



Entanglement without dissipation: A touchstone for an exact comparison of entanglement measures

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Dedicated to the memory of Krzysztof Wodkiewicz.

ABSTRACT

Entanglement, which is an essential characteristic of quantum mechanics, is the key element in potential practical quantum information and quantum communication systems. However, there are many open and fundamental questions (relating to entanglement measures, sudden death, etc.) that require a deeper understanding. Thus, we are motivated to investigate a simple but non-trivial correlated two-body continuous variable system in the absence of a heat bath, which facilitates an *exact* measure of the entanglement at all times. In particular, we find that the results obtained from all well-known existing entanglement measures agree with each other but that, in practice, some are more straightforward to use than others.

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1. Introduction

Entanglement, which is an essential characteristic of quantum mechanics, plays a key role in all applications related to information science [1–4]. Entanglement describes correlations between two or more particles or subsystems (qubits, oscillators, etc.). Despite the fact that much insight has already been obtained, much remains to be done as underlined by the title of a recent book on the subject [5]. For example, whereas there is a plethora of entanglement measures, there is also a consensus that a more unifying fundamental measure needs to be developed. Also, the discovery of “entanglement sudden death” (ESD) [6–8], in contrast to the well-known exponential decay of decoherence, requires a deeper understanding. Since investigations of ESD have also incorporated heat bath effects, master equations have been the tool of choice, despite their inherent limitations [9,10]. Thus, we are motivated to analyze a simple but non-trivial correlated system which displays entanglement in the absence of a heat bath. For this system, an *exact* measure of entanglement exists which provides us with a touchstone for judging some of the various entanglement measures discussed in the literature. Since some of these measures involve entropy considerations, this should also throw some light on whether there is a close relationship between the entanglement of a system and its entropy.

Thus, in Section 2, we will consider a system of two free particles in an initially entangled state and we calculate its time dependence. In Sections 3–5, we examine the same state using various

entanglement criteria and demonstrate explicitly the various steps needed to demonstrate entanglement at all times. In Section 6, we present our conclusions that the various entanglement criteria lead to the same results but that the logarithmic negativity entanglement criterion is the simplest to use.

2. Entangled wave function

We consider two free particles, each of mass m , at positions x_1 and x_2 , in an initially entangled Gaussian state. Thus, we are dealing with a system with continuous degrees of freedom (as distinct from a system of discrete variables such as qubits), applicable to particle position or momenta or to the field modes of light (of interest in connection with linear optical quantum computing).

The most general initial Gaussian wave function is

$$\psi(x_1, x_2; 0) = \frac{(a_{11}a_{22} - a_{12}^2)^{1/4}}{\sqrt{2\pi}} \exp\left\{-\frac{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2}{4}\right\}. \quad (2.1)$$

In order that this state be square-integrable we must of course assume that a_{11} and a_{22} are positive and that $a_{11}a_{22} - a_{12}^2 > 0$. We specialize to the symmetric case, by choosing

$$a_{11} = a_{22} = \frac{1}{\sigma^2} + \frac{1}{4d^2}, \quad a_{12} = \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right) < 0, \quad (2.2)$$

so that $2d > \sigma$ (the opposite choice would lead to similar conclusions). As we shall see presently, d corresponds to the width of the center-of-mass system. Thus, a_{12} provides a measure of entanglement at $t = 0$. In fact, since we are dealing with free particles

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we expect that the entanglement will not change in time. However, as we shall see, the time-dependent coefficient of x_1x_2 will *not* serve as a measure of entanglement.

The above equations enable us to write (2.1) in the form

$$\psi(x_1, x_2; 0) = \frac{1}{\sqrt{2\pi\sigma d}} \exp \left\{ -\frac{(x_1 - x_2)^2}{4\sigma^2} - \frac{(x_1 + x_2)^2}{16d^2} \right\}. \quad (2.3)$$

Next, we transform to center-of-mass and relative coordinates [11]

$$x = x_1 - x_2; \quad X = (x_1 + x_2)/2, \quad (2.4)$$

$$m_x = m/2; \quad M = 2m \quad (2.5)$$

to give the result

$$\psi(x_1, x_2; 0) = \phi(x)\Psi(X) = \frac{1}{\sqrt{2\pi\sigma d}} \exp \left\{ -\frac{x^2}{4\sigma^2} - \frac{X^2}{4d^2} \right\}. \quad (2.6)$$

It is clear that σ is the width of the relative coordinate system whereas d is the width of the center-of-mass coordinate system.

For future reference, we also note that the corresponding relative and center-of-mass momenta are

$$p = \frac{1}{2}(p_1 - p_2); \quad P = p_1 + p_2, \quad (2.7)$$

respectively. It is clear that both (x, p) and (X, P) satisfy the usual commutation relations for conjugate canonical variables.

Since our wave function in the transformed coordinates now behaves as the product of two independent Gaussian wave packets, we can now apply the exact propagation method for a free particle [11] to obtain the wave function at a time t , with the result

$$\begin{aligned} \psi(x_1, x_2; t) &= \sqrt{\frac{2\sigma d}{\pi(\sigma^2 + \frac{i\hbar t}{m})(4d^2 + \frac{i\hbar t}{m})}} \\ &\times \exp \left\{ -\frac{(x_1 - x_2)^2}{4(\sigma^2 + \frac{i\hbar t}{m})} - \frac{(x_1 + x_2)^2}{4(4d^2 + \frac{i\hbar t}{m})} \right\} \\ &= \sqrt{\frac{2\sigma d}{\pi(\sigma^2 + \frac{i\hbar t}{m})(4d^2 + \frac{i\hbar t}{m})}} \\ &\times \exp \left\{ -\frac{x^2}{4(\sigma^2 + \frac{i\hbar t}{m})} - \frac{X^2}{4(d^2 + \frac{i\hbar t}{4m})} \right\} \\ &= \sqrt{\frac{2\sigma d}{\pi(\sigma^2 + \frac{i\hbar t}{m})(4d^2 + \frac{i\hbar t}{m})}} \\ &\times \exp \left\{ -\frac{x^2}{4\sigma^2(t)} \left[1 - \frac{i\hbar t}{m\sigma^2} \right] - \frac{X^2}{4\sigma_d^2(t)} \left[1 - \frac{i\hbar t}{4md^2} \right] \right\}, \end{aligned} \quad (2.8)$$

where

$$\sigma^2(t) = \sigma^2 \left\{ 1 + \left(\frac{\hbar t}{m\sigma^2} \right)^2 \right\} \quad (2.9)$$

and

$$\sigma_d^2(t) = d^2 \left\{ 1 + \left(\frac{\hbar t}{4md^2} \right)^2 \right\}. \quad (2.10)$$

