta_{ij} - \frac{\partial_i \partial_j}{\Delta})\delta^3(x-y)\right)$$

b) **Dirac quantization**

The method of quantization introduced by Dirac in 1966 has been generalized:
Ashtekar et. al. J. Math Phys. 36, 6456 (1995)
Giulini and Marolf Class. Quan. Grav. 16, 2479 (1999).

Schematically:

1)One chooses a set S of classical variables such that any quantity in phase space is given by a sum of products of elements of S and their Poisson brackets belong in S. An example of a set S are the canonical variables themselves q^a, p_a .

2)To every F in S we associate an operator in an algebra that act in an auxiliary ("kinematical") Hilbert space H_{aux} and such that the commutator of two such operators F, G is given by,

$$\left[\hat{F},\hat{G}\right] = i\hbar\widehat{\{F,G\}}$$

3) The realization of the elements of S is such that

$$R(A^*) = \left(R(A)\right)^+ \text{ in } \mathrm{H}_{\mathrm{aux}}$$

4)The first class constraints are promoted to self-adjoint operators in H_{aux}. Operators in H_{aux} are in general gauge dependent and do not commute with the first class constraints. The idea is to define a *physical Hilbert space* in which the Dirac observables are well defined operators.

The elements $|\Psi\rangle_{phys}$ of H_{phys} are annihilated by the constraints, $\hat{\phi}_\alpha \,|\, \Psi >_{\mathrm{phys}} = 0.$

And if Q is a Dirac observable, $\hat{\phi}_\alpha \hat{Q} \,|\, \Psi >_{\mathrm{phys}} = 0.$

And therefore its action keeps H_{phys} invariant $\hat{Q} \,|\, \Psi >_{\mathrm{phys}} \in \mathrm{H}_{\mathrm{phys}}$

Generically, the physical states are distributional in H_{aux} and belong in the dual of a subspace of H_{aux}.

Example: $q^a, p_a \in S \quad \hat{\phi} = \hat{p}_1$

$$\Psi(q^1, \ldots, q^N)\ \text{square integrable} \in \mathrm{H}_{\mathrm{aux}}$$

$$\Psi_{\mathrm{phys}}(p_1, \ldots, p_N) = \delta(p_1)\Psi(p_2, \ldots, p_N)$$

And Ψ_{phys} is not normalizable in $\mathrm{H}_{\mathrm{aux}}$

The *Algebraic Quantization procedure* (group averaging) for the construction of an inner product in the physical Hilbert space leads to,

$$< \varphi | \psi >_{\text{phys}} = \int dp_1 \cdots dp_N \delta(p_1) \varphi^*(p_2,\ldots,p_N) \psi(p_2,\ldots,p_N) \quad = \int_{\mathbb{R}^3} dx dy dz \int_{\mathbb{R}} dz' g_0^*(x,y,z+z') f_0(x,y,z) \ .$$

And the procedure also ensures that the observables are self-adjoint.

Difficulties with the algebraic quantization procedure:

Consistency

The system of constraints $\hat{\phi}_\alpha | \Psi >_{\text{phys}}$ must satisfy

the integrability conditions $[\hat{\phi}_\alpha, \hat{\phi}_\beta] | \Psi >_{\text{phys}} = 0$

Classically, one has that $\{\phi_\alpha, \phi_\beta\} = C_{\alpha\beta}{}^{\gamma} \phi_\gamma$

But at a quantum level one may encounter corrections:

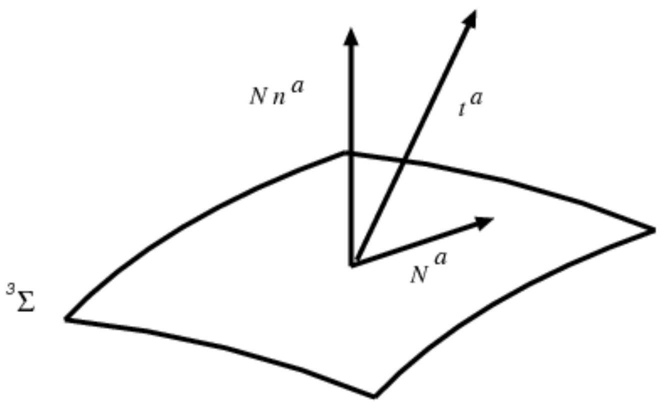
$$[\hat{\phi}_\alpha, \hat{\phi}_\beta] = i\hbar C_{\alpha\beta}{}^{\gamma} \hat{\phi}_\gamma + \hbar^2 \hat{D}_{\alpha\beta}$$

And the original invariances may be lost unless the operator $D_{\alpha\beta}$ annihilates the elements of $|\Psi>_{\text{phys}}$. The additional terms are known as gauge anomalies.

Finally, the group averaging technique used to define the inner product does not always work. Some constraints are not group generators.

Canonical quantization of general relativity

1)Metric variables:



We consider a manifold M with metric g_{ab}. We decompose M=$^3\Sigma$xR where $^3\Sigma$ is a spatial 3-surface.

The foliation is generated by a function t on M that is constant on each $^3\Sigma_t$.

The vector field t^a : $t^a \nabla_a t = 1$ defines a diffeomorphism from $^3\Sigma_0$ to $^3\Sigma_t$.

This allows to describe the evolution in terms of functions of t on a given Σ.

Let $n^a \perp \Sigma$, and the induced metric on Σ is given by,

$$q_{ab} = g_{ab} + n_a n_b$$

$$t^a = Nn^a + N^a$$

We introduce coordinates x^i on ${}^3\Sigma_t$:

$$ds^2 = -N^2 dt^2 + q_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

The *extrinsic curvature* of ${}^3\Sigma$ is defined by,

$$K_{ab} = q_a{}^c q_b{}^d \nabla_c n_d ,$$

$$K_{ab} = K_{ba}, \quad K_{ab} n^b = 0.$$

and it is a measure of how ${}^3\Sigma$ curves in M. It also contains information about the time derivative of the metric,

$$\dot{q}_{ab} \equiv \mathcal{L}t\, q_{ab} = 2N K_{ab} + D_a N_b + D_b N_a$$

where $\quad D_c q_{ab} = 0 \quad$ and $\quad D_c N_b = q_c{}^a q_b{}^d \nabla_a N_d$

The Einstein field equations of general relativity can be derived from the Einstein-Hilbert action,

$$S = \int d^4 x \sqrt{-\det(g)} R$$

And given that $\sqrt{-\det(g)} = N\sqrt{\det(q)}$ one has that,

$$S = \int N \sqrt{\det(q)} \left({}^3 R - K_{ab} K^{ab} + K^2 \right)$$

The action depends on $q_{ab}, \dot{q}_{ab}, N, N^a$ but not on $\dot{N}, \dot{N}^a$.

One defines the canonical momenta,

$$\tilde{\pi}^{ab} = \frac{\delta L}{\delta \dot{q}_{ab}} = \sqrt{\det(q)} \left(K^{ab} - K q^{ab} \right)$$

$$\tilde{\pi}^{N} = 0, \quad \tilde{\pi}^{N^a} = 0$$

Which constitute primary constraints.

Preservation in time of the primary constraints leads to secondary constraints,

$$\tilde{C}_a(\tilde{\pi},q) = 2D_b\tilde{\pi}^b{}_a = 0,$$

$$\tilde{H}(\tilde{\pi},q) = -\sqrt{\det(q)}\ {}^3R + \frac{1}{\sqrt{\det(q)}}\left(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi}^2\right).$$

which are first class and therefore there are no tertiary constraints.

And general relativity is a totally constrained theory,

$$H = \int d^3x\,[N\tilde{H} + N^a\tilde{C}_a]$$

With the *lapse and shift* *N*, N^a arbitrary functions (Lagrange multipliers) and canonical pair,

$$\{q_{ab}(x),\tilde{\pi}^{cd}(y)\} = \delta^c{}_{(a}\delta^d{}_{b)}\,\delta(x,y)$$

It is convenient to introduce smeared versions of the constraints,

$$C(\vec{N}) \equiv \int d^3x\, N^a \widetilde{C}_a = 0, \quad H(M) \equiv \int d^3x\, M\, \widetilde{H} = 0.$$

C is the generator of diffeomorphisms on the three-manifold and H is related to time reparametrization invariance and does not have a natural geometric action in the three manifold.

$$\left\{C(\overrightarrow{N}), f(\widetilde{\pi}, q)\right\} = \mathcal{L}_{\overrightarrow{N}} f(\widetilde{\pi}, q)$$

The constraint algebra is first class but has *structure functions*, an additional difficulty at the time of quantizing the theory.

$$\{H(N), H(M)\} = \int d^3x q^{ab} (N\partial_a M - M\partial_a N)\widetilde{C}_a$$

2) **Ashtekar variables**

In any manifold the metric is diagonalizable at each point by a local change of coordinates, $x^I(x)$: $x^I(A)=0$. The matrix of change of coordinates is called tetrad in four dimensions,

$$e_a^{\;I} = \left.\frac{\partial x^I(x)}{\partial x^a}\right|_{x=x(A)}$$

The knowledge of the tetrad allows to reconstruct the metric

$$g_{ab}(x) = \eta_{IJ} e^I{}_a(x) e^J{}_b(x)$$

Where η_{IJ} is the Minkowski metric, and there is an additional symmetry in that the "internal" indices I,J that can be changed by Lorentz transformations (the coordinates $x^I(x)$ are not unique).

