Lorentz transformation of blackbody radiation

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We present a simple calculation of the Lorentz transformation of the spectral distribution of blackbody radiation at temperature T. Here we emphasize that T is the temperature in the blackbody rest frame and does not change. We thus avoid the confused and confusing question of how temperature transforms. We show by explicit calculation that at zero temperature the spectral distribution is invariant. At finite temperature we find the well-known result familiar in discussions of the 2.7 K cosmic radiation.

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Here we present a straightforward derivation of the spectral distribution of blackbody radiation as seen in a moving frame. A key feature of our discussion is the notion of the temperature in the blackbody rest frame: the frame in which the energy-momentum tensor is diagonal and the spectral distribution is isotropic. This temperature is fixed, i.e., it does not change under Lorentz transformation. In this way, we avoid the confused and confusing question of how temperature transforms [1]. An important result is that at zero temperature the spectral distribution is invariant under Lorentz transformation. This is not an entirely new result. It has long been recognized that in quantum electrodynamics the vacuum state is invariant, but our derivation is explicit. Finally, at finite temperature the result agrees with that obtained nearly half a century ago in connection with the problem of detection of the earth's motion through the 2.7 K cosmic radiation [2]. However, as discussed in detail in [1], there is a wide variety of opinions among the different authors in [2], in addition to many previous authors, as to how T transforms, if at all. Thus, our resolution of this question is an important part of our presentation.

We begin with the familiar expressions of quantum electrodynamics for the free electric and magnetic field operators [3]:

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha}\sqrt{\frac{2\pi\hbar\omega}{V}}(ia_{\mathbf{k},\alpha}\hat{\mathbf{e}}_\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - ia^\dagger_{\mathbf{k},\alpha}\hat{\mathbf{e}}^*_\alpha e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}),$$
$$\mathbf{B}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha}\sqrt{\frac{2\pi\hbar\omega}{V}}(ia_{\mathbf{k},\alpha}\hat{\mathbf{k}}\times\hat{\mathbf{e}}_\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - ia^\dagger_{\mathbf{k},\alpha}\hat{\mathbf{k}}\times\hat{\mathbf{e}}^*_\alpha e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}), \quad (1)$$

where $a_{\mathbf{k},\alpha}$ and $a^\dagger_{\mathbf{k},\alpha}$ are the usual lowering and raising operators for the field oscillators. Here we should keep in mind that $\omega = ck$. The expressions (1) were obtained by quantizing in a box of volume V. We have in mind that the box is at rest with walls at temperature T. The radiation within comes to equilibrium at that temperature and we obtain the familiar result [4]

$$\langle a_{\mathbf{k},\alpha}a^\dagger_{\mathbf{k}',\alpha'} + a^\dagger_{\mathbf{k}',\alpha'}a_{\mathbf{k},\alpha}\rangle = \coth\left(\frac{\hbar\omega}{2kT}\right)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}, \quad (2)$$

where the braces $\langle\cdots\rangle$ indicate the thermal equilibrium expectation value. We shall need the correlation functions of the field fluctuations:

$$C^{(\text{el-el})}_{jm}(\mathbf{r}-\mathbf{r}',t-t') = \tfrac{1}{2}\langle E_j(\mathbf{r},t)E_m(\mathbf{r}',t') + E_m(\mathbf{r}',t')E_j(\mathbf{r},t)\rangle,$$
$$C^{(\text{mag-mag})}_{jm}(\mathbf{r}-\mathbf{r}',t-t') = \tfrac{1}{2}\langle B_j(\mathbf{r},t)B_m(\mathbf{r}',t') + B_m(\mathbf{r}',t')B_j(\mathbf{r},t)\rangle,$$
$$C^{(\text{el-mag})}_{jm}(\mathbf{r}-\mathbf{r}',t-t') = \tfrac{1}{2}\langle E_j(\mathbf{r},t)B_m(\mathbf{r}',t') + B_m(\mathbf{r}',t')E_j(\mathbf{r},t)\rangle. \quad (3)$$