It follows that [12]

$$\langle x(t) \rangle = 0; \quad \langle X(t) \rangle = 0; \quad \langle p(t) \rangle = 0; \quad \langle P(t) \rangle = 0 \quad (2.11)$$

$$\langle x^2(t) \rangle = \sigma^2(t); \quad \langle X^2(t) \rangle = \sigma_d^2(t) \quad (2.12)$$

and

$$\langle p^2(t) \rangle = \frac{\hbar^2}{4\sigma^2}; \quad \langle P^2(t) \rangle = \frac{\hbar^2}{4d^2}. \quad (2.13)$$

We also note that

$$a_{11}(t) = a_{22}(t) = \frac{1}{\sigma^2(t)} + \frac{1}{4\sigma_d^2(t)}. \quad (2.14)$$

In addition, we denote the coefficient of the real part of the $(-\frac{1}{2}x_1x_2)$ power in the exponential in (2.8) by $a_{12}(t)$ to obtain

$$a_{12}(t) = \left\{ -\frac{1}{\sigma^2(t)} + \frac{1}{4\sigma_d^2(t)} \right\}. \quad (2.15)$$

However, it is now clear that $a_{12}(t)$ cannot be used as a measure of entanglement because it predicts that entanglement decreases in time, eventually falling to zero [see the following section, especially (3.16)] whereas we expect that entanglement does not change in time. On the other hand, we have the tools to investigate the results obtained by use of various entanglement measures discussed in the literature.

More generally, we point out that the starting point for all investigations is the well-known separability condition

$$\rho = \sum_j c_j \rho_j^{(1)} \otimes \rho_j^{(2)}, \quad (2.16)$$

where ρ is the density matrix of the quantum state which is written as a convex combination of tensor product states for the j states and where the individual terms are normalized so that $\sum_j c_j = 1$. If this decomposition is not possible, then we say that the state is entangled. We now turn to some specific entanglement criteria which have been proposed.

3. Duan et al. [13] criterion

Using the uncertainty principle, Duan et al. [13] derived a sufficient criterion for inseparability for a pair of EPR type operators for continuous variable systems. Further work on this topic appears in [14,15]. In particular, this approach applies to our problem. Following [13], we write

$$u = \frac{1}{L} \left(|a|x_1 + \frac{1}{a}x_2 \right), \quad v = \frac{L}{\hbar} \left(|a|p_1 - \frac{1}{a}p_2 \right). \quad (3.1)$$

(except that we have introduced the parameter L which has the dimension of length), where a is an arbitrary, non-zero real number. Use of the uncertainty relation [13] leads to the result

$$\frac{1}{L^2} \langle \Delta u^2 \rangle + \frac{L^2}{\hbar^2} \langle \Delta v^2 \rangle \geq \left(a^2 + \frac{1}{a^2} \right), \quad (3.2)$$

for any L . For $L = 1$, this is the Duan et al. result. However, by minimizing with respect to a and L , we will obtain an improved version given by (3.11) below.

For a state that is *not* entangled, form

$$\langle u^2 \rangle + \langle v^2 \rangle = \frac{a^2}{L^2} \langle x_1^2 \rangle + \frac{L^2 a^2}{\hbar^2} \langle p_1^2 \rangle + \frac{1}{L^2 a^2} \langle x_2^2 \rangle + \frac{L^2}{a^2 \hbar^2} \langle p_2^2 \rangle. \quad (3.3)$$

Here, we have used the fact that for a non-entangled state (2.16) the quantities

$$\langle x_1 x_2 \rangle = \langle x_1 \rangle \langle x_2 \rangle = 0, \quad (3.4)$$

$$\langle p_1 p_2 \rangle = \langle p_1 \rangle \langle p_2 \rangle = 0, \quad (3.5)$$

because we have restricted our discussion to states for which

$$\langle x_1 \rangle = \langle x_2 \rangle = \langle p_1 \rangle = \langle p_2 \rangle = 0. \quad (3.6)$$

(Duan et al. do not make this restriction but come to the same conclusion for what they call Δu and Δv .) Next, we use the uncertainty principle,

$$\langle p_1^2 \rangle \geq \frac{\hbar^2}{4\langle x_1^2 \rangle}, \quad \langle p_2^2 \rangle \geq \frac{\hbar^2}{4\langle x_2^2 \rangle}, \quad (3.7)$$

to get

$$\langle u^2 \rangle + \langle v^2 \rangle \geq \left(\frac{\langle x_1^2 \rangle}{L^2} + \frac{L^2}{4\langle x_1^2 \rangle} \right) a^2 + \left(\frac{\langle x_2^2 \rangle}{L^2} + \frac{L^2}{4\langle x_2^2 \rangle} \right) \frac{1}{a^2}. \quad (3.8)$$