One can introduce a covariant derivative on objects with internal indices,

$$\nabla_a v^I = \partial_a v^I + \omega_a{}^I{}_J v^J$$

With $\omega_a{}^I{}_J$ a one-form belonging in the algebra of so(3,1) with $\omega_a{}^{IJ} = -\omega_a{}^{JI}$.

This connection can be related with the Christoffel one demanding that it annihilate the triad, which in turn it implies it annihilates the metric,

$$\nabla_a e_b{}^I = \partial_a e_b{}^I + \omega_a{}^I{}_J e_b{}^J - \Gamma_{ab}{}^c e_c{}^I = 0$$

The torsion is given by $T^I_{ab} = \Gamma^c_{[ab]}\, e^I_c = \partial_{[a} e^I_{b]} + \omega^I_{[a\,J} e^J_{b]}$

And the curvature of the connection ω is related with the Riemann tensor by,

$$R_{ab}{}^I{}_J \equiv \partial_a \omega_b{}^I{}_J + \omega_a{}^I{}_K \omega_b{}^K{}_J$$

$$R_{abc}{}^d e_{dJ} e^{cI} = R_{ab}{}^I{}_J$$

where we have used the inverse tetrad $e_a{}^I e^a{}_J = \delta^I{}_J$

The Ricci tensor is $$R_a{}^I = R_{ab}{}^{IJ} e^b{}_J = R_a{}^b e_b{}^I$$

And the Einstein equations with cosmological constant are,

$$R_a{}^I - \frac{1}{2} R\, e_a{}^I + \Lambda e_a{}^I = 0$$

The equations can be derived from a first order action (“Palatini action”),

$$S[e,\omega] = \frac{1}{8\pi G}\int d^4x \left(e\, e^a{}_I\, e^b{}_J\, R_{ab}{}^{IJ}(\omega) + \Lambda e \right)$$

where e is the determinant of the tetrad.

Variation with respect to ω leads to the condition that the connection be torsion free (unless one couples the theory to Fermions), and variation with respect to $e^a{}_I$ leads to the Einstein equations.

One can add to the previously considered action a term that does not change the field equations to get the so-called “Holst” action.

$$S_H[e,\omega] = S_P[e,\omega] + \frac{1}{8\pi G\gamma}\int d^4x\, e\, e^a{}_I e^b{}_J \overline{R}_{ab}{}^{IJ}$$

where $\overline{R}_{ab}{}^{IJ} = R_{abKL}\varepsilon^{IJKL}$.

The additional term is a total divergence when the torsion free condition holds. γ is known as the Barbero-Immirzi parameter. It plays a role similar to that of the Θ parameter in Yang-Mills theory. Classically all values of γ are equivalent, at a quantum level it has other implications we will discuss.

Immirzi, Class. Quan. Grav. 14, 177 (1997).
Holst, Phys. Rev D53, 3966 (1996)

The Holst action written in the language of differential forms is

$$S_H = \frac{1}{16\pi G}\int\left\{\varepsilon_{IJKL}\left(e^I \wedge e^J \wedge R^{KL}(\omega) + \Lambda e^I \wedge e^J \wedge e^K \wedge e^L\right) + \frac{1}{\gamma} e^I \wedge e^J \wedge R_{IJ}\right\}$$

The original Ashtekar variables correspond to the Holst action with $\gamma=\pm 1$ for the Euclidean theory or $\gamma=\pm\, i$ for the Lorentzian theory ($\det(\eta)=-1$). For the Lorentzian case they are complex.

In the Euclidean case,

$$S_{Eucl} = \frac{1}{8\pi G}\int\left\{\frac{1}{2}\varepsilon_{IJKL}\left(e^I \wedge e^J \wedge R^{KL}(\omega)\right) - e^I \wedge e^J \wedge R_{IJ}\right\} \qquad \gamma = -1$$

Defining $\omega^+{}_{IJ} = \frac{1}{2}\left(\omega_{IJ} + \frac{1}{2}\varepsilon_{IJ}{}^{KL}\omega_{KL}\right)$, $\qquad \omega^+{}_{IJ} = {}^*\omega^+{}_{IJ}$ is self-dual.

$$S = -\frac{1}{8\pi G}\int \Sigma^{+IJ} \wedge R^+{}_{IJ} \quad \text{with} \quad \Sigma^{+IJ} = \frac{1}{2}\left(e^I \wedge e^J - \frac{1}{2}\varepsilon^{IJ}{}_{KL} e^K \wedge e^L\right)$$

$$R^+_{IJ} = d\omega^+_{IJ} - \omega^+_{IK} \wedge \omega^{+K}_J$$

Ashtekar variables arise in the Hamiltonian version of the previous action. In particular, the Ashtekar connection will be related with $A^i{}_a = \omega^{+}{}_{a0}{}^{i}$

The tetrad formalism is the natural language for the inclusion of Fermionic fields,

$$D_a\psi = \partial_a\psi + \omega_a{}^I{}_J L^J{}_I\psi + A_a{}^\alpha X^\alpha\psi$$

Gauge algebra

Lorentz Algebra

$$S_\psi = \int d^4x\, e\,(\overline{\psi}\gamma^I e_I{}^a D_a\psi + m\overline{\psi}\psi)$$

Canonical quantization: Asthekar’s variables
Barbero Phys. Rev. D51, 5507 (1995), Ashtekar, Lewandowski gr-qc/0404018 (2004)

Defining,

$$A_a^{\ IJ} = \omega_a^{\ +IJ}, \qquad \text{and} \qquad \Pi^a{}_{IJ} = \frac{1}{16\pi G}\eta^{abc}\Sigma^+{}_{bcIJ}, \qquad \mathrm{I,J} = 0..3, \mathrm{abc} = 1..3$$

The Holst action can be written in terms of the self dual variables as,

$$S_H = \int dt \int d^3x \left(\Pi^a{}_{JI}\dot{A}_a^{\ IJ} + \omega_{aIJ}t^a G^{IJ} - N^a C_a - NC\right)$$

Where the Gauss law, diffeomorphism and Hamiltonian constraints are given by,

$$G_{IJ} = \partial_a \Pi^a{}_{IJ} + A_{aI}{}^J \Pi^a{}_{KJ} + A_{aJ}{}^K \Pi^a{}_{IK},$$

$$C_a = \Pi^b{}_{IJ} F_{ab}{}^{IJ},$$

$$C = -\frac{8\pi G}{\sqrt{\det(q)}}\Pi^a{}_I{}^J \Pi^b{}_J{}^K F_{abK}{}^I.$$

Where $\quad F = dA + A \wedge A$

The canonical formulation takes a simpler form if one chooses a gauge and works with real variables.

$$n_a = e_a{}^0 = n_I e_a{}^I \Rightarrow n_I = (1,0,0,0)$$

The remaining components of the tetrad –the triad- are $E_i^A = q_i^a e_a^A$ with A,B,C = 1..3

The extrinsic curvature is $K_i^A = q_i^a \omega_a^{AI} n_I$

and the connection on Σ is $\Gamma_i^A = \varepsilon^A{}_{BC}\omega_b{}^{BC} q_i^b$

That satisfies $dE^A + \varepsilon^A{}_{BC}\Gamma^B \wedge E^C = 0$

Then introducing the connection

$$A_i^B = \Gamma_i^B + \gamma K_i^B$$

and the densitized inverse triad

$$\tilde{E}_B{}^i = \frac{1}{16\pi G\gamma} E_j^C E_k^D \tilde{\eta}^{ijk} \varepsilon_{BCD}$$

Which satisfy

$$\tilde{E}_B{}^i = \frac{1}{8\pi G\gamma}\sqrt{|\det(q)|}\, E_B^i$$

$$\tilde{E}_B{}^i \tilde{E}^{jB} = \frac{1}{(8\pi G\gamma)^2} |\det(q)|\, q^{ij}$$

One has the generalized Ashtekar variables.

In terms of the Ashtekar variables the action takes the form,

$$S_H = \int d^3x \left\{ \widetilde{E}^a{}_B \dot{A}_a{}^B - \frac{1}{2}\omega_{aBC}\varepsilon^{BCD} t^a G_D - N^a C_a - NH \right\}$$

$$\{A_a^B(x), \underset{\sim}{E}{}^b{}_A(y)\} = \delta_a^b \delta_A^B \delta(x,y)$$

Where the constraints are:

Gauss law: $G_A = D_a \widetilde{E}^a{}_A$

Diffeomorphism: $C_a = \widetilde{E}^b{}_A F_{ab}{}^A + \dfrac{(1+\gamma^2)}{\gamma} K_a{}^A G_A$

Hamiltonian: $H = \dfrac{8\pi G\gamma^2}{\sqrt{|\det(q)|}} \widetilde{E}^a{}_A \widetilde{E}^b{}_B \left[\varepsilon^{AB}{}_C F_{ab}{}^C - \dfrac{2(1+\gamma^2)}{\gamma} K_a{}^A K_b{}^A \right]$

$A_a{}^A$ is a connection in Σ and $F_{ab}{}^B$ the associated curvature.