With the expressions (1) for the fields and (2) for the thermal expectation of the operators, we use the prescription $\sum_k \to \frac{V}{(2\pi)^3}\int d\mathbf{k}$ to form the limit $V\to\infty$ and obtain the explicit results

$$C^{(\text{el-el})}_{jm}(\mathbf{r},t) = C^{(\text{mag-mag})}_{jm}(\mathbf{r},t) = \frac{\hbar}{(2\pi)^2}\int d\mathbf{k}\,\omega\coth\frac{\hbar\omega}{2kT}\times(\delta_{jm}-\hat{\mathbf{k}}_j\hat{\mathbf{k}}_m)\cos(\mathbf{k}\cdot\mathbf{r}-\omega t),$$
$$C^{(\text{el-mag})}_{jm}(\mathbf{r},t) = \frac{\hbar}{(2\pi)^2}\int d\mathbf{k}\,\omega\coth\frac{\hbar\omega}{2kT}\times e_{jml}\hat{\mathbf{k}}_l\cos(\mathbf{k}\cdot\mathbf{r}-\omega t). \quad (4)$$

As a first application of these results, we calculate the spectral distribution of blackbody radiation. The energy density of the electromagnetic field in thermal equilibrium is given by

$$W = \frac{\langle E^2\rangle + \langle B^2\rangle}{8\pi} = \frac{C^{(\text{el-el})}_{jj}(0,0)}{4\pi}. \quad (5)$$

We can write

$$W = \int_0^\infty d\omega\int d\Omega\,\rho(\omega,\hat{\mathbf{k}}), \quad (6)$$

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where $\rho(\omega,\hat{\mathbf{k}})d\omega\, d\Omega$ is the energy density of radiation with frequency in the interval $d\omega$ and propagating in solid angle $d\Omega$ about the direction $\hat{\mathbf{k}}$. With the explicit results (4) we find that the spectral distribution in the rest frame is given by

$$\rho(\omega,\hat{\mathbf{k}}) = \frac{\hbar}{(2\pi c)^3}\omega^3 \coth \frac{\hbar\omega}{2kT}. \tag{7}$$

Of course, this is independent of the direction of propagation and, except for a factor of 4π, is just the Planck spectrum with the inclusion of the zero-point fluctuations.

We now consider the question of how this spectral distribution transforms under a Lorentz transformation. Here we should remind ourselves that the above discussion is understood to be in the rest frame of the blackbody radiation: the frame in which the energy-momentum tensor is diagonal and the spectral distribution is isotropic. The temperature T is the temperature in this rest frame and *does not transform*. We begin with the well-known expressions for the Lorentz transformation of the fields from a frame at rest to a frame moving with velocity $\mathbf{v}$ [5]:

$$\begin{aligned} \mathbf{E}'(\mathbf{r}',t') &= \hat{\mathbf{v}}\cdot\mathbf{E}(\mathbf{r},t)\hat{\mathbf{v}} \\ &\quad + \gamma\left[\mathbf{E}(\mathbf{r},t) - \hat{\mathbf{v}}\cdot\mathbf{E}(\mathbf{r},t)\hat{\mathbf{v}} + \frac{\mathbf{v}}{c}\times\mathbf{B}(\mathbf{r},t)\right], \\ \mathbf{B}'(\mathbf{r}',t') &= \hat{\mathbf{v}}\cdot\mathbf{B}(\mathbf{r},t)\hat{\mathbf{v}} \\ &\quad + \gamma\left[\mathbf{B}(\mathbf{r},t) - \hat{\mathbf{v}}\cdot\mathbf{B}(\mathbf{r},t)\hat{\mathbf{v}} - \frac{\mathbf{v}}{c}\times\mathbf{E}(\mathbf{r},t)\right], \end{aligned} \tag{8}$$

where, as usual, $\gamma = 1/\sqrt{1-v^2/c^2}$. With this we find for the energy density in the moving frame:

$$\begin{aligned} W' &= \frac{\langle E'^2\rangle + \langle B'^2\rangle}{8\pi} \\ &= \frac{1}{4\pi}\Big\{C_{jj}^{(\text{el-el})}(0,0) + 2(\gamma^2-1) \\ &\quad \times \big[C_{jj}^{(\text{el-el})}(0,0) - \hat{v}_j\hat{v}_m C_{jm}^{(\text{el-el})}(0,0)\big] \\ &\quad + 2\gamma^2\frac{v_l}{c}e_{ljm}C_{jm}^{(\text{el-mag})}(0,0)\Big\}. \end{aligned} \tag{9}$$