Now the quantity $y + \frac{1}{4y}$, $0 \leq y < \infty$ has a minimum value of 1 at $y = \frac{1}{2}$. We conclude that

$$\langle u^2 \rangle + \langle v^2 \rangle \geq a^2 + \frac{1}{a^2}, \quad (3.9)$$

independent of L . This is the result of Duan et al. Since the inequality is independent of L , we can minimize the left hand side with respect to L to get

$$2\sqrt{\left\langle \left(|a|x_1 + \frac{1}{a}x_2 \right)^2 \right\rangle \left\langle \left(|a|p_1 - \frac{1}{a}p_2 \right)^2 \right\rangle} \geq \left(a^2 + \frac{1}{a^2} \right) \hbar. \quad (3.10)$$

This is our improved inequality. It is a sufficient condition that the state is separable (not entangled). If it fails, the state must be entangled. We note that $a^2 + \frac{1}{a^2}$ has a minimum value of 2 at $a = \pm 1$. Hence

$$\sqrt{\left\langle \left(x_1 + \frac{a}{|a|}x_2 \right)^2 \right\rangle \left\langle \left(p_1 - \frac{a}{|a|}p_2 \right)^2 \right\rangle} \geq \hbar. \quad (3.11)$$

This is the necessary condition that a two-particle state be separable. Thus, we have two necessary conditions, corresponding to choosing a to be positive or negative. Using (2.4) and (2.7), we may write the two conditions in the succinct forms

$$\langle X^2 \rangle \langle P^2 \rangle \geq \frac{1}{4} \left(\frac{\hbar^2}{4} \right) \quad (3.12)$$

and

$$\langle x^2 \rangle \langle p^2 \rangle \geq 4 \left(\frac{\hbar^2}{4} \right). \quad (3.13)$$

We now wish to apply these results to the particular state discussed in Section 2. Thus, using the results given in (2.13) and (2.15), together with (2.9) and (2.10), these conditions take the explicit forms

$$d^2 + \left(\frac{\hbar t}{4md} \right)^2 \geq \frac{\sigma^2}{4} \quad (3.14)$$

and

$$\sigma^2 + \left(\frac{\hbar t}{m\sigma} \right)^2 \geq 4d^2. \quad (3.15)$$

Since we assumed $2d > \sigma$, it follows that (3.14) (which corresponds to the choice of positive a) is automatically fulfilled, implying separability. However, (3.15) (which corresponds to the choice of negative a) is only fulfilled if

$$t \geq \left(\frac{m}{\hbar} \right) \left\{ \sigma^2 (4d^2 - \sigma^2) \right\}^{1/2} = \left(\frac{2m\sigma d}{\hbar} \right) \left(1 - \frac{\sigma^2}{4d^2} \right)^{1/2} \equiv t_d. \quad (3.16)$$

Thus, for $t < t_d$, the separability condition is violated and the state is entangled. However at $t = t_d$, we encounter ESD [6–8], despite the fact that we know from our exact analysis in Section 3 that the state is entangled for all times. The solution to this apparent contradiction stems from the fact that the Duan et al. condition is a sufficient criterion for inseparability (entanglement) but it is not necessary. Recognizing this, these authors were led to develop a necessary and sufficient condition for entanglement by using a variety of local linear unitary transformations (consisting of various rotations and squeezing transformations of x_1 and p_1 that preserve the commutation relations and similarly for x_2 and p_2), and referred to as LOCC, to

map any Gaussian state into what they refer to as Standard forms I and II, which eventually leads them to a state for which their separability criterion is both necessary and sufficient. It is known that these local operations do not affect the entanglement of the state i.e. we have a family of states all with the same entanglement [16,4].

Guided by the fact that our results are satisfactory at $t = 0$, supplemented by the detailed results which we already obtained for the motion of a free particle [12], especially Eq. (9) of the latter reference, we make the following local canonical transformations:

$$\begin{aligned} \bar{x}_1(t) &= \exp\left(-\frac{i}{\hbar}H_1t\right)x_1(t)\exp\left(\frac{i}{\hbar}H_1t\right); \\ \bar{x}_2(t) &= \exp\left(-\frac{i}{\hbar}H_2t\right)x_2(t)\exp\left(\frac{i}{\hbar}H_2t\right), \end{aligned} \quad (3.17)$$

where

$$H_1 = p_1^2/2m; \quad H_2 = p_2^2/2m. \quad (3.18)$$

It follows that [since $p_1(t) = p_1(0)$ and $p_2(t) = p_2(0)$]

$$\bar{x}_1(t) = x_1(t) - \frac{t}{m}p_1(0) = x_1(0), \quad (3.19)$$

$$\bar{x}_2(t) = x_2(t) - \frac{t}{m}p_2(0) = x_2(0). \quad (3.20)$$

In fact, the second equality in the latter two equations readily follows from the Heisenberg equation of motion. These transformations lead to the results

$$\bar{x}(t) = x(t) - \frac{t}{m_x}p(0) = x(0), \quad (3.21)$$

$$\bar{X}(t) = X(t) - \frac{t}{M}P(0) = X(0). \quad (3.22)$$

It follows that the necessary conditions for separability now becomes

$$\langle \bar{X}^2 \rangle \langle P^2 \rangle \geq \frac{1}{4} \left(\frac{\hbar^2}{4} \right) \quad (3.23)$$

and

$$\langle \bar{x}^2 \rangle \langle p^2 \rangle \geq 4 \left(\frac{\hbar^2}{4} \right) \quad (3.24)$$

which takes the explicit forms

$$2d \geq \sigma \quad (3.25)$$

and

$$\sigma \geq 2d, \quad (3.26)$$

which are only compatible for $\sigma = 2d$. However, since we assumed that $2d > \sigma$, it is clear that the above analysis leads to the conclusion that the system is entangled for all times. We note that the unitary transformations given in (3.17) led to the elimination of terms depending on t in the separability conditions.

A similar analysis may be carried out using the Peres–Horodecki criterion [17,18], namely that a state is separable if the partial transpose of the density matrix is a positive operator. It can be shown explicitly (See Appendix A) that, for our model, it leads to the same result (3.11), that was obtained from the Duan et al. criterion.

