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Equations of Motion in Relativistic Gravity
Fundamental Theories of Physics
179
10.1007/978-3-319-18335-0_7

General Relativistic Two-Body Problem: Theory and Experiment and the Role of Hidden Momentum

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Abstract

Emphasis will be placed on unusual aspects of our derivation, particularly the use of techniques from the more developed realm of QED (quantum electrodynamics). We obtain explicit results for the precession of the spin (rotation) and orbital effects and we discuss the relatively recent (2008) verification of our spin precession result. Unusual features of our orbital precession analysis is the appearance of hidden momentum effects whose origins may be traced to a special relativistic effect.

1 Introduction

The work of Schiff [*1*] in 1960 led to a resurgence of interest in analyzing and measuring the Lense-Thirring effect due to spin (rotation) effects. In particular, Barker and I [*2*], using a potential which we derived using techniques from QED, obtained the classical motion of a gyroscope in the gravitational field of a much larger mass with a quadrupole moment. Our method of derivation was much shorter and more straightforward than conventional methods especially as we made use of familiar Lagrangian concepts. Also, our results agreed with Schiff except that our equations of orbital motion appeared to be different than those of Schiff (although they both led to the same orbital precession results). Later, we traced this difference to the use of a different choice of coordinates [3], which was not due to the freedom enjoyed by the use of general relativity but due to a special relativistic effect, which we will discuss below in connection with hidden momentum.

When the first binary pulsar was discovered [4], we were motivated to extend our results to the two-body domain [5, 6]. However, due to the nature of their system, Hulse and Taylor were unable to observationally verify our results. However, with the discovery of the double pulsar, our results were verified to an accuracy of 13 % [7]. In Sect. 2, we discuss the derivation of our results and their experimental verification. Then, in Sect. 3, we discuss the origin of hidden momentum.

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2 Two-Body Results: Theory and Experimental Verification

QED theory generally deals with the electromagnetic interaction of elementary particles whereas gravitation for the most part is confined to macroscopic systems. Thus, we decided to start with the gravitational interaction of two electrons, given by a one-graviton exchange interaction [8]. Next, making use of the universality of the gravitational interaction, we obtain the classical macroscopic result by letting

$$\begin{aligned}\frac{1}{2}\,\hbar\vec{\sigma}^{(1)} &\rightarrow \vec{S}^{(1)},\\ \frac{1}{2}\,\hbar\vec{\sigma}^{(2)} &\rightarrow \vec{S}^{(2)},\end{aligned} \tag{1}$$

where $\vec{\sigma}^{(1)}$ and $\vec{\sigma}^{(2)}$ are the Pauli spin matrices associated with electrons 1 and 2 and where $\vec{S}^{(1)}$ and $\vec{S}^{(2)}$ are the classical spin angular momenta of the macroscopic masses m_1 and m_2. Thus, we obtain the gravitational potential energy $V_1 + V_2$, correct to order c^{-2} (first post-Newtonian order). This leads to the spin-dependent Lagrangian and Hamiltonian. In order to obtain the spin-independent Hamiltonian H_0 in the center-of-mass system, we start with an expression give by Hiida and Okamura [9] which contains a quantity α which is an arbitrary dimensionless parameter which can take a number of different values, one of which $(\alpha = 0)$ corresponds to the EIH Hamiltonian. However, we found it convenient to select a value in order to obtain a Hamiltonian without a G^2 term. Further work enabled us to include quandrupole moment terms for both masses so that we finally obtained, in an obvious notation, the total Hamiltonian

$$H_t(\alpha) = Mc^2 + H(\alpha) + V_{S1} + V_{S2} + V_{S1,S2} + V_{Q1} + V_{Q2}, \tag{2}$$

from which the total Lagrangian readily followed. The Euler-Lagrange equations were then used to obtain the precessions of the spins and the precession of the orbit. Some general comments on our results are as follows:

(1) If we make the replacement $e^2 \rightarrow Gm^2$ in the result for the electromagnetic interaction for positronium [10], p. 286], we obtain terms of the same structure as our gravitational results. The

numerical coefficients are generally different excepting the case for the spin-spin interaction term $V_{S1,S2}$ where they are the same.