Using the expressions (4) for the correlation functions together with the expression (7) for the spectral distribution, we can write this in the form

$$\begin{aligned} &\int_0^\infty d\omega' \int d\Omega' \rho'(\omega',\hat{k}') \\ &\quad = \int_0^\infty d\omega \int d\Omega \gamma^2\left(1 - \hat{\mathbf{k}}\cdot\frac{\mathbf{v}}{c}\right)^2 \rho(\omega,\hat{\mathbf{k}}). \end{aligned} \tag{10}$$

That is,

$$\rho'(\omega',\hat{\mathbf{k}}') = \gamma^2\left(1 - \hat{\mathbf{k}}\cdot\frac{\mathbf{v}}{c}\right)^2\rho(\omega,\hat{\mathbf{k}})\frac{d\omega}{d\omega'}\frac{d\Omega}{d\Omega'}. \tag{11}$$

To proceed, we introduce the Lorentz transformation of the propagation vector,

$$\begin{aligned} \omega' &= \gamma(\omega - \mathbf{k}\cdot\mathbf{v}), \\ \mathbf{k}' &= \mathbf{k} - \hat{\mathbf{v}}\cdot\mathbf{k}\hat{\mathbf{v}} + \gamma\left(\hat{\mathbf{v}}\cdot\mathbf{k}\hat{\mathbf{v}} - \omega\frac{\mathbf{v}}{c^2}\right). \end{aligned} \tag{12}$$

For photons $\omega = ck$, which implies $\omega' = ck'$. In this case, we get from the first Lorentz transform equation the formula for the Doppler shift,

$$\omega' = \gamma\left(1 - \hat{\mathbf{k}}\cdot\frac{\mathbf{v}}{c}\right)\omega, \tag{13}$$

and from the second the aberration formula,

$$\hat{\mathbf{k}}'\cdot\hat{\mathbf{v}} = \frac{\hat{\mathbf{k}}\cdot\hat{\mathbf{v}} - \frac{v}{c}}{1 - \hat{\mathbf{k}}\cdot\frac{\mathbf{v}}{c}}. \tag{14}$$

It will be useful to have the inverse of these formulas, obtained by solving the aberration formula for $\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}$ and then putting the result in the Doppler shift formula. The result is

$$\hat{\mathbf{k}}\cdot\hat{\mathbf{v}} = \frac{\hat{\mathbf{k}}'\cdot\hat{\mathbf{v}} + \frac{v}{c}}{1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{c}}, \quad \omega = \gamma\left(1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{c}\right)\omega'. \tag{15}$$

From these formulas, we find

$$\begin{aligned} d\omega &= \gamma\left(1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{c}\right)d\omega', \\ d\Omega &= \frac{d\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}}{d\hat{\mathbf{k}}'\cdot\hat{\mathbf{v}}}d\Omega' = \frac{d\Omega'.}{\gamma^2\left(1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{c}\right)^2}. \end{aligned} \tag{16}$$

With these results on the right-hand side of the identity (11), we find that the spectral distribution in the moving frame is given by

$$\rho'(\omega',\hat{\mathbf{k}}') = \frac{\rho\left[\gamma\left(1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{c}\right)\omega',\hat{\mathbf{k}}\right]}{\gamma^3\left(1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{\mathbf{c}}\right)^3}. \tag{17}$$

Finally, with the expression (7) for the spectral distribution in the rest frame, we find that in the moving frame it takes the explicit form

$$\rho'(\omega',\hat{\mathbf{k}}') = \hbar\left(\frac{\omega'}{2\pi c}\right)^3 \coth\left(\frac{\hbar\gamma\left(1 + \hat{\mathbf{k}}'\cdot\frac{\mathbf{v}}{c}\right)\omega'}{2kT}\right). \tag{18}$$

The expression (18) is our key result. First, we note that at zero temperature this spectral distribution in the moving frame is exactly of the form of that in the rest frame. That is, the spectral distribution at zero temperature is invariant under Lorentz transformations. Moreover, at finite temperature our result is exactly of the form long known in discussions of motion through the 2.7 K cosmic radiation [2]. However, our derivation has made it clear that T is the invariant temperature in the blackbody rest frame. There has therefore been no need to get into the question of how temperature transforms under Lorentz transformations: our T is the temperature in the rest frame and does not change.

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