Also, $\det(q) = (\hbar\gamma)^3 \det(\widetilde{E})$

For $\gamma = \pm i$ we get the original Ashtekar variables, which simplify the constraints but make the connections complex.

In spite of the apparent complication of the constraints if one keeps the Ashtekar variables real, techniques have been developed to treat them satisfactorily.

The Gauss constraint is associated with the gauge freedom in the choice of the tetrad.

The physical phase space of gravity is subspace of that of an SO(3) Yang-Mills theory.

The constraints G and C_a generate gauge transformations and diffeomorphisms.

Smearing the constraints,

$$G(\lambda) = \int d^3x\, \lambda^A D_a \tilde{E}^a{}_A, \qquad C(\vec{N}) = \int d^3x\, N^a \tilde{E}^b{}_A F_{ab}{}^A,$$

one has that,

$$\delta A_a{}^A = \{G(\lambda), A_a^A\} = D_a \lambda^A \quad \text{and} \quad \delta A_a{}^A = \{C(\vec{N}), A_a^A\} = \mathcal{L}_{\vec{N}} A_a{}^A,$$

Quantization

We need to pick a polarization, for instance, to work in a space of functionals Ψ[A], and we proceed to promote the constraints to operators in a space of "square integrable functionals" of the connection.

Determining the integration measure in an infinite dimensional case like general relativity is delicate. To illustrate this point, let us consider what happens if one ignores this point as in the finite dimensional case.

$$\hat{A}^B{}_a \Psi[A] = A^B{}_a \Psi[A], \qquad \hat{E}^a{}_B \Psi[A] = i \frac{\delta \Psi[A]}{\delta A_a{}^B}.$$

Two of the constraints do not pose problems to be promoted to operators. The Gauss law,

$$\hat{G}^B \Psi[A] = D_a \frac{\delta \Psi[A]}{\delta A_a{}^B},$$

$$\left(1 + \varepsilon\, G(\lambda)\right)\Psi[A] = \Psi[A_a{}^B + \varepsilon\, D_a \lambda^B]$$

So imposing the constraint is equivalent to demanding that the states be gauge invariant functions of the connection.

Similarly for the diffeomorphism constraint,

$$(1+\varepsilon C(\vec{N}))\Psi[A] = \Psi[A + \mathcal{L}_{\vec{N}} A]$$

And imposing the diffeomorphism constraint implies that the state is a function that is invariant under diffeomorphisms.

The Hamiltonian constraint, however, is considerably harder to implement as a quantum operator. It requires regularizing ill defined products of distributions. The regularization will in principle break gauge and diffeomorphisms invariance Furthermore without an integration measure on the space of connections and an inner product, one will not have a true Hilbert space on which define the constraint operators.

The integration measure

As we pointed out, computing integrals in infinite dimensional spaces is delicate. To illustrate how it is done let us consider the simpler case of a scalar field.

One starts by considering projections of the scalar field on test functions e(x) that are real and smooth. Ashtekar and Lewandowski gr-qc/04040180.

$$h_e(\phi) = \int d^3x e(x)\phi(x)$$

Given a set α of test functions $e_1(x)\ldots e_n(x)$, we define,

$$\Psi_\alpha(\phi) = f\left(h_{e_1}(\phi)\ldots h_{e_n}(\phi)\right)$$

And we will say that functionals of the scalar field that can be written in this way are *cylindrical* and we will call Cyl_α the space of such functions.

The inner product in Cyl_α is,

$$<\Psi_1|\Psi_2>_\alpha = \int_{\Re^n} d\mu_{(n)} \bar{f}_1 f_2$$

The idea is to go from *Cyl*$_\alpha$ to *Cyl*, the space of cylindrical functions defined for some α.

To do this one needs to show that Ψ is cylindrical for two different sets α and β. There exist consistent measures $\mu_{(n)}$ that yield the same result. It can be shown that the Gaussian measures are good for this purpose.

The idea is to include in the Hilbert space states that are limits of Cauchy sequences of normalizable cylindrical states.

Measures in the space of connections

$$\phi(x) \quad \rightarrow \quad \mathrm{A}_{\mathrm{a}}{}^{B}(x)$$

Since connections are gauge dependent the test functions one chooses to smear it should be such that the resulting projections are well behaved under gauge transformations.

One such smearing is the *path ordered exponential along a line*, (in the case of the line being a closed loop they are called *holonomies)*,

$$U(A,\gamma) = P\exp\left(-i\int_{\gamma} A\right)$$

$$\equiv 1 + \sum_{n=1}^{\infty}(-i)^n \int_0^1 dt_1 \int_{t_1}^1 dt_2 \cdots \int_{t_{n-1}}^1 dt_n \dot{\gamma}^{a_1}(t_1) A_{a_1} \dot{\gamma}^{a_2}(t_2) A_{a_2} \ldots \dot{\gamma}^{a_n}(t_n) A_{a_n}$$

Under gauge transformations $V = e^{-i\lambda}$

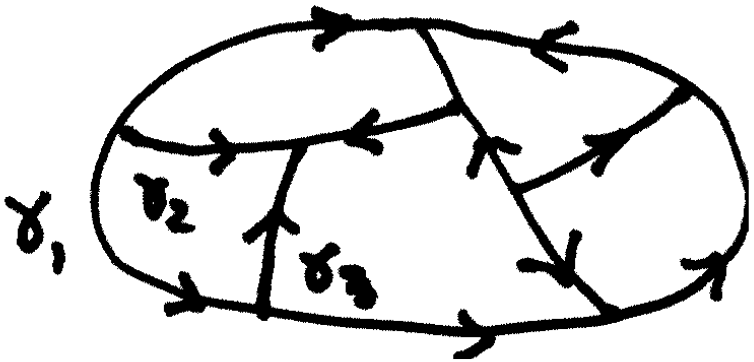
$A \rightarrow VAV^{-1} + VdV^{-1}$

$U(A,\gamma) \rightarrow V(i(\gamma))U(A,\gamma)V^{-1}(f(\gamma))$

with $i(\gamma)$, $f(\gamma)$ the initial and final point of the curve γ

Notice that holonomies are not linear in A. However, one can proceed in a similar way as in the scalar case. One considers an ordered oriented graph Γ with paths $\gamma_1,\ldots,\gamma_L$ and group dependent functions

$$f(U_1,U_2 \ldots U_L)$$



For the gravity case, elements of SU(2).

We now introduce cylindrical functions Cyl_{Γ}

$$\Psi_{\Gamma}[A] = f\big(U(A,\gamma_1),\ldots,U(A,\gamma_L)\big)$$

And define the space *Cyl* of cylindrical functions for some Γ.
This space is endowed with an inner product,

$$< \Psi_{\Gamma f_1} \mid \Psi_{\Gamma f_2} >= \int dU_1 \ldots dU_L \bar{f}_1 f_2$$

Where dU is the Haar measure on SU(2). This product coincides with the standard product in lattice Yang-Mills theories. However here we are working with a continuous theory where the states live on all possible lattices. This inner product corresponds to a space of distributional connections where the holonomies along two loops γ and γ' that differ by an infinitesimal amount are not necessarily "close" in the topology of the group.

The kinematical space K of loop quantum gravity is defined by the Cauchy completion of the space of normalizable cylindrical functions.

We are also interested in the space of distributions in *Cyl, Tcyl.*

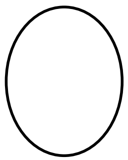
$Cyl \subset K \subset Tcyl$

LOST Theorem

Jerzy Lewandowski, Andrzej Okolow, Hanno Sahlmann, Thomas Thiemann

Ashtekar, Lewandowski, Marolf, Mourao, Thiemann, J. Math. Phys. 36, 6456 (1995).

The idea of representing wavefunctions depending on a connection in terms of holonomies was already used in the Yang-Mills context. For a closed loop α, the state,



α

$$\Psi_\alpha[A] = \mathrm{Tr}\left[U(A,\alpha)\right] = \mathrm{Tr}\left[P\exp\left(-i\oint_\alpha A\right)\right]$$

is gauge invariant and therefore automatically solves Gauss law.

$$\Psi_\alpha[A] \to \mathrm{Tr}[\mathrm{V}_\lambda U V_\lambda^{-1}] = \mathrm{Tr}[U]$$

One can then consider states on a graph Γ composed by sets of loops. Such states provide an (overcomplete) basis of the gauge invariant space of states of any gauge theory.
RG and A. Trias, Phys. Rev. D22, 1380 (1980); Nucl. Phys. B278, 436 (1986);
R.Giles Phys.Rev.D24:2160,1981

C. Rovelli, L. Smolin, Nucl. Phys. B331, 80 (1990).

The kinematical space K

What we just discussed was the origin of the loop quantum gravity approach.

