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BOOK REVIEW

QUANTUM MECHANICS IN PHASE SPACE: AN OVERVIEW WITH SELECTED PAPERS, EDITED BY C. K. ZACHOS, D. B. FAIRLIE, T. L. CURTRIGHT. WORLD SCIENTIFIC, SINGAPORE (2005)

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In Quantum Mechanics in Phase Space, by Zachos, Fairlie and Curtright, the bulk of the book (of 551 pages) consists of reprints of a selection of 23 papers dealing with the phase space formalism. In addition, there is a very useful overview, consisting of an introduction of 30 pages with an emphasis on the Wigner quasi-probability classical-like distribution function (27 pages), followed by a brief historical outline and discussion of the selected papers, as well as a comprehensive bibliography. My overall impression is that the authors have struck the right note in their choice of presentation and also their decision as to what to omit, since the subject matter covers a very broad range. In particular, there are many other quasi-probability distribution functions (such as the Husimi, Kirkwood-Rihaczek, Glauber-Sudarshan and Born-Jordan functions) which are briefly mentioned. These are found to be useful in certain applications, but I feel that the authors have made a wise choice in concentrating on W since, apart from the fact that it was the first such function, its relative simplicity has resulted in it being applied extensively to calculations in a wide variety of areas, as outlined in the authors' Introduction. The authors have also concentrated on fundamentals as distinct from applications, which is again understandable in view of the size of the book (551 pages). For example, applications to the area of Quantum Optics alone has led to a book of nearly 700 pages (Quantum Optics in Phase Space by W. P. Schleich). Whereas some readers may bemoan the omission of their favorite paper compared to some of the 23 selected reprints, I feel that there will be universal agreement that the vast majority of those papers selected would have to be on everyone's list.

In making the transition from classical to quantum physics, calculations generally become more difficult because quantum mechanics involves operators and wave functions (or, more generally, density operators). One ubiquitous feature of this transition is Heisenberg's uncertainty principle such that the coordinates q and momenta p cannot be simultaneously measured. Wigner recognized that this led to a difficulty in making the transition from classical statistical mechanics to the quantum regime since, in particular, classical statistical equilibrium is given by the Gibbs-Boltzmann formula, which involves both the kinetic and potential energies, that is p and q. Thus, in 1932, Wigner was motivated to introduce a quasi-classical probability function (which is not an operator but which does depend on Planck's constant) $W(q, p)$. As Wigner emphasized, W cannot be regarded as giving the simultaneous probability for p and q (because, in particular, it can take negative values) but it is a very useful auxiliary function such that the marginals of $W(q, p)$ yield the correct quantum probabilities of q and p separately. At that time, Wigner's derivation of $W(q, p)$ stemmed from its potential usefulness in calculating quantities such as virial coefficients and chemical reaction rates. As a result, as is clear from the book of Zachos *et al.* it was subsequent authors who extended these ideas to show how classical equivalents could be obtained for all quantum mechanical operators (W being the classical equivalent of the density matrix), leading to a complete re-expression of quantum mechanics in terms of classical concepts, so that quantum mechanics expectation values are now expressed as averages over phase-space distribution functions. In other words, statistical information is transferred from the density operator to a quasi-classical (distribution) function.

Zachos *et al.* proceed by first giving an overview of the subject, with emphasis on the pioneering papers and a list of the various review articles. They point out that the phase space formulation of quantum mechanics has proven to be very useful in a variety of different areas, not just in physics but also in engineering, chemical physics and mathematical physics. In addition, they are particularly eager to present their viewpoint that Wigner's function provides a third formulation of quantum mechanics, independent of the Hilbert space approach (which incorporates, as emphasized by Dirac, both the Heisenberg matrix path and the Schrödinger wave function path) and the approach based on the path integral method of Dirac and Feynman. Whereas this is a legitimate point of view, I would argue otherwise since, as the authors themselves point out, it is first necessary to ignore the fact that the development of W and its applications have always relied heavily on the principles underlying the Hilbert-space formulation. Moreover, one could also argue that all of the other quasi-probability distribution functions could qualify for consideration as alternative formulations of quantum mechanics. On the other hand, one could argue, with some justification, that in many respects, the Wigner function is more fundamental and, of course, it is by far the most commonly used distribution. However, departing from these philosophical musings, we return to the thrust of the overview, which is an excellent survey of the fundamentals underlining the Wigner function, as well as a detailed presentation of how it may be applied.