Although the model we are considering here is a pure state, it is instructive to see how it fits into the general framework of mixed states which are best considered using Wigner distributions [13,21].

It should be emphasized at the outset that not all Wigner functions are permissible distribution functions since the corresponding density matrix elements must be positive definite [22] and the uncertainty relations must be satisfied. We now briefly review the work of Duan et al. [13] and Simon [21], which will result in bringing M into the “standard form” [13] given in (3.30) below.

Recalling that Gaussian states are completely characterized by their first and second moments (and here we have arranged that the former are zero), it follows that the Wigner characteristic function for a Gaussian state of a pair of particles can be written in the general form

$$\widetilde{W}(Q_1, P_1; Q_2, P_2; t) = \exp\left\{-\frac{\mathbf{Q} \cdot \mathbf{M} \cdot \mathbf{Q}}{2}\right\}, \quad (3.27)$$

where

$$\mathbf{Q} = \begin{pmatrix} \frac{LP_1}{h} \\ \frac{Q_1}{L} \\ \frac{LP_2}{h} \\ \frac{Q_2}{L} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{G} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{H} \end{pmatrix}. \quad (3.28)$$

Here \mathbf{M} is the correlation (variance) matrix and \mathbf{G} and \mathbf{C} are 2×2 matrices given by

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} \frac{\langle x_1^2 \rangle}{L^2} & \frac{\langle x_1 p_1 + p_1 x_1 \rangle}{2h} \\ \frac{\langle x_1 p_1 + p_1 x_1 \rangle}{2h} & \frac{L^2 \langle p_1^2 \rangle}{h^2} \end{pmatrix}, \\ \mathbf{H} &= \begin{pmatrix} \frac{\langle x_2^2 \rangle}{L^2} & \frac{\langle x_2 p_2 + p_2 x_2 \rangle}{2h} \\ \frac{\langle x_2 p_2 + p_2 x_2 \rangle}{2h} & \frac{L^2 \langle p_2^2 \rangle}{h^2} \end{pmatrix}, \\ \mathbf{C} &= \begin{pmatrix} \frac{\langle x_1 x_2 \rangle}{L^2} & \frac{\langle x_1 p_2 \rangle}{h} \\ \frac{\langle x_2 p_1 \rangle}{h} & \frac{L^2 \langle p_1 p_2 \rangle}{h^2} \end{pmatrix}. \end{aligned} \quad (3.29)$$

In these expressions L and h are constants introduced to make the matrix variance (correlation) M dimensionless. However, as far as the subsequent analysis is concerned, the L may be ignored with impunity, as we will do henceforth.

Making use of a series of local linear canonical transformations (rotations and squeezings), it was shown [13,21] that it is possible to bring \mathbf{M} to the special form:

$$\mathbf{M}' = \begin{pmatrix} g & 0 & c & 0 \\ 0 & g & 0 & c' \\ c & 0 & h & 0 \\ 0 & c' & 0 & h \end{pmatrix}. \quad (3.30)$$

Since determinants are invariant under these transformations we have the following simple relations for determining the quantities g, h, c and c' , in terms of four invariants,

$$\begin{aligned} \det \mathbf{G} &= g^2, & \det \mathbf{H} &= h^2, \\ \det \mathbf{C} &= cc', & \det \mathbf{M} &= (gh - c^2)(gh - c'^2). \end{aligned} \quad (3.31)$$

We now turn to the special case of interest here, that is the pure Gaussian state given in (2.1). Using the techniques developed in [23], or, since we are dealing with a free particle, from the Wigner function given in (A5), together with the results given in (2.14) and (2.15), it follows that

$$\begin{aligned} \langle x_1^2 \rangle &= \langle x_2^2 \rangle = \left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right) \left[\sigma^2 d^2 + \left(\frac{\hbar t}{2m}\right)^2\right], \\ \langle x_1 x_2 \rangle &= \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right) \left[-\sigma^2 d^2 + \left(\frac{\hbar t}{2m}\right)^2\right], \\ \langle p_1^2 \rangle &= \langle p_2^2 \rangle = \left(\frac{1}{2}\right)^2 \left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right), \\ \langle p_1 p_2 \rangle &= \left(\frac{1}{2}\right)^2 \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right), \\ \langle x_2 p_1 \rangle &= \langle x_1 p_2 \rangle = \frac{\hbar^2 t}{4m} \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right), \\ \frac{\langle x_1 p_1 + p_1 x_1 \rangle}{2} &= \frac{\langle x_2 p_2 + p_2 x_2 \rangle}{2} = \frac{\hbar^2 t}{4m} \left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right). \end{aligned} \quad (3.32)$$

Hence

$$\mathbf{G} = \mathbf{H} = \begin{pmatrix} \frac{1}{\sigma^2} + \frac{1}{4d^2} & \\ & \frac{\hbar^2 t}{4m} \\ & \frac{\hbar^2 t}{4m} & \left(\frac{1}{2}\right)^2 \end{pmatrix}, \quad (3.33)$$

$$\mathbf{C} = \begin{pmatrix} -\frac{1}{\sigma^2} + \frac{1}{4d^2} & \\ & \frac{\hbar^2 t}{4m} \\ & \frac{\hbar^2 t}{4m} & \left(\frac{1}{2}\right)^2 \end{pmatrix}. \quad (3.34)$$

It follows that

$$\det \mathbf{G} = \det \mathbf{H} = \left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right)^2 \sigma^2 d^2, \quad (3.35)$$

$$\det \mathbf{C} = -\frac{1}{4} \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right)^2 \sigma^2 d^2, \quad (3.36)$$

$$\det \mathbf{M} = \det(\mathbf{G} + \mathbf{C}) \det(\mathbf{G} - \mathbf{C}). \quad (3.37)$$