The spin networks based on general graphs Γ provide a simpler basis of the space Cyl.
The elements of the basis are labeled by

$$|\Gamma, j_i, \alpha_i, \beta_i > \qquad i = 1 \ldots L$$

$$< A | \Gamma, j_i, \alpha_i, \beta_i >= R^{j_1 \alpha_1}{}_{\beta_1}(U(\gamma_1, A)) \ldots R^{j_L \alpha_L}{}_{\beta_L}(U(\gamma_L, A))$$

Under a gauge transformation $V = e^{i\lambda}$ the basis vectors transform as,

$$| j, \alpha, \beta > \quad \rightarrow \quad R_\lambda{}^{j\alpha}{}_{\alpha'} \,| j, \alpha', \beta' > R^+{}_\lambda{}^{j\beta'}{}_\beta \qquad\qquad j = \frac{1}{2}, 1, \frac{3}{2}, \ldots \qquad\qquad \alpha, \beta : \quad - j \le \alpha \le j$$

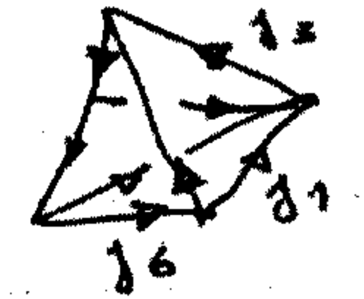
Gauge invariant and diffeomorphism invariant states

The kinematical space K is a space of arbitrary wave functionals of the connection. Recall that two of the quantum gravity constraints imply wave functionals are invatiant under local gauge transformations and diffeomorphisms.

To impose the quantum constraints is equivalent to reducing the Hilbert space

$$\mathrm{K} \xrightarrow{G^B} \mathrm{K}_0 \xrightarrow{C_a} \mathrm{K}_{\mathrm{diff}} \xrightarrow{H} H_{ph}$$

The K_0 space; spin network states



Let us consider a set of nodes and a set of oriented lines γ connecting the nodes. We assume that the paths γ only intersect at the nodes. We assign an irreducible representation j_i to each link j_i=1/2,1,3/2,… To each node we assign an intertwiner i_n that is an invariant tensor of SU(2).

A valence N intertwiner satisfies,

$$i_N{}^{\alpha_1\ldots\alpha_N} V^{j_1}{}_{\alpha_1}{}^{\beta_1} \ldots V^{j_N}{}_{\alpha_N}{}^{\beta_N} = i_N{}^{\beta_1\ldots\beta_N}$$

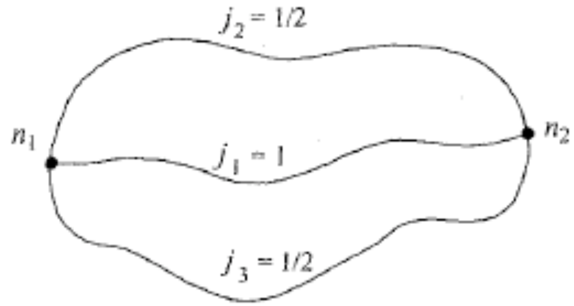
For instance, the 3-j symbols are the 3-valent intertwiners up to a factor,

$$i_3{}^{m_1 m_2 m_3} \propto \begin{Bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{Bmatrix}$$

Given a spin network S={G,j_l,i_N}, a *spin network state* is given by,

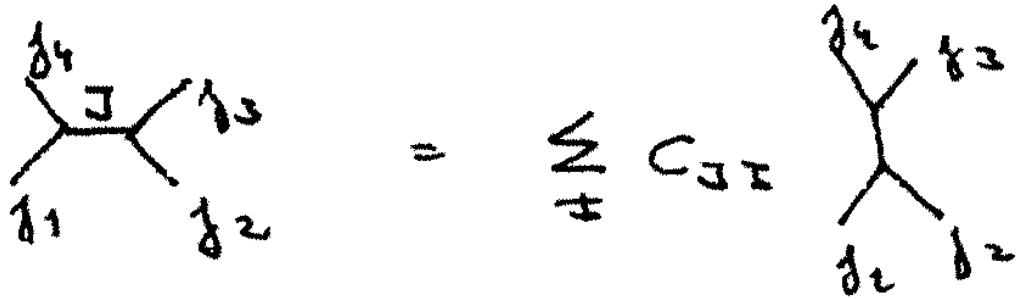
$$| s >= \sum_{\alpha,\beta} i_1{}^{\beta_1\ldots\beta_{n_1}}{}_{\alpha_1\ldots\alpha_{n_1}} \, i_2{}^{\beta_{n_1+1}\ldots\beta_{n_2}}{}_{\alpha_1\ldots\alpha_{n_2}} \ldots i_N{}^{\beta_{n_{N-1}}\ldots\beta_L}{}_{\alpha_{n_{N-1}}\ldots\alpha_L} \, | \Gamma, j_i, \alpha_i, \beta_i >$$

And $< A | s >$ is gauge invariant.

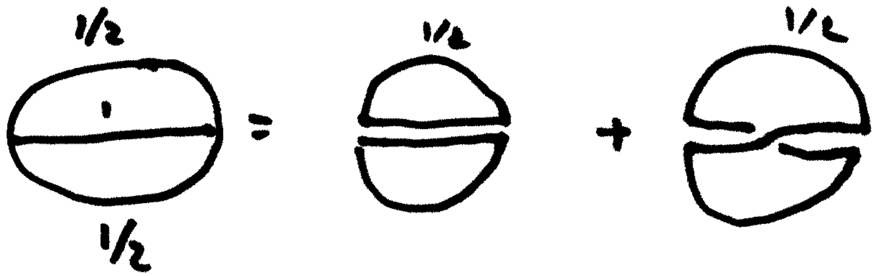
S = 

Some comments about spin networks:

1) The spin network basis is also overcomplete, since intertwiners of valence greater or equal to four can be written in terms of 3j symbols in several forms,



2) Spin networks can be represented in terms of loops. Any representation R^j is a tensor product of the fundamental representation of SU(2), for example



3) States with different graph G or different irreducible representations j are orthogonal.

$$< s \mid s >= 1 \qquad < s \mid s' >= 0$$

The K_{diff} space

Let |s> belonging to K_0 and ϕ_n a diffeomorphism

$$\Psi_S(A) = < A \mid s >, \qquad \Psi_{S_n} = U_{\phi_n} \Psi_S$$

Let us define a map from *Cyl* to *Tcyl:* $$P_{diff} \Psi_S = \sum_{\phi_n} \Psi_{S_n}$$

It is an infinite sum and it does not belong in Cyl, however,

$$< P_{diff} s \mid s' > = \sum_{\phi_n} < s_n \mid s' >$$

Is well defined and different from zero only if $\Gamma_{s_n} = \Gamma_{s'}$

$$U_\phi P_{diff} \mid s > = \sum_{\phi_n} U_\phi \mid s_n > = \sum_{\phi'_n} U_{\phi'_n} \mid s > = P_{diff} \mid s >$$

And the states Pdiff|s> are invariants under diffeomorphisms.

The elements of K_{diff} given by P_{diff} |s> are not in a subspace of K_0, they are distributional on K_0. The inner product is given by,

$$< P_{diff} s \mid P_{diff} s' >_{K_{diff}} =< P_{diff} s \mid s' >_{K_0}$$

The elements of K_{diff} are “spin knots”

This concludes the construction of the kinematical space of loop quantum gravity. In order to gain some insight into the geometric meaning of the states we will introduce operators associated with the geometry.

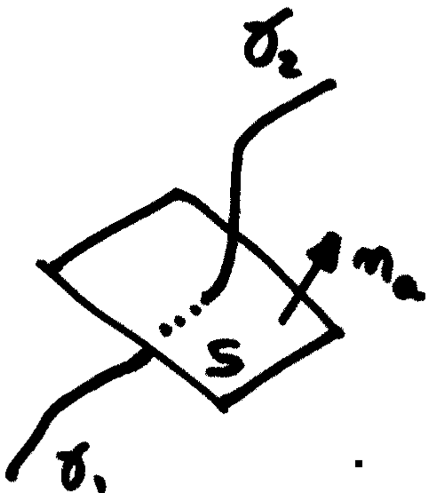
Operators on K

Neither the triads nor the connections are well defined operators on K. However, holonomies are well defined,

$$U_\alpha{}^\beta(A,\gamma)\,|\,\Gamma, j_i, \alpha_i, \beta_i >=|\,\gamma \cup \Gamma, j_i, \frac{1}{2}, \alpha_i, \alpha, \beta_i, \beta >$$

$$\hat{E}_A^{\;i}(x)U(A,\gamma) = i\hbar \frac{\delta}{\delta A_i^A(x)} U(A,\gamma) = \int_\gamma ds\gamma^i(s)\delta^3(x,\gamma(s))[U(A,\gamma_1)\tau_A U(A,\gamma_2)]$$

The electric field smeared on a two surface is also well defined



$$n_a = \eta_{abc} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2}$$

$$\hat{E}_A(S)U(A,\gamma) = \int_S d\sigma^1 d\sigma^2 n_a \tilde{E}^a{}_A(x(\sigma))U(A,\gamma)$$

$$\hat{E}_A(S)U(A,\gamma) = \pm i\hbar U(A,\gamma_1)\tau_A U(A,\gamma_2)$$

Where the sign depends on the relative orientation of the surface and the curve. One needs to smear on a surface since η_{abc} E^a is a two-form.