(2) In the large mass approximation $(m_2 \gg m_1)$, the results reduce to the Lense-Thirring results.

(3) Our procedure for calculating the precession of the orbit was greatly facilitated by making use of the fact that the Runge-Lenz vector $\vec{A}$ is a constant in the non-relativistic case. In addition, $\vec{A}$ and $\vec{L}$ are always perpendicular, where $\vec{L}$ is the orbital angular momentum and thus although both $\vec{A}$ and $\vec{L}$ precess in the more general case they must do so in such a way as to ensure that $\vec{A} \cdot \vec{L} = 0$. In other words, the secular results for the precession of the orbit are of the form

$$\begin{aligned} \dot{\vec{L}}_{av} &= \vec{\Omega}^* \times \vec{L}, \\ \dot{\vec{A}}_{av} &= \vec{\Omega}^* \times \vec{A}. \end{aligned} \tag{3}$$

The fact that $\vec{\Omega}^*$ appears in both equations provided a beautiful check on the results.

(4) Defining the total angular momentum as

$$\vec{J} \equiv \vec{L} + \vec{S}^{(1)} + \vec{S}^{(2)}, \tag{4}$$

we found that

$$\dot{\vec{J}}_{av} = \vec{\Omega}^* \times \vec{L} + \vec{\Omega}_{av}^{(1)} \times \vec{S}^{(1)} + \vec{\Omega}_{av}^{(2)} \times \vec{S}^{(2)} \equiv 0, \tag{5}$$

where $\vec{\Omega}^*$ is the precession of $\vec{L}$ and where $\vec{\Omega}_{av}^{(2)}$ is the precession of the spin of body 2, which can be obtained from the result for $\vec{\Omega}_{av}^{(1)}$ by interchanging the indices 1 and 2. In other words, the total angular momentum is conserved, which provides another excellent check on our results.

(5) The orbital precession $\vec{\Omega}^*$ was broken down into time derivatives of the longitude of the ascending node, the argument of the perihelion and the inclination of the orbit, following astronomical and space physics practise [5]. This is clearly displayed in Figs. 2 and 3 of [*11*]. In addition, we note that the first verification of the one-body Lense-Thirring results by Ciufolini and Pavlis [*12*] made use of nodal precession.

(6) One notable result which emerges is that, in the case of arbitrary masses m_1 and m_2, the dominant <u>spin-orbit contribution</u> to the *spin* precession of body 1 is a factor $(m_2 + \mu/3)/(m_1 + m_2)$ times what it would be for a test body moving in the field of a fixed central mass $(m_1 + m_2)$. Here μ denotes the reduced mass $m_1\, m_2/(m_1 + m_2)$. This contrasts with the result of Robertson for the *periastron* procession where the corresponding factor is unity. This result, which was published in 1975 [*5*, *6*] had to await until 2008 (a period of 33 years) for its verification by the work of Breton et al. [*7*] utilizing the double pulsar PSR J0737-3039A/B [*13*].

This pulsar consists of two neutron stars in a highly relativistic 2.45 h orbit, which displays an eclipse of pulsar A of the order of 30 s. when pulsar A passes behind pulsar B, enabling a measurement of the relativistic precession of the spin axis of pulsar B around the total orbital angular momentum $\vec{L}$.

The double pulsar system, is the only system observed so far in which both components are pulsing neutron stars and has the added advantage that the system is observed nearly perfectly edge-on. Its periastron precession of 16.9° /yr has been measured with enormous accuracy (to 6 significant figures) which enabled the total mass $M = m_1 + m_2 = 2.2587\ M_\odot$ to be measured also very accurately. However, in order to obtain the spin-orbit precession it is necessary to determine both m_1 and m_2. These were determined from the projected semi-major values and the Shapiro time delay with the results $m(A) = m_2 = 1.3381\ M_\odot$ and $m(B) = m_1 = 1.2489\ M_\odot$ so that the measured precession rate of pulsar B is $4.77 \pm 0.66^\circ$ /yr, in agreement, to an accuracy of 13 % [7], with the theoretical rate [5] of $5.0734 \pm 0.0007^\circ$ /yr. Further details on our theoretical work and recent experiments appear in [*11*, *14*].