We note that the latter two equations are independent of t . Thus, the transformed matrix M has the form (3.30) with

$$g^2 = \frac{1}{4} \left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right)^2 \sigma^2 d^2, \quad (3.38)$$

$$cc' = -\frac{1}{4} \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right)^2 \sigma^2 d^2, \quad (3.39)$$

$$(g^2 - c^2)(g^2 - c'^2) = \frac{1}{16}. \quad (3.40)$$

The solution of these equations is

$$g = \frac{1}{2} \sigma d \left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right), \quad (3.41)$$

$$c = -c' = -\frac{1}{2} \sigma d \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right). \quad (3.42)$$

In terms of these quantities, the inequality (3.11) becomes

$$\sqrt{(g \mp c)(g \pm c')} \geq 1. \quad (3.43)$$

In terms of the above expressions, this becomes

$$\sigma d \left(\left(\frac{1}{\sigma^2} + \frac{1}{4d^2}\right) \pm \left(-\frac{1}{\sigma^2} + \frac{1}{4d^2}\right)\right) \geq 1, \quad (3.44)$$

which implies

$$\sigma \geq 2d, \quad \text{or} \quad 2d \geq \sigma. \quad (3.45)$$

In other words, the condition for separability only holds when

$$2d = \sigma \quad (3.46)$$

that is when the center-of-mass coordinate width and the relative coordinate width are equal. As a consequence,

$$a_{12} = 0, \quad (3.47)$$

as expected. As indicated above, these conclusions hold at all times.

It is also of interest to note that, within the present context, the Peres–Horodecki criterion implies that a Gaussian state is separable if and only if the minimum value of its symplectic spectrum of M^{T_2} is greater than $1/2$ [19,21] which leads to a good measure of entanglement for all Gaussian states given by

$$E = \max\{0, -\log(2\nu_{\min})\}, \quad (3.48)$$

where ν_{\min} is the smallest symplectic eigenvalue of M^{T_2} . The equation determining the symplectic spectrum is [19]

$$\nu^4 + (\det \mathbf{G} + \det \mathbf{H} - 2 \det \mathbf{C})\nu^2 + \det \mathbf{M} = 0, \quad (3.49)$$

with solutions $\pm i\nu_\alpha$, $\alpha = 1, 2$ where ν_α is the symplectic spectrum. Hence, using (3.35) and (3.36), we obtain $\nu_1 = (d/\sigma)$ and $\nu_2 = (\sigma/4d)$. Since we assume $2d > \sigma$, we see that $\nu_{\min} = \nu_2 < \frac{1}{2}$ and hence

$$E = \max \left\{ 0, \log \left(\frac{2d}{\sigma} \right) \right\} = \log \left(\frac{2d}{\sigma} \right), \quad (3.50)$$

in agreement with the result (4.10) arising from the log negativity criterion, as discussed in the next section.

4. Logarithmic negativity criterion

The logarithmic negativity is defined as

$$E_N(\rho) = \log \{2N(\rho) + 1\}, \quad (4.1)$$

where $N(\rho)$ is the negativity of the state and is given by the absolute sum of the negative eigenvalues of the partial transpose of ρ [19,20].

We want to solve the eigenfunction equation:

$$\int dx_1 \int dx_2 \langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle \phi(x_1, x_2) = \lambda \phi(x'_1, x'_2), \quad (4.2)$$

especially in order to obtain the negative eigenvalues. After some algebra (See Appendix B), we find that the eigenvalues are given by

$$\lambda_{mn} = \begin{cases} \pm \beta^{m+n} \lambda_0 & \text{for } m \neq n \\ \beta^{2n} \lambda_0 & \text{for } m = n, \end{cases} \quad (4.3)$$

where $m, n = 0, 1, 2, \dots$ and λ_0 is a positive eigenvalue given by

$$\lambda_0 = \frac{2\sqrt{a_{11}^2 - a_{12}^2}}{a_{11} + \sqrt{a_{11}^2 - a_{12}^2}}. \quad (4.4)$$

In addition,

$$\beta = \sqrt{\frac{a_{11} - \sqrt{a_{11}^2 - a_{12}^2}}{a_{11} + \sqrt{a_{11}^2 - a_{12}^2}}}. \quad (4.5)$$

For the symmetric case, these reduce to

$$\lambda_0 = \frac{8\sigma d}{(2d + \sigma)^2} \quad (4.6)$$

and

$$\beta = \frac{2d - \sigma}{2d + \sigma} \quad (4.7)$$

recalling that we have assumed that $2d > \sigma$. As a check, we note that

$$\sum_{mn} \lambda_{mn} = (1 + \beta^2 + \beta^4 + \dots) \lambda_0 = \left(\frac{1}{1 - \beta^2} \right) \lambda_0 = 1, \quad (4.8)$$

verifying that $\text{Tr} \rho^{T_2} = 1$. In addition,

$$N(\rho) = \sum_{m>n} |\lambda_{mn}| = \left(\frac{\beta}{1 - \beta} \right) = \left(\frac{2d - \sigma}{2\sigma} \right). \quad (4.9)$$

Hence

$$E_N(\rho) = \log \{2N(\rho) + 1\} = \log \left\{ \frac{2d}{\sigma} \right\}. \quad (4.10)$$

Thus, the greater $2d$ is compared to σ , the larger the negativity and hence the greater the entanglement. In addition, since $\rho(t) = \exp(-iHt)\rho(0)\exp(iHt)$, where $H = (p_1^2 + p_2^2)/2m$, it is clear, from (4.2), that the eigenvalues of ρ^{T_2} are invariant under this local unitary transformation. Hence, the result (4.10) is valid for all times.