Geometric operators on K_0

The area operator
C. Rovelli, L. Smolin Nucl. Phys. B442, 593 (1995).

Neither $U(A,\gamma)$ nor $E_A(S)$ are gauge invariant operators. If γ is a closed loop then the $Tr(U(A,\gamma))$ is gauge invariant and f(A, s) can be similarly constructed.

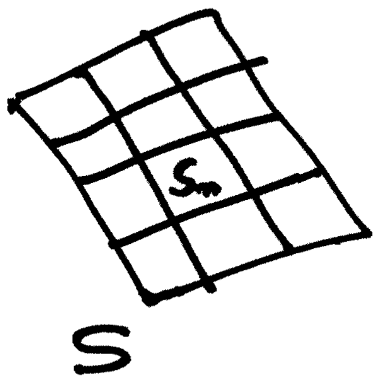
Neither E^A nor $E^A E_A$ are gauge invariant. The fields on S have different gauge transformations at each point on S. Geometric operators like the area operator will be gauge invariant. The area of a two dimensional surface can be written, classically, as,

$$A[S] = \int_S d^2\sigma \sqrt{\det(g_{ij})}, \qquad \mathrm{i,j} = 1,2$$

$$= \int_S d^2\sigma \sqrt{\det\left(g_{ab}\frac{\partial x^a}{\partial \sigma^i}\frac{\partial x^b}{\partial \sigma^j}\right)},$$

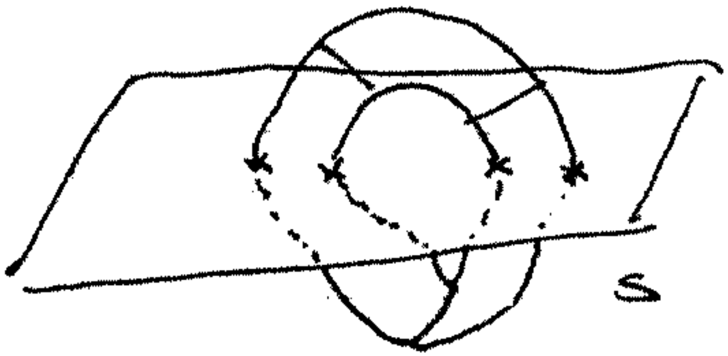
$$= \int_S d^2\sigma \sqrt{\det\left(e_a{}^A e_{bA}\frac{\partial x^a}{\partial \sigma^i}\frac{\partial x^b}{\partial \sigma^j}\right)},$$

$$= \frac{8\pi\gamma G}{c^3}\int_S d^2\sigma \sqrt{n_a \tilde{E}^a{}_A n_b \tilde{E}^{bA}}$$



$$\hat{A}(S) = \lim_{N\to\infty}\sum_n \sqrt{E^A(S_N)E_A(S_N)}$$

One has that, $$\hat{A}(S)\Psi_S[A] = \frac{8\pi G\gamma\hbar}{c^3}\sum_i \sqrt{j_i(j_i+1)}\,\Psi_S[A]$$



The area operator has a discrete spectrum that depends on the Barbero-Immirzi parameter.

$\frac{4\pi G\gamma\hbar}{c^3} \approx 10^{-66} cm^2\gamma$ is the quantum of area.

If the spin network has nodes on the surface,

$$\hat{A}(S)\Psi_S[A] = \frac{4\pi G\gamma\hbar}{c^3}$$
$$\times \sum_i \sqrt{2\underbrace{j^u{}_i(j^u{}_i+1)}_{\text{Outgoing}} + 2\underbrace{j^d{}_i(j^d{}_i+1)}_{\text{Incoming}} - \underbrace{j^t{}_i(j^t{}_i+1)}_{\text{Tangent}}}\,\Psi_S[A]$$

One can show that the area operators do not commute. The commutator has non-trivial action on vertices of valence 4 or higher.

The volume operator

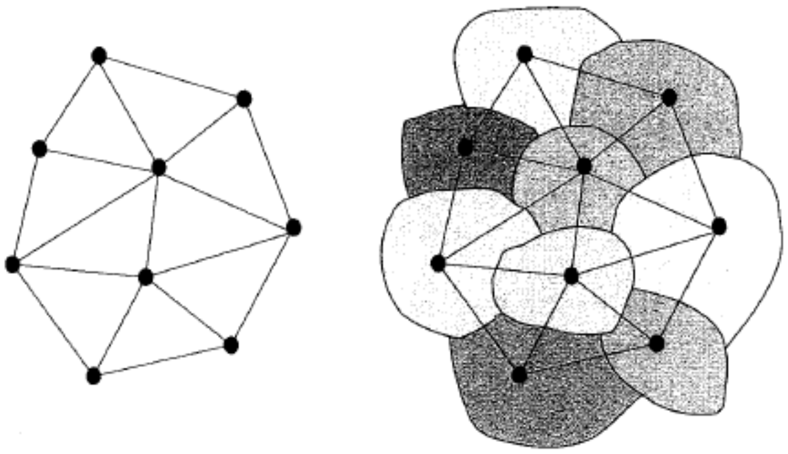
$$V(R) = \int_R d^3x \sqrt{|\det(g)|} = (8\pi G\gamma)^{\frac{3}{2}} \int_R d^3x \sqrt{\frac{1}{3!} |\eta_{abc}\varepsilon^{ABC} \tilde{E}^a{}_A \tilde{E}^b{}_B \tilde{E}^c{}_C|}$$

The quantum operator associated with V is defined by considering a partition of the region R in small volumes ε^3. In each elementary volume the integral may be rewritten in terms of field operators associated to the faces of the "cubical" elementary volume. Non-vanishing contributions only arise in the vertices of the spin networks.

$$\hat{V}(R)|s> = \hat{V}(R)|\Gamma, j_i, i_1 \ldots i_N>$$

$$= \left(\frac{8\pi\gamma G\hbar}{c^3}\right)^{3/2} \sum_{n\subset s\cap R} M_{i_n}{}^{i'_n} |\Gamma, j_i, \ldots i'_n \ldots>$$

The recoupling matrices M are non-vanishing for non planar vertices of valence higher or equal than 4 when acting on K_0. On K the first non-vanishing contribution is for 3 valent vertices.
A. Ashtekar, J. Lewandowski, Adv. Theor. Math. Phys 1,388 (1998).



The graph of an abstract spin network and the ensemble of “chunks of space”, or quanta of volume, it represents. Chunks are adjacent when the corresponding nodes are linked. Each link cuts one elementary surface separating two chunks.

Weaves. It is possible to define families of spin nets such that for $l >> l_{Planck}$ reproduce the classical space with metric g_{ab}.

$$\hat{V}(R)\,|\,s> = \left[V_g(R) + O\left(\frac{l_{\text{Planck}}}{l}\right)\right]|\,s>,$$

$$\hat{A}(S)\,|\,s> = \left[A_g(S) + O\left(\frac{l_{\text{Planck}}}{l}\right)\right]|\,s>.$$

There is an absolute limit to the approximation. Additional refinements of the spin network do not improve the approximation. If one adds lines or nodes, V and A would increase and the error is always l_p/l.

The issue of the dynamics

The physical state space $K \xrightarrow{G^B} K_0 \xrightarrow{C_a} K_{\text{diff}} \xrightarrow{H} H$

$$H = \frac{8\pi G\gamma^2}{\sqrt{|\det(q)|}} \tilde{E}^a{}_A \tilde{E}^b{}_B \left[\varepsilon^{AB}{}_C F_{ab}{}^C - \frac{2(1+\gamma^2)}{\gamma} K_a{}^A K_b{}^A \right] = H_E + H'$$

In the Euclidean case, choosing $\gamma=\pm1$ makes the factor $(\gamma^2-\sigma)$ vanish, with σ the determinant of the flat metric in Cartesian coordinates.

In spite of its very complicated non-polynomial appearance, Thiemann was able to write this constraint in a way that makes the theory amenable to a quantum treatment.

The non-polynomiality has two origins: 1) The Lorentzian term H', 2) the square root of the determinant of q that appears in the denominator. This factor is crucial for the constraint algebra to be consistent. If one ignores the factor there does not exist a regularization of the operator that is background independent of the structure function that appears on the right hand side of algebra of two Hamiltonians. Furthermore, eliminating the factor produces a constraint that is a double density, and we do not have at hand any natural double density on a manifold.