(7) There is also a precession of $\vec{L}$ in the double pulsar system due to spin precession effects, as is clear from Eq. (5). The dominant effect is due to pulsar A [*15*] since its spin frequency is 122 times larger than that of pulsar B. However, the effect is only about $4.06''$ /yr and thus may be difficult to observe.

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3 Hidden Momentum

In the introduction, we stated that, even in the 1-body case, our results for the equations of orbital motion differed from these of Schiff but that they both led to the same orbital precession results. As we pointed out [*3*], the resolution of this problem may be traced to the work of Moller [*16*] and [*17*], p. 176]. He pointed out that in special relativity, a particle with structure and "spin" (its angular momentum vector in the rest system) is subject to a spin supplementary condition, which "— expresses in a covariant way that the proper center of mass is the center of mass in its own rest system (K^0)—" and that "—the difference between simultaneous positions of the centre of mass in K (obtained from K^0 by a Lorentz transformation with velocity $(\vec{v})$ and the proper centre of mass (in K^0)—" is

$$\Delta\vec{r} = \frac{\vec{S} \times \vec{v}}{mc^2}, \tag{6}$$

where $(\vec{S})$ is the spin and m is the rest mass. In essence, it is related to the fact that, in special relativity, there are two rest systems for the particle, zero velocity and zero momentum, reflecting the choice of spin supplementary conditions and the fact that only in special cases are the velocity and momentum proportional to each other [*18*] and [*19*], see Eq. (7) which for ms the basis of a more rigorous quantum mechanical calculation]. We refer to an extensive review for more details [*20*].

It is clear that an increase in the coordinate by an amount of $\Delta\vec{r}$ will give rise to an increase in the momentum, which is generally referred to as hidden momentum $\Delta\vec{P}$. An expression for $\Delta\vec{P}$ was obtained in two extensive papers [*21*, *22*] but a simple derivation is possible as we shall now demonstrate. Thus, taking the time derivative of Eq. (6), and neglecting the very small second order terms in $\vec{S}$ which arise in the relation between the velocity $\vec{v}$ and momentum $\bar{P}$, we obtain

$$\Delta\bar{P} = \frac{\vec{S} \times \vec{a}}{c^2} , \tag{7}$$

where $\vec{a}$ is the acceleration. This is a general expression for the hidden momentum.

In particular, in the realm of electrodynamics, since the expression for a magnetic dipole moment $\vec{M}$ is derived from either the spin of a particle or from a steady current (bodies with structure in both cases), it is clear that

$$\Delta\vec{P} = k_1 \frac{\vec{M} \times \vec{a}}{c^2} , \tag{8}$$

where k_1 is a constant.

In the particular case where the particle with a magnetic moment $\vec{M}$ is interacting with a pointlike electric charge e then, the electric field $\vec{E}$ created gives rise to an acceleration that is given by $e\vec{E}/m$ so that

$$\Delta\vec{P} = k_2 \frac{\vec{M} \times \vec{E}}{c^2} , \tag{9}$$

where k_2 is a constant. This is essentially the expression used in all discussions in electromagnetism [*23–29*] for the hidden momentum. Thus, the hidden momentum of electrodynamics is a particular consequence of the motion of a spinning body (or, alternatively, a magnetic moment) for which, due to special relativistic effects, the simultaneous position of the center of mass in the moving system is different [by an amount given in Eq. (6)] than the proper center of mass.

Acknowledgments

This work was partially supported by the National Science Foundation under grant no. ECCS-1125675.

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