5. Entanglement of formation

For bipartite pure states, the entanglement of formation is given by [24]

$$E_F = S_1(\rho_1) = S_2(\rho_2), \quad (5.1)$$

where ρ_1 and ρ_2 are the reduced density matrices [defined in (B15)] and

$$S_1 = -\text{Tr}_2[\rho_1 \log \rho_1] \quad (5.2)$$

is the von Neumann entropy. Thus, from Appendix B, we have

$$S = -\sum_{n=0}^{\infty} \lambda_n \log \lambda_n, \quad (5.3)$$

where the eigenvalues of ρ_1 are given by

$$\lambda_n = \lambda_0 \beta^{2n} \quad (n = 0, 1, 2, \dots), \quad (5.4)$$

where, from (4.6) and (4.7), we have (with $R \equiv 2d/\sigma$)

$$\lambda_0 = \frac{4R}{(1+R)^2} = 1 - \beta^2 \quad (5.5)$$

and

$$\beta = \frac{R-1}{R+1}. \quad (5.6)$$

Hence

$$\begin{aligned} S_1 &= -\lambda_0 \sum_{n=0}^{\infty} \beta^{2n} [\log \lambda_0 + 2n \log \beta] \\ &= -\lambda_0 \log \lambda_0 \sum_{n=0}^{\infty} \beta^{2n} - 2\lambda_0 \log \beta \sum_{n=0}^{\infty} n(\beta^2)^n \\ &= -\log \lambda_0 - \frac{2\beta^2}{1-\beta^2} \log \beta \\ &= \log R + \left[2 \log \frac{1+R}{2R} + \frac{(R-1)^2}{2R} \log \frac{R+1}{R-1} \right]. \end{aligned} \quad (5.7)$$

We recall, from (4.10), that the logarithmic negativity $E_N(\rho)$ is given by $\log R$. Also it can be shown that $0 \leq S_1 \leq \log R$ for $R \geq 1$ and the equalities hold for $R = 1$ and $R \rightarrow \infty$, respectively. This is consistent with the result that the entanglement of formation is always less than logarithmic negativity, and they are equal for maximally entangled pure states. Thus, the entanglement exists if $2d > \sigma$.

6. Conclusions

We examined a simple but non-trivial model of entanglement which enabled us to carry out an *exact* analysis. We analyzed various entanglement criteria, arising especially from the work of Duan et al. [13]; Peres–Horodecki [17,18]; Vidal and Werner [19], who considered both the logarithmic negativity and that arising from a determination of the smallest symplectic eigenvalue of the Peres transform of the transformed variance matrix and Bennett et al. [25] on the entanglement of formation. We found that all of these various entanglement criteria led to the same results but that some are more straightforward than others. In particular, it was clear that the logarithmic criterion is the simplest to use since the procedure is straightforward, that is obtain the eigenvalues of the Peres transform of the density matrix.

After this paper was completed, we became aware (courtesy of the referee) of various papers that have closely related themes. Our work is an example of “entanglement without dissipation,” which apparently was initially discussed by Chan and Eberly [25] who also investigated a Gaussian state but used a Schmidt-state analysis as a measure of entanglement. Next, Yonac et al. [26] considered two isolated atoms each in their own lossless Jaynes–Cummings cavity. They showed that, due to the interaction with the local lossless cavities, ESD occurs for atom–atom entanglement due to information loss to the cavity modes but that entanglement

is resurrected in a periodic manner following each ESD event due to the fact that the time evolution is lossless. The same system was analyzed by Sainz and Bjork [27] who concluded that the atoms simply transfer their entanglement to the cavity fields and that an entanglement measure exists that is constant under the time evolution. A different system, photoionization in a lossless environment, was considered by Fedorov et al. [28], who found narrowing of electron and ion wave packets due to electron–ion entanglement.

Entanglement of formation [24] is one of the measures we have discussed (see Section 5) and this quantity is referred to by Munro et al. [29] as “... the canonical measure of entanglement ...,” who then go on to present a class of states that have the maximum amount of entanglement for a given linear entropy. All of this work is leading to a better understanding but, to quote from the recent general overview of Yu and Eberly [30], “... there is still no deep understanding of sudden death dynamics.”

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Appendix A. Density matrix elements and Wigner functions

In general, the Peres partial transpose of the density matrix is

$$\langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle = \langle x'_1, x_2 | \rho | x_1, x'_2 \rangle. \quad (\text{A1})$$

However, in practice, it is often more convenient to consider the corresponding result for the Wigner function [21], that is

$$\mathcal{W}^{T_2}(q_1, p_1; q_2, p_2) = \mathcal{W}(q_1, p_1; q_2, -p_2). \quad (\text{A2})$$

The corresponding transpose of the Wigner Characteristic function is

$$\widetilde{W}^{T_2}(Q_1, P_1; Q_2, P_2) = \widetilde{W}(Q_1, P_1; -Q_2, P_2). \quad (\text{A3})$$

We recall that the most general Gaussian *pure* state corresponds to the wave function

$$\psi(x_1, x_2) = \frac{(a_{11}a_{22} - a_{12}^2)^{1/4}}{\sqrt{2\pi}} \exp \left\{ -\frac{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2}{4} \right\}, \quad (\text{A4})$$

where a_{11} and a_{22} are positive and $a_{11}a_{22} - a_{12}^2 > 0$ (to ensure integrability). It is straightforward to obtain the Wigner function

$$\mathcal{W}(q_1, p_1; q_2, p_2) = \frac{1}{(\pi\hbar)^2} \exp \left\{ -\frac{a_{11}q_1^2 + 2a_{12}q_1q_2 + a_{22}q_2^2}{2} \right\} \times \exp \left\{ -2\frac{a_{22}p_1^2 - 2a_{12}p_1p_2 + a_{11}p_2^2}{\hbar^2(a_{11}a_{22} - a_{12}^2)} \right\}, \quad (\text{A5})$$

and the Wigner characteristic function (Fourier transform of the Wigner function)

$$\widetilde{W}(Q_1, P_1; Q_2, P_2) = \exp \left\{ -\frac{a_{11}Q_1^2 + 2a_{12}Q_1Q_2 + a_{22}Q_2^2}{8} \right\} \times \exp \left\{ -\frac{a_{22}P_1^2 - 2a_{12}P_1P_2 + a_{11}P_2^2}{2(a_{11}a_{22} - a_{12}^2)\hbar^2} \right\}. \quad (\text{A6})$$