Thiemann starts from the following observation. Let us assume that the following quantities can be promoted to well defined operators on K_{diff}.

1) The total volume on Σ, $V(\Sigma) = \int_\Sigma d^3x\sqrt{|\det(q)|}$

2) The integrated trace of the extrinsic curvature

$$K = \int_\Sigma d^3x\sqrt{|\det(q)|}K_{ab}q^{ab} = \int_\Sigma d^3x K_a{}^A \tilde{E}^a{}_A$$

Starting from them one can define

$$\frac{\tilde{E}^a{}_A \tilde{E}^b{}_B \varepsilon^{ABC}}{\sqrt{|\det(q)|}} = 2\eta^{abc}\frac{\delta V}{\delta \tilde{E}^c{}_C} = 2\eta^{abc}\{A_c^C, V\}$$

$$K_a{}^A = \frac{\delta K}{\delta \tilde{E}^a{}_A} = \{A_c^C, K\}.$$

Which in turn allows to write,

$$H_E = \frac{1}{\sqrt{|\det(q)|}}\mathrm{Tr}\left([\tilde{E}^a, \tilde{E}^b]F_{ab}\right) = 2\eta^{abc}\mathrm{Tr}\left(\{A_a, V\}F_{bc}\right),$$

$$H' = \frac{2}{\sqrt{|\det(q)|}}\mathrm{Tr}\left([K_a, K_b][\tilde{E}^a, \tilde{E}^b]\right) = 2\eta^{abc}\mathrm{Tr}\left([K_a, K_b]\{A_c, V\}\right)$$

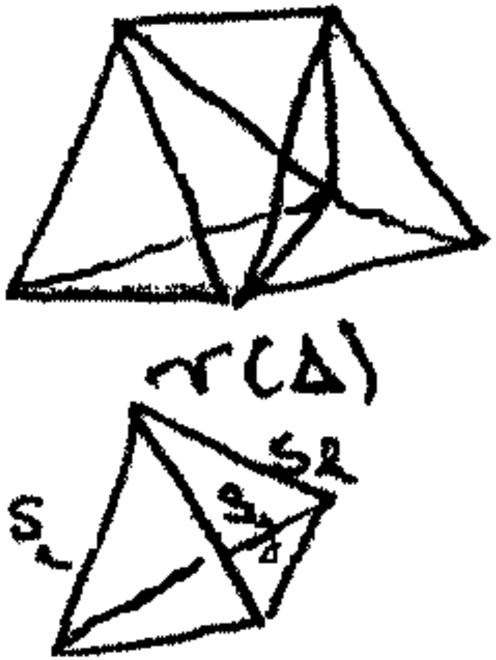
$$= 8\eta^{abc}\mathrm{Tr}[\{A_a, K\}\{A_b, K\}\{A_c, V\}]$$

To promote those expressions to quantum operators, the Poisson brackets get promoted to commutators and as we shall see, all the quantities involved have well defined operators. We have already discussed the volume V.
To quantize K one begins by noticing that,

$$K = -\{V, \int d^3x H_E\},$$

where we have used that the integrated densitized trace of the extrinsic curvature is the “time derivative of the volume”.

We need to regularize H_E. Recall that neither A nor F are well defined in K. We need to represent them using holonomies. To do that we define a triangulation of the spatial manifold Σ.



Let us first consider the regularized constraint at a classical level. We start by considering a tetrahedral decomposition of Σ. Take $v(\Delta)$ to be a vertex of the tetrahedron Δ and $s_i(\Delta)$ the links that converge in v.

Then,

$$H_E^{\Delta}[N] = -\frac{2}{3} N_v \varepsilon^{ijk} \mathrm{Tr}\left(h_{\alpha_{ij}}(\Delta) h_{s_k}(\Delta) \left\{h^{-1}{}_{s_k}(\Delta), V\right\}\right)$$

This is gauge invariant and therefore well defined in K_0.

One can then write the Hamiltonian as a sum over the triangulation of the elementary contribution we just discussed,

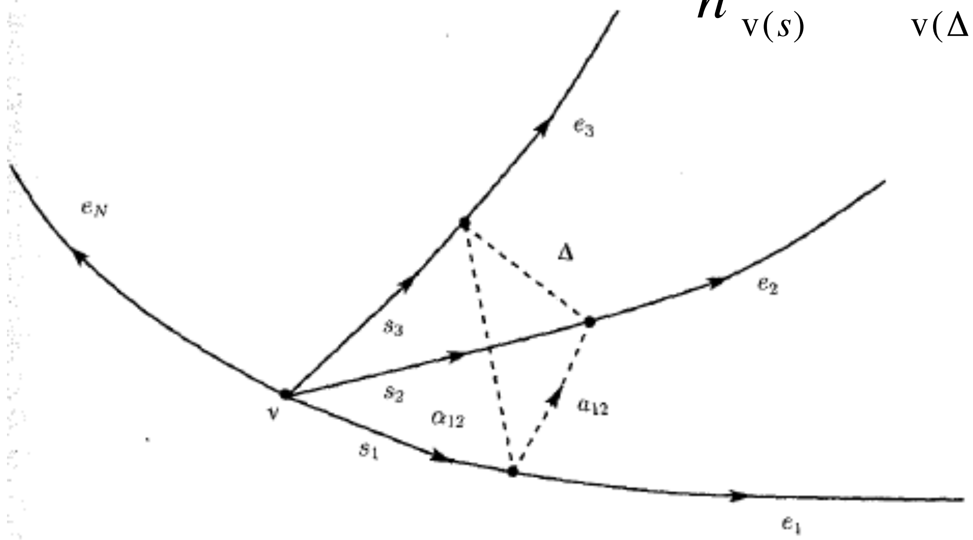
$$H_E = \lim_{\Delta \to 0} \sum_{\Delta \in T} H^{\Delta}{}_E[N] = \lim_{\Delta \to 0} \sum_{\Delta \in T} N_{\mathrm{v}} H^{\mathrm{v}}{}_E$$

Quantization

One is interested in computing $\hat{H}_E \mid s >$

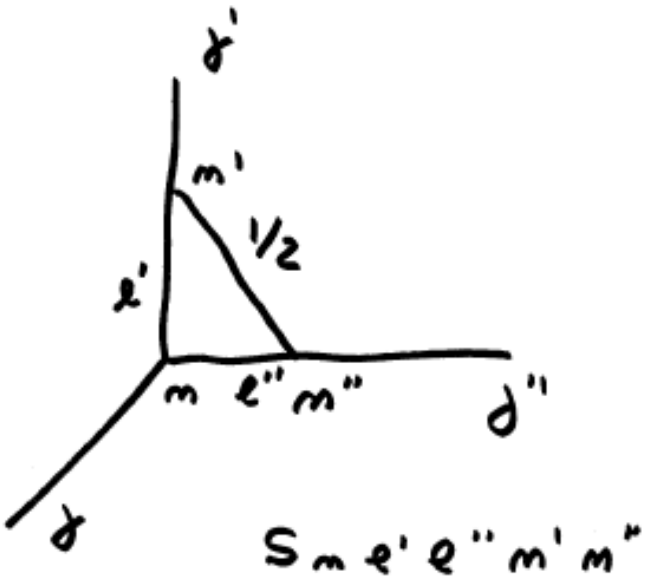
To do it, it is good to consider an adapted triangulation such that some of its vertices coincide with the vertices of the spin network and some of its links coincide with the links of the spin network.

$$H_E{}^{\Delta}[\mathrm{v}] \,|\, s >= -\frac{i}{\hbar} \sum_{\mathrm{v}(s)} N_{\mathrm{v}} \sum_{\mathrm{v}(\Delta)=\mathrm{v}} \varepsilon^{ijk} \mathrm{Tr}\Big(\hat{h}_{\alpha_{ij}}(\Delta) \hat{h}_{s_k}(\Delta) \Big[\hat{h}^{-1}{}_{s_k}(\Delta), \hat{V} \Big] \Big) \,|\, s >$$



Due to the properties of the volume operator the Hamiltonian constraint acts non-trivially at intersections of valence 3 or higher.

Action of H



It introduces a new “extraordinary” link in the fundamental representation. n’ and n” are “extraordinary” vertices. The volume combines spin networks with different l’, l”. These links are added at all vertices in all possible ways.

$$H^E \mid s >= \sum_n N_n \sum_{l',l''} M_{n\,l'\,l''} \mid S_{n\,l'\,l''\,n'\,n''} >$$

The action of H’ can be obtained from the definition of H^E.
The limit Δ->0 in K_{diff} only implies that the additional links are adjacent to the vertex.

We have therefore obtained a well defined finite Hamiltonian constraint.
T. Thiemann, Class. Quan. Grav. 15, 839 (1998).

