The Wigner Distribution

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In contrast to classical physics, the language of quantum mechanics involves operators and wave functions (or, more generally, density operators). However, in 1932, Wigner formulated quantum mechanics in terms of a distribution function $W(q,p)$, the marginals of which yield the correct quantum probabilities for q and p separately [1]. Its usefulness stems from the fact that it provides a re-expression of quantum mechanics in terms of classical concepts so that quantum mechanical expectation values are now expressed as averages over phase-space distribution functions. In other words, statistical information is transferred from the density operator to a quasi-classical (distribution) function.

Wigner [1] presented a specific form for $W(p,q)$, while recognizing that other possibilities exist, depending on the conditions which are imposed on W. Wigner's choice has the virtue of mathematical simplicity but it has the feature that it may take negative values, with the result that several authors have investigated non-negative distribution functions. However, we regard negative values of W as a manifestation of its quantum nature and the fact that it " - - - cannot be really interpreted as the simultaneous probability for coordinates and momenta - - -". [1] Wigner's original paper was concerned with using W for the specific purpose of calculating the quantum correction for thermodynamic equilibrium. The recognition of its more general applicability stems mainly from the work of Groenewold [2] and Moyal [3], who investigated the correspondence between physical quantities and quantum operators and showed, in particular, that the correspondence is not unique and moreover, that the distribution functions obtained by the Weyl correspondence [4] are the Wigner functions. Moyal also showed how the time dependence of W and other such functions (which arise from alternative association rules other than Wigner-Weyl but which lead to the same physical results) may be determined without using the Schrödinger equation. In fact, Moyal's paper was a landmark contribution as, in essence, " - - it establishes an independent formulation of quantum mechanics in phase space". [5] As for all quantum formulations, Ballentine [6] has shown that the development of the classical limit of the Wigner distribution is a subtle process, especially in view of the fact that, in general, $W(q,p)$ has negative parts. Turning to specifics, we present some basic results developed in the original pioneering papers [1–4] but conveniently presented in a comprehensive review by Hillery et al. [7]. Thus, in one dimensional space (generalization to n dimensions being straightforward), for a mixed state represented by a densty matrix $\hat{\rho}$,

$$W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy \langle q-y|\hat{\rho}|q+y\rangle e^{2ipy/\hbar}, \tag{1}$$

whereas, for a pure state represented by a wave function $\psi(q)$,

$$W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy \psi^{*}(q+y)\psi(q-y) e^{2ipy/\hbar}. \tag{2}$$

However, in order to calculate correct expectation values and ensemble averages, it is also necessay to specify the classical function $A(q,p)$ corresponding to a quantum operator $\hat{A}$ as

$$A(q,p) = \int dz\, e^{ipz/\hbar}\langle q-\frac{1}{2}z|\hat{A}|q+\frac{1}{2}z\rangle, \tag{3}$$

so that $\int\int\ dq\, dp\, A(q,p) = 2\pi\hbar\ \ Tr(\hat{A})$. This ensures that

$$\int dq \int dp\, A(q,p)B(q,p) = (2\pi\hbar)\, Tr(\hat{A}\hat{B}), \tag{4}$$

and

$$\int dq \int dp\, A(q,p)W(q,p) = Tr(\hat{\rho}\hat{A}(\hat{q},\hat{p})), \tag{5}$$

so that, in particular, we see that $W(q,p)$ derived from the density matrix, is $(2\pi\hbar)^{-1}$ times the phase space operator which corresponds to the same matrix.

Following these original papers, [1–4] there were many papers devoted to extending the framework and overall understanding of distribution functions. In addition, distributions other than those of Wigner were introduced, notable those of Kirkwood, Cahill and Glauber, Glauber, Sudarshan and Husimi (all of which are reviewed in Ref. [8], where it is noted that some of these are everywhere non-negative) and Cohen [9] and all require classical functions different from that given in (3) in order to ensure consistency. It is clear that all distribution functions are not measurable, despite some claims to the contrary in the literature, where in fact what is observed are the marginal q probabilities from which values of $W(q,p)$ are inferred but one could equally have inferred values for other distribution functions.