Also, the corresponding density matrix is

$$\begin{aligned} \langle x'_1, x'_2 | \rho | x_1, x_2 \rangle &= \psi(x'_1, x'_2) \psi^*(x_1, x_2) \\ &= \frac{(a_{11}a_{22} - a_{12}^2)^{1/2}}{2\pi} \\ &\times \exp \left\{ -\frac{a_{11}(x_1^2 + x_1'^2) + 2a_{12}(x_1x_2 + x_1'x_2') + a_{22}(x_2^2 + x_2'^2)}{4} \right\}. \end{aligned} \quad (\text{A7})$$

The Peres partial transpose of this density matrix is

$$\begin{aligned} \langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle &= \psi(x'_1, x_2) \psi^*(x_1, x'_2) \\ &= \frac{(a_{11}a_{22} - a_{12}^2)^{1/2}}{2\pi} \\ &\times \exp \left\{ -\frac{a_{11}(x_1^2 + x_1'^2) + 2a_{12}(x_1x_2' + x_1'x_2) + a_{22}(x_2^2 + x_2'^2)}{4} \right\}. \end{aligned} \quad (\text{A8})$$

The corresponding transpose of the Wigner characteristic function is

$$\begin{aligned} \widetilde{W}^{T_2}(Q_1, P_1; Q_2, P_2) &= \exp \left\{ -\frac{a_{11}Q_1^2 - 2a_{12}Q_1Q_2 + a_{22}Q_2^2}{8} \right\} \\ &\times \exp \left\{ -\frac{a_{22}P_1^2 - 2a_{12}P_1P_2 + a_{11}P_2^2}{2(a_{11}a_{22} - a_{12}^2)\hbar^2} \right\}. \end{aligned} \quad (\text{A9})$$

In addition, the corresponding transpose of the Wigner function is

$$\begin{aligned} \mathcal{W}^{T_2}(q_1, p_1; q_2, p_2) &= \frac{1}{(\pi\hbar)^2} \exp \left\{ -\frac{a_{11}q_1^2 + 2a_{12}q_1q_2 + a_{22}q_2^2}{2} \right\} \\ &\times \exp \left\{ -2\frac{a_{22}p_1^2 + 2a_{12}p_1p_2 + a_{11}p_2^2}{\hbar^2(a_{11}a_{22} - a_{12}^2)} \right\}. \end{aligned} \quad (\text{A10})$$

Consider the symmetric case, for which

$$a_{22} = a_{11} = \frac{1}{\sigma^2} + \frac{1}{4d^2}, \quad a_{12} = -\frac{1}{\sigma^2} + \frac{1}{4d^2}. \quad (\text{A11})$$

Then

$$\begin{aligned} \langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle &= \frac{(a_{11}^2 - a_{12}^2)^{1/2}}{2\pi} \\ &\times \exp \left\{ -\frac{a_{11}(x_1^2 + x_1'^2) + 2a_{12}(x_1x_2' + x_1'x_2) + a_{11}(x_2^2 + x_2'^2)}{4} \right\}, \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} \widetilde{W}^{T_2}(Q_1, P_1; Q_2, P_2) &= \exp \left\{ -\frac{(Q_1 + Q_2)^2}{8\sigma^2} - \frac{(Q_1 - Q_2)^2}{32d^2} \right\} \\ &\times \exp \left\{ -\frac{d^2}{2\hbar^2}(P_1 + P_2)^2 - \frac{\sigma^2}{8\hbar^2}(P_1 - P_2)^2 \right\}, \end{aligned} \quad (\text{A13})$$

$$\begin{aligned} \mathcal{W}^{T_2}(q_1, p_1; q_2, p_2) &= \frac{1}{(\pi\hbar)^2} \exp \left\{ -\frac{(q_1 - q_2)^2}{2\sigma^2} - \frac{(q_1 + q_2)^2}{8d^2} \right\} \\ &\times \exp \left\{ -\frac{2d^2}{\hbar^2}(p_1 - p_2)^2 - \frac{\sigma^2}{2\hbar^2}(p_1 + p_2)^2 \right\}. \end{aligned} \quad (\text{A14})$$

If \mathcal{W}^{T_2} is to be a Wigner function we must require that the uncertainty relation be satisfied. In particular we must require

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \quad (\text{A15})$$

where

$$\begin{aligned} (\Delta x)^2 &= \langle (x_1 - x_2)^2 \rangle^{T_2} \\ &= \int dq_1 \int dp_1 \int dq_2 \int dp_2 (q_1 - q_2)^2 \mathcal{W}^{T_2}(q_1, p_1; q_2, p_2) \\ &= \int dq_1 \int dp_1 \int dq_2 \int dp_2 (q_1 - q_2)^2 \mathcal{W}(q_1, p_1; q_2, p_2) \\ &= \langle (x_1 - x_2)^2 \rangle, \\ 4(\Delta p)^2 &= \langle (p_1 - p_2)^2 \rangle^{T_2} \\ &= \int dq_1 \int dp_1 \int dq_2 \int dp_2 (p_1 - p_2)^2 \mathcal{W}^{T_2}(q_1, p_1; q_2, p_2) \\ &= \int dq_1 \int dp_1 \int dq_2 \int dp_2 (p_1 + p_2)^2 \mathcal{W}(q_1, p_1; q_2, p_2) \\ &= \langle (p_1 + p_2)^2 \rangle. \end{aligned} \quad (\text{A16})$$

Therefore, if we require that the Peres transpose corresponds to a Wigner function, we see from (A15) and (A16) that the following inequality must be satisfied

$$\sqrt{\langle (x_1 - x_2)^2 \rangle \langle (p_1 + p_2)^2 \rangle} \geq \hbar. \quad (\text{A17})$$

This is just the refined condition of Duan et al.