Open problems:

1) Constraint algebra

The Hamiltonian constraint satisfies the correct algebra on K_{diff}. Recall that classically,

$$\{H(N), H(M)\} = \int d^3x q^{ab} \left(N\partial_a M - M\partial_a N\right)\widetilde{C}_a$$

So as operators o K_{diff} one should have that, $\left[\hat{H}(N), \hat{H}(M)\right] = 0$

One can easily verify that this relation is indeed verified. However, on K_{diff} the rich structure of the right hand side of the equation is lost.

If one had a definition of the Hamiltonian constraint on K_0, then one could verify the full algebra. However, the diffeomorphism constraint is not well defined in K_0.

2) Uniqueness

H is not unique. There exist ambiguities in,
a)The representation of the additional edge ½->j
b)Other forms of adding the small loop.

3) Ultralocality

H acts only on the original vertices of the spin network, not on the ones added by H since the latter are exceptional.

4) Excessive solutions

States that approximate any classical geometry are solutions. In fact any geometry may be approximated by spin nets without extraordinary links and vertices $|s_i>$

$$\text{Let } \; | g >= \sum_i | s_i >,$$

$$< g | \hat{H} | s >= 0 \qquad \forall s,$$

$$\text{so} < g | \; \text{is solution of } \hat{H}.$$

Coupling to matter:

Matter is naturally incorporated in the spin network framework. One needs to extend the kinematical space,

$$|s>=|\Gamma, j_l, k_l, F_n, S_n, i_n, i'_n>$$

The key dynamical result is that Thiemann's construction may be extended and the total Hamiltonian one constructs is a well defined operator on K_{diff} encoding the entire dynamics of the standard model.

The fact that the total Hamiltonian is finite, consistent and well defined is remarkable and is one of the central results of loop quantum gravity.

Incomplete developments:

Inner product on physical states?

Classical limit?

As we have seen, Thiemann's proposal for the Hamiltonian has several open issues.

Other approaches to the dynamics of loop quantum gravity:

Spin foams: Reisenberger, Rovelli, Krasnov, Freidel, Pérez, many others.

Spin foam models for quantum gravity. Alejandro Pérez Jan 2003.
80pp. Class.Quant.Grav.20:R43,2003. e-Print: gr-qc/0301113

Master Constraint: Thiemann and collaborators.

The Phoenix project: Master **constraint program for loop quantum gravity.**
Thomas Thiemann. **Class.Quant.Grav.23:2211-2248,2006**. **gr-qc/0305080**

Uniform discretizations: RG, Pullin.

Uniform **discretizations: A New approach for the quantization of totally constrained systems.**
M. Campiglia, C Di Bartolo, R.G J.Pullin. **Phys.Rev.D74:124012,2006**.
gr-qc/0610023

An application: Cosmology

In addition to the conceptual difficulties we mentioned, one has all the intrinsic complications of a non-linear field theory when dealing with full quantum gravity in 3+1 dimensions.

In order to simplify things one can study approximate models with high degree of symmetry.

Cosmological models approximate space time at large scales by a homogeneous and isotropic space.

If one quantizes the homogeneous, isotropic theory the problem simplifies quite a bit since one is left with a mechanical problem with a finite number of degrees of freedom.The Robertson-Walker metric:

$$ds^2 = -dt^2 + \frac{a(t)^2}{1-kr^2}dr^2 + a(t)^2 r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right)$$

$$k = 1,0,-1, \quad \text{spatial section} = \text{sphere, flat, hyperbolic}$$

Einstein equations for such metric are known as the Friedmann equations:

$$3\frac{\dot{a}^2}{a^2} = 8\pi G\rho - \frac{3k}{a^2},$$

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3p).$$

Equation of state:
Dust: p=0, radiation p=ρ/3
The conservation of energy leads:

$$\text{Dust}: \quad \rho a^3 = const.$$

$$\text{Radiation}: \rho a^4 = const.$$

$$\ddot{a} < 0, \text{no static solutions.}$$

For quantum discussions it is good to introduce fields as matter sources, for instance a scalar field,

$$H_\phi = \frac{1}{2a^3} P_\phi^{\;2} + a^3 W(\phi)$$

$$a^3 \propto \text{volume.}$$

$$\text{Friedmann equation holds for}: \quad \rho(a) = \frac{H_\phi}{a^3}.$$

For small values of ***a*** the kinetic term in the Hamiltonian dominates, which diverges for *a*=0, and leads to a singular solution (Big Bang).

The traditional quantization of these equations does not eliminate the singularity.

Starting from the Hamiltonian constraint,

$$H = -\frac{P_a^{\,2}}{2a} - 2ka + 16\pi G H_\phi = 0$$

One can quantize, $\hat{H}\Psi_E = 0$ $\Psi_E(a,\phi)$

and one gets the Wheeler-DeWitt equation, which is defined up to factor orderings.
One can take a as a time variable and describe the evolution in a.
The energy is still unbounded and the singularity is not removed through the quantization, volume goes to zero and the energy density diverges.

Loop quantum cosmology

We consider Ashtekar variables with gauge and diffeomorphism symmetry fixed to take advantage of the symmetries,

$$A_a{}^B = \frac{1}{2}\left(k - \gamma \dot{a}\right)\delta_a{}^B = c\delta_a{}^B,$$

$$\tilde{E}^a{}_B = p\delta_B{}^a, \qquad |p| = a^2.$$

$$\text{Poisson brackets:} \qquad \{c, p\} = \frac{8\pi G}{3}\gamma.$$

$$\text{Hamiltonian constraint:}$$

$$\mathrm{H} = -\frac{6}{\gamma^2}c^2\sqrt{|p|} + 8\pi G\frac{P_\phi{}^2}{|p|^{3/2}}.$$

We will now proceed to quantize this Hamiltonian “in loop space”.

In a homogeneous isotropic space one can divide space into cells and study the dynamics in one cell. By homogeneity, it will be the same as in any other cell. In the loop representation the fundamental variables are the holonomies h_λ along the sides of the elementary cell and fluxes of the triad field E along the surfaces.

$$h_\lambda = e^{\frac{i\lambda c}{2}}$$

We describe the gravitational part of the kinematical Hilbert space by

$$|\Psi> = \sum_i c_i \, | p_i >,$$

$$< p_i \, | \, p_j > = \delta_{ij}.$$

Notice the sum and the Kronecker delta. One also has that,

$$\Psi(p) = < p \, | \, \Psi >,$$

$$\hat{p}\Psi(p) = p\Psi(p) \quad \text{and} \quad \hat{h}_\lambda \Psi(p) = \Psi(p+\lambda)$$

The derivative of h_λ with respect to λ, and therefore c are not well defined as operators, just like the connection in loop quantum gravity.

In order to write Thiemann's Hamiltonian one needs to realize the volume $V=|p|^{3/2}$ as a quantum operator, which is straightforward in the |p> basis.

$$\hat{H}_{\mathrm{grav}} = \frac{24i\, sg(p)}{8\pi\gamma^3 \mu_0{}^3 l_{Planck}^2} \sin(\mu_0 c)\left(\sin\left(\frac{\mu_0 c}{2}\right)\hat{V}\cos\left(\frac{\mu_0 c}{2}\right) - \cos\left(\frac{\mu_0 c}{2}\right)\hat{V}\sin\left(\frac{\mu_0 c}{2}\right)\right)$$

Just like Thiemann's Hamiltonian introduced finite holonomies to represent the curvature $F_{ab}{}^{C}$. Here μ_0 is finite and the limit μ_0->0 does not exist. This is a manifestation of the discrete structure of space-time. The area operator has a discrete spectrum and μ_0 is determined by the minimum area eigenvalue.

In the |p> representation, H leads to a finite difference equation. Its solution approximates excellently the Wheeler-DeWitt equation far away from the Big Bang. However, in loop quantum gravity the dynamics differs at the Big Bang (and Big Crunch) and both are replaced by quantum bounces.