The earliest applications of the Wigner function were in the arena of statistical mechanics but, more recently, among the diverse areas in which the W function was found to be useful we mention hydrodynamics [10], plasmas [11], quantum corrections for transport coefficients [12], collision theory [13] and signal analysis [14]. However, we feel that the overwhelming majority of applications are to be found in quantum systems where fluctuations and dissipation are playing an important role. In this context, the 1984 review of the W function by Hillery et al. [7] made extensive reference to its relevance in quantum optics, which is underlined by the more recent books of Scully and Zubairy [15] and Schleich [16]. Complementary to this work is the application of the W function to a variety of problems in quantum statistical mechanics, where effects associated with the analysis of quantum systems in a heat bath (including the radiation field heat bath) are of the essence. As examples of the usefulness of the W function in this context we note its role in obtaining the simplest approach to solving the initial value quantum Langevin equation and, concomitantly, the solution to an exact master equation [17] and also its role in the investigation of Schrödinger cat superpositions [18]. However there are limitations to the usefulness of the W function (some of which were discussed by Moyal [3]), notably for particles with spin and for relativistic particles. Finally, we mention the excellent and comprehensive overview of selected papers on quantum mechanics in phase space, with emphasis on the Wigner function [5].

[1] E. Wigner, On the Quantum Correction for Thermodynamic Equilibrium, Phys. Rev. **40**, 749-759 (1932).
[2] H. Groenewold, On the Principles of Elementary Quantum Mechanics, Physica **12**, 405-460 (1946).
[3] J. Moyal, Quantum Mechanics as a Statistical Theory, Proc. Camb. Phil. Soc. **45**, 99-124 (1949).
[4] H. Weyl, Quantenmechanik und Gruppentheorie, Z. Phys. **46**, 1-46 (1927).
[5] C. K. Zachos, D. B. Fairlie, T. L. Curtright, Quantum Mechanics in Phase Space, World Scientific (2005).
[6] L. E. Ballentine, Quantum Mechanics: a modern development, World Scientific (2005), Chap. 15.
[7] M. Hillery, R. O'Connell, M. Scully, and E. Wigner, Distribution Functions in Physics: Fundamentals, Phys. Rep. **106**, 121-167 (1984).
[8] R. F. O'Connell, Quantum Distribution Functions in Non-Equilibrium Statistical Mechanics, in Frontiers of Nonequilibrium Statistical Physics 83-95, Plenum Publishing Corporation, 1986.
[9] L. Cohen, Generalized Phase-Space Distribution Functions, J. Math. Phys. **7**, 781-786 (1966).
[10] J. H. Irving and R. W. Zwanzig, The Statistical Mechanical Theory of Transport Processess. V. Quantum Hydrodynamics, J. Chem. Phys. **19**, 1173-1180 (1951).
[11] W. E. Brittin and W. R. Chappell, The Wigner Distribution Function and Second Quantization in Phase Space, Rev. Mod. Phys. **34**, 620-627 (1962).
[12] S. Choi and J. Ross, Quantum Corrections for Transport Coefficients, J. Chem. Phys. **33**, 1324 (1960).
[13] R. Carruthers and F. Zachariasen, Quantum collision theory with phase-space distributions, Rev. Mod. Phys. **55**, 245-285 (1983).
[14] L. Cohen, Time-Frequency Analysis, Prentice-Hall 1995.
[15] M. O. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press, Cambridge, England, 1997
[16] W. P. Schleich, Quantum Optics in Phase Space, Wiley, England, 2001.
[17] G. W. Ford and R. F. O'Connell, Exact solution of the Hu-Paz-Zhang master equation, Phys. Rev. D **64**, 105020, 1-13 (2001).
[18] G. W. Ford and R. F. O'Connell, Wigner Distribution Analysis of a Schrödinger Cat Superposition of Displaced Equilibrium Coherent States, Acta Physica Hungarica Quantum Electronics B **20**, 91-94 (2004).