Appendix B. Eigenvalues and eigenfunctions of the density matrix and the reduced density matrix

For the case of the pure Gaussian state given by (2.1), and specializing to the symmetric case, where $a_{22} = a_{11}$, we found that the Peres transpose is [see (A8)]

$$\langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle = \psi(x'_1, x_2) \psi^*(x_1, x'_2), \quad (\text{B1})$$

where [see (2.1)]

$$\psi(x_1, x'_2) = \frac{(a_{11}^2 - a_{12}^2)^{1/4}}{\sqrt{2\pi}} \exp \left\{ -\frac{a_{11}(x_1^2 + x_2'^2) + 2a_{12}x_1x_2'}{4} \right\}, \quad (\text{B2})$$

with a corresponding result for $\psi(x'_1, x_2)$. We want to solve the eigenfunction equation:

$$\int dx_1 \int dx_2 \langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle \Phi(x_1, x_2) = \lambda \Phi(x'_1, x'_2). \quad (\text{B3})$$

The form of (B2) suggests use of the Mehler formula [11], which is written in terms of the Hermite functions H_n . However, we find it is more useful to modify this formula so that it is now written in terms of the related orthogonal function ϕ_n , the eigenstate of the quantum harmonic oscillator. Thus, for arbitrary variables x and y , the modified Mehler formula may be written in the form

$$\begin{aligned} &\sqrt{\frac{\gamma}{\pi}} \exp \left\{ \frac{\gamma}{1 - \beta^2} \left[2\beta xy - \frac{1 + \beta^2}{2} (x^2 + y^2) \right] \right\} \\ &= \sqrt{1 - \beta^2} \sum_{n=0}^{\infty} \beta^n \phi_n(x) \phi_n(y), \end{aligned} \quad (\text{B4})$$

where the function $\phi_n(x)$ is related to the Hermite polynomial $H_n(x)$ by

$$\phi_n(x) = \sqrt{\frac{1}{2^n n!}} \left(\frac{\gamma}{\pi}\right)^{1/4} e^{-\gamma x^2/2} H_n(\sqrt{\gamma}x) \quad (\text{B5})$$

and

$$\int \phi_m(x) \phi_n(x) dx = \delta_{mn}. \quad (\text{B6})$$

Thus, applying (B4) to both $\psi(x_1, x'_2)$ and $\psi(x'_1, x_2)$, we obtain

$$\langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle = \lambda_0 \sum_{m,n} \beta^{m+n} \phi_m(x_1) \phi_m(x'_2) \phi_n(x'_1) \phi_n(x_2). \quad (\text{B7})$$

Here, the parameters γ , β , and γ_0 are given by

$$\gamma = \sqrt{a_{11}^2 - a_{12}^2}/2, \quad (\text{B8})$$

$$\beta = \frac{a_{11} - \sqrt{a_{11}^2 - a_{12}^2}}{a_{11} + \sqrt{a_{11}^2 - a_{12}^2}} \quad (\text{B9})$$

$$\lambda_0 = \frac{2\sqrt{a_{11}^2 - a_{12}^2}}{a_{11} + \sqrt{a_{11}^2 - a_{12}^2}} = 1 - \beta^2. \quad (\text{B10})$$

The advantage of writing the matrix elements in terms of orthogonal functions, as given in (B7), immediately leads to solutions of the eigenfunction Eq. (B3), in the form

$$\int dx_1 \int dx_2 \langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle \phi_m(x_1) \phi_n(x_2) = \lambda_0 \beta^{m+n} \phi_m(x'_1) \phi_n(x'_2). \quad (\text{B11})$$

If $m = n$, $\Phi_{mn}(x_1, x_2) = \phi_n(x_1) \phi_n(x_2)$ is the desired eigenfunction with eigenvalue $\lambda_{mn} = \beta^{2n} \lambda_0$. On the other hand if $m \neq n$, we find that the eigenfunction is given by

$$\Phi_{mn}(x_1, x_2) = \frac{1}{\sqrt{2}} [\phi_m(x_1) \phi_n(x_2) \pm \phi_n(x_1) \phi_m(x_2)] \quad (\text{B12})$$

with eigenvalues $\pm \lambda_0 \beta^{m+n}$. In summary, the complete eigenvalues are

$$\lambda_{mn} = \begin{cases} \pm \beta^{m+n} \lambda_0 & \text{for } m \neq n \\ \beta^{2n} \lambda_0 & \text{for } m = n, \end{cases} \quad (\text{B13})$$

with $m, n = 0, 1, \dots$

Next, we consider reduced density matrices. Thus, we define

$$\rho_1 = \text{Tr}_2 \langle x'_1, x'_2 | \rho^{T_2} | x_1, x_2 \rangle. \quad (\text{B14})$$

This is obtained by setting $x_2 = x'_2$ in (B7) and integrating over x_2 to get

$$\rho_1 = \lambda_0 \sum_m \beta^{2m} \phi_m(x_1) \phi_m(x'_1). \quad (\text{B15})$$

Thus, the eigenvalue equation for the reduced density matrix is

$$\int dx_1 \rho_1 \Phi^{(1)}(x_1) = \lambda \Phi^{(1)}(x'_1). \quad (\text{B16})$$

It immediately follows that this leads to solutions $\phi_m(x_1)$ with corresponding eigenvalues $\lambda_0 \beta^{2m}$ ($m = 0, 1, 2, \dots$). Moreover, it is clear that the corresponding eigenvalues of ρ_2 are the same.

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