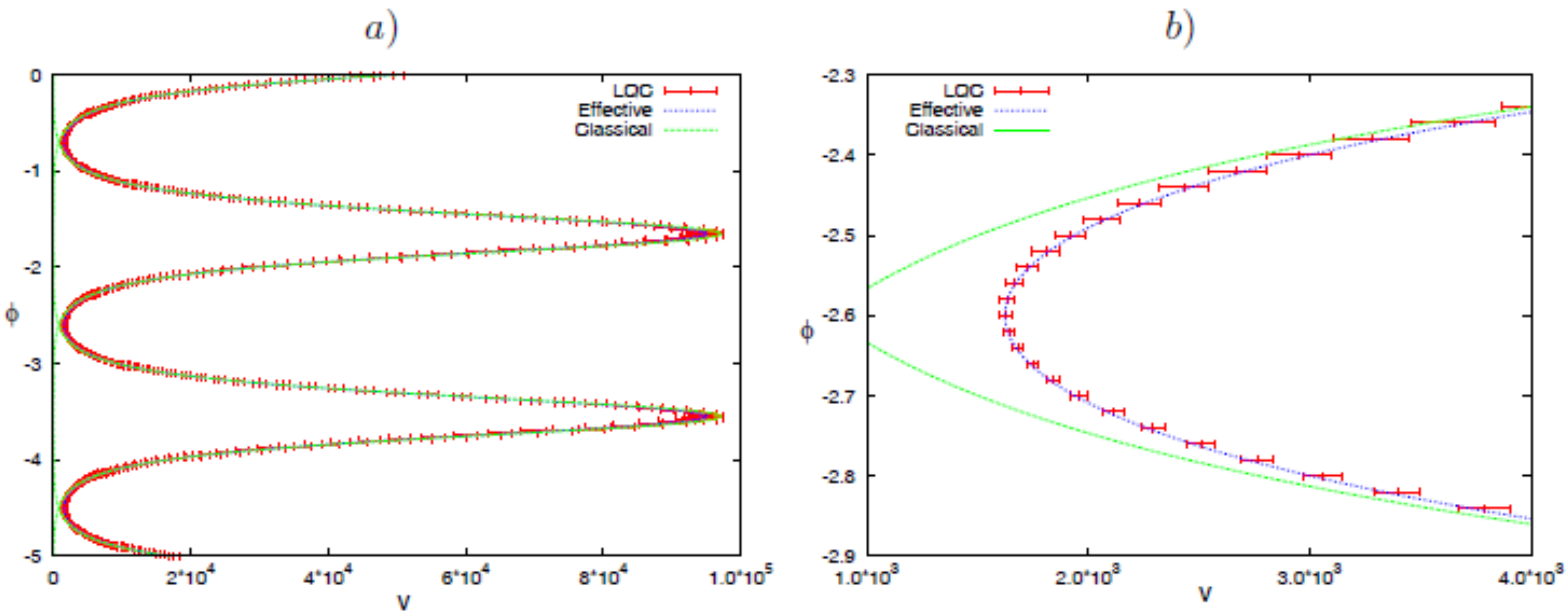


FIG. 2: In the LQC evolution the big bang and big crunch singularities are replaced by quantum bounces. a) Expectation values and dispersion of $|\hat{v}|_\phi$ are compared with the classical trajectory and the trajectory from effective Friedmann dynamics (see (3.27)). The classical trajectory deviates significantly from the quantum evolution at Planck scale and evolves into singularities. The effective trajectory provides an excellent approximation to quantum evolution at all scales. b) Zoom on the portion near the bounce point of comparison between the expectation values and dispersion of $\hat{v}|_\phi$, the classical trajectory and the solution to effective dynamics. At large values of $|v|_\phi$ the classical trajectory approaches the quantum evolution. In this simulation $p_\phi^\star = 5 \times 10^3$, $\Delta p_\phi / p_\phi^\star = 0.018$, and $v^\star = 5 \times 10^4$.

Black hole thermodynamics

In 1972 Hawking noted that the area of a black hole always grows and proved a theorem about it. Bekenstein suggested that the area of a black hole may play a role as an entropy, estimating the entropy of a collapsed object and using the second law of thermodinamics,

$$S_{\mathrm{BH}} = a \frac{k}{\hbar G} A$$

Here a is a dimensionless constant and we have set c=1. The sum of the entropy of the environment the black hole is in, plus S_{BH} always grows.

Initially the idea was received with skepticism. In thermodynamics one has that,

$$T^{-1} = \frac{\partial S}{\partial E} = \frac{32 a \pi k M G}{\hbar}$$

And therefore black holes should have a temperature and emit radiation, something not observed in the classical theory.

However, Hawking, studying quantum fields on the background of a black hole found that

$$T = \frac{\hbar}{8\pi k G M}, \quad \text{and therefore, } a = \frac{1}{4}, \quad \text{and } S_{\mathrm{BH}} = \frac{1}{4}\frac{\hbar}{kG} A$$

This opens several new questions:

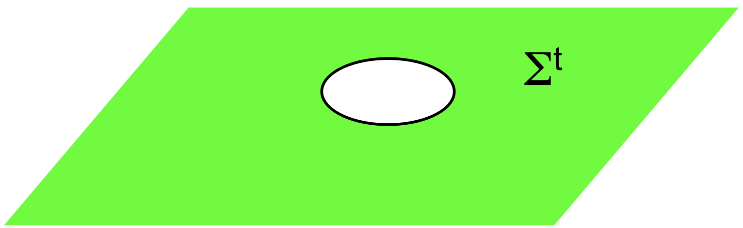
Do this results hold in quantum gravity?

What is the statistical origin of S?

The presence of ħ confirms that quantum gravity must play a role.

We will see that loop quantum gravity leads to this result and does so for black holes of different type.

Let us consider a black hole described approximately by a Schwarzschild metric. It will not be exactly spherical due to quantum fluctuations at the horizon. It will absorb matter and light.



The horizon at a given instant *t* fluctuates, We consider the statistical ensemble of metrics with a given value of the energy or equivalently with a given area.

All the information we have about a black hole is on its horizon, we can cannot get information about what happens behind the horizon through the fluctuations of the latter.

Other types of systems trapped in a box S, one can know what happens in the interior and therefore increase the information and lower the entropy. This does not occur in black holes.

We must therefore determine the ensemble of microstates g_t. In statistical mechanics a microcanonical ensemble has states of a given energy E. In this case we are interested in the state of area A, proportional to M^2. If there are N(A) states with area A,

$$S(A) = k\ln(N(A))$$

In order to determine N(A) we observe that classically such number is infinite but quantum mechanically it will correspond to the number of orthogonal states with area A.

The number of different quantum configurations in the spin net basis will correspond to the number of possible spin nets with area A at the horizon.

We need to take into account a) lines that cross the horizon, b) vertices lying on the surface and c) lines living on the surface. Here we consider the dominant contribution, more precise calculations have been done recently by Lewandowski.

Let i=1,….,n the indices that identify the links of the spin network that cross the horizon and $j_1,\ldots j_n$ their valences.

A line of valence j can be in 2j+1 states. The Hilbert space in the horizon will be,

$$H = \otimes_i H_{ji}$$

I present here a heuristic computation. For a more rigorous calculation see Quantum geometry and black hole entropy. Phys. Rev. Lett. 1997
A. Ashtekar, J.Baez, A.Corichi, K.Krasnov.
The calculation was initially done estimating the number of states of area A dominated by j=1/2 (the case that corresponds more to the classical case, since it maximizes the lines crossing the horizon). Every j=1/2 contributes an area,

$$A_{\frac{1}{2}} = 4\pi\gamma\, l^2_{\text{Planck}}\sqrt{3}$$

Given that we had $\frac{\mathrm{A}}{\mathrm{A}_{\frac{1}{2}}} = \frac{A}{4\pi\gamma l^2{}_{\text{Planck}}\sqrt{3}}$ lines.

This implies that $S_{BH} = k\ln(N) = \frac{k\ln(2)A}{4\pi\gamma\sqrt{3}G\hbar}$

and if $\gamma = \frac{\ln(2)}{\pi\sqrt{3}}$ one recovers the result of Bekenstein and Hawking.

This calculation was initially done for Schwarzschild black holes and was later extended to charged and Kerr black holes, deformed black holes and nonminimally coupled scalar field black holes. In all cases the same value of the Barbero-Immirzi parameter is needed to recover the Bekenstein-Hawking result. Unfortunately we do not know of another means of determining the value of the parameter to check for compatibility.

More recent calculations for the parameter in order to get the Bekenstein Hawking result are Lewandowski and Domagala gr-qc/0407051, Meissner gr-qc/0407052

$$\frac{\ln(2)}{\pi\sqrt{3}} \le \gamma \le \frac{\ln(3)}{\pi\sqrt{3}}$$

Some have attempted to link these values with classical quasinormal modes, but the argument is not compelling enough.

Potentially observable effects

Some possible low energy effects of loop quantum gravity have been studied using rudimentary approaches to the semiclassical approximation. An example is the arrival time of gamma ray bursts (RG, Pullin)

If one studies the propagation of matter fields on weave states one is led to consider possible quantum gravitational effects on the dispersion relations, which may lead to Lorentz violating terms,

$$E^2 = p^2 + m^2 + f(p, l_{\text{Planck}})$$

The calculation is subject to severe limitations:

1)it is a kinematical calculation, the constraints are not enforced.

2)The weave is not the best semiclassical state.

3)In order to have first order corrections in E/E_{Planck} one has to assume that the weave violates parity.

So the calculation is an illustration of a possible effect, not a prediction of the theory that can be used to validate it. But it is fascinating that one may have low energy effects of loop quantum gravity.

In fact the first order effects have been ruled out using radioastronomy and gamma ray bursts.

An important conceptual open question is if LQG violates Lorentz invariance or not.

Conclusions

1) We have a very satisfactory and robust description of the kinematics of canonical quantum gravity.
2) Important conceptual progress, for instance in the problem of time
3) It leads to a very satisfactory form of quantum cosmology

We still do not have a complete understanding of significant aspects:

1) The optimum way of discussing the dynamics.
2) The recovery of general relativity in the semi-classical limit.
3) A complete description of quantum field theory on a quantum space-time.
4) Reliable predictions that can be tested experimentally.