After presenting the basic properties of the Wigner function, the authors then point out that Wigner's choice corresponds to a particular case of the earlier (1927)

Weyl association rule which maps the c-number phase space functions to operators in a given ordering prescription (a completely symmetrized expression in q and p in the Wigner case). There are a large number of viable choices for the Weyl correspondence rule which lead to many choices of quasi-probability phase space functions other than the choice W used by Wigner. The time dependence of W was already calculated by Wigner using Schrödinger wave functions, but the authors present an alternative proof due to Moyal (using so-called Moyal brackets and Bopp operators) which does not use wave functions explicitly. This is followed by a discussion of the uncertainty principle and Ehrenfest's theorem and a demonstration of how the use of Bopp operators, as distinct from the use of wave functions, can lead to a solution of the harmonic oscillator. However, there is an implicit caveat regarding the fact that atomic physics problems, such as the solution of the hydrogen atom, are solved much more simply using wave functions. Next, it is shown that consideration of mixed states, which conventionally require the use of a density matrix, can be treated by the use of so-called nondiagonal Wigner functions. This is followed by a discussion of stationary perturbation theory, propagators and canonical transformations. The overview of phase-space quantization is brought to a conclusion with a detailed discussion of the Weyl correspondence, showing explicitly the particular "Wigner map" which leads to W and also how other alternative rules of association lead to a variety of other quasi-classical distribution functions.

The authors then go on to present a brief historical outline, with useful remarks on the 23 selected papers (the first 14, covering the years 1927–1982, being chronologically ordered). The first selected paper is that of Weyl (written in German), in which the association rule mentioned above is presented. The second paper (also in German) by von Neuman, expands on Weyl's ideas. The third paper is the famous Wigner paper which was clearly not influenced by either of the preceeding papers but, in a manner not untypical of Wigner modus operandi, was based on the recognition of and the desirability of solving an important physical problem, followed by a powerful mathematical solution of the problem with clear physical applications in mind. The intimate relationship between the Weyl correspondence and the Wigner function is due to Groenewold and Moyal (papers 4 and 5) and, as emphasized by Zachos *et al.*, the Moyal paper is a "grand synthesis" in that it clearly establishes that quantum mechanics may be re-expressed in terms of quasi-classical quantities corresponding to both quantum mechanical operators and expectation values. In particular, the Moyal paper was very influential and led, in papers 6 to 12, to further elaborations and clarifications of the basic formalism and especially his conclusion that "phase-space distributions are not unique for a given state". In particular, paper 10 by Cartwright highlights the fact that W can be negative, and discusses how smoothing can lead to a positive distribution. However, it is now recognized that this smoothed W actually corresponds in essence to a different distribution with a different Weyl correspondence. In paper 11, Royer writes the expectation value of the parity operator in terms of W. Application to atomic structure is presented by Dahl and Springborg in paper 14, in the course of which they resolve a pedagogical dilemma related to the fact that the angular momentum for the ground state of hydrogen is zero, whereas in Bohr theory, it is $\hbar$. Representations of quantum canonical transformations of W functions is the subject of papers 13 and 16. A comprehensive review of W functions is given in paper 15, which also reviews extensions to a system of identical particles, as well as expressing W functions in second-quantized form, facilitating the consideration of both bosons and fermions. Other distribution functions are also considered. Feynman's thoughts on negative probabilities are expressed in paper 17. The rest of the papers discuss deformation theory (18 and 19), general nonlinear canonical transformations (20), the uncertainty principle (21), techniques for perturbation theory (22) and techniques for numerical solutions on the basis of Chebyshev polynomials (23). In summary, the authors have performed an excellent job in presenting a timely and very useful resource for investigators, in potentially many areas requiring quantum physics, who wish to use quasi-probability functions, particularly the Wigner function. I highly recommend it.