

# Measured quantum probability distribution functions for Brownian motion

G. W. Ford

*Department of Physics, University of Michigan, Ann Arbor, Michigan 48109-1040, USA*

R. F. O'Connell

*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA*

(Received 1 April 2007; published 22 October 2007)

The quantum analog of the joint probability distributions describing a classical stochastic process is introduced. A prescription is given for constructing the quantum distribution associated with a sequence of measurements. For the case of quantum Brownian motion this prescription is illustrated with a number of explicit examples. In particular, it is shown how the prescription can be extended in the form of a general formula for the Wigner function of a Brownian particle entangled with a heat bath.

DOI: [10.1103/PhysRevA.76.042122](https://doi.org/10.1103/PhysRevA.76.042122)

PACS number(s): 03.65.Ta, 05.40.-a, 05.40.Jc

## I. INTRODUCTION

The notion of a joint probability distribution is basic to the description of classical stochastic processes. The purpose here is to describe the extension to the quantum regime, giving the prescription for constructing the quantum joint probability distribution associated with a sequence of measurements. The prescription is illustrated with a number of examples. These are an important part of this work, since they show how the prescription can be used to calculate a variety of quantities of practical interest.

The idea is simple: The system is initially in thermal equilibrium. A first measurement prepares a state that then develops in time according to the underlying dynamics. Then a second measurement prepares a new state and so on until the final measurement in the sequence.

In this connection it is necessary to consider the description of quantum measurement. In its most naive form, found in many textbooks, a quantum measurement of a dynamical variable is described as a projection of the system into an eigenstate of the variable, with no memory of the previous state. In particular, for a variable with a continuous spectrum, such as the position of a Brownian particle, with no square-integrable eigenstate, this naive description is unsatisfactory. In an earlier publication of Ford and Lewis [1], the description of measurement as applied to quantum stochastic processes was addressed in some detail. Since that earlier publication may not be accessible to all readers, in Sec. III a summary is given of the essential features of quantum measurement as they apply to the definition of the distribution functions. The reader will observe that the description given there involves no new theory of quantum measurement. Rather, a prescription is adopted based on that used by many authors, making practical calculations related to real experiments.

The plan of the paper is as follows. To begin, in a brief section II the joint distribution functions of classical mechanics are described. The quantum joint distribution functions, which are a close analog of the classical quantities, are introduced in Sec. III. There the key result is the prescription (3.29) for the joint distribution function associated with  $n$  successive measurements. For the case of quantum Brownian motion this prescription can be readily evaluated to give ex-

PLICIT closed-form expressions. Therefore in Appendix A a review is given of those aspects of the theory of quantum Brownian motion that will be useful in the applications. There the key quantities needed for the later discussion are the commutator and the mean-square displacement, given by the general expressions (A7) and (A11). Later in Appendix A these expressions are evaluated explicitly for the Ohmic and single relaxation time models and the results compared with approximate expressions obtained by master equation methods. In Sec. IV the results of Appendix A are used to evaluate the characteristic function associated with the distribution function describing  $n$  successive measurements. There the important result is the expression (4.2) for the characteristic function, where it is seen explicitly how in classical mechanics, where the commutator vanishes, the effects of measurement can be separated from the underlying stochastic process. As an application of this result, an explicit expression is constructed for the pair distribution function associated with wave packet spreading. In Sec. V we discuss the probability distribution, which corresponds to what in elementary quantum mechanics is the “square of the wave function.” A general expression is obtained that is illustrated first with the example of wave packet spreading and then with the example of a macroscopic quantum interference (Schrödinger cat) state. In either case the discussion includes the case of a free particle as well as that of a particle in a harmonic well. Finally, in Sec. VI the Wigner function is introduced. There the key result is the simple formula (6.5) for the Wigner characteristic function (the Fourier transform of the Wigner function). The Wigner function corresponds to the phase space distribution of classical mechanics, but is definitely not a probability distribution (it is seen explicitly in the examples that the Wigner function need not be positive) and cannot be the result of direct quantum measurement. Nevertheless, the Wigner function is useful for describing the results of the measurement. In particular, the probability distribution in coordinate or momentum is obtained by integration over the conjugate variable.

## II. DISTRIBUTION FUNCTIONS IN CLASSICAL BROWNIAN MOTION

In the theory of classical Brownian motion a stochastic process is completely described by a hierarchy of probability

distributions. Here the standard reference for the physicist is the review article by Wang and Uhlenbeck [2], reprinted in the "Noisebook" [3]. For a stochastic variable  $y(t)$  one introduces

$W(y_1, t_1)dy_1$  as probability of finding  $y(t_1)$  in the interval  $dy_1$  about  $y_1$ ,

$W(y_1, t_1; y_2, t_2)dy_1dy_2$  as probability of finding  $y(t_1)$  in the interval  $dy_1$  about  $y_1$ ,

and  $y(t_2)$  in the interval  $dy_2$  about  $y_2$ ,

and so on. (2.1)

This hierarchy must satisfy the following more or less obvious conditions:

(1) Positivity

$$W(y_1, t_1; y_2, t_2; \dots; y_n, t_n) \geq 0.$$

(2) Symmetry

$W(y_1, t_1; y_2, t_2; \dots; y_n, t_n)$  is a symmetric function of the set of variables  $y_1, t_1; y_2, t_2, \dots, y_n, t_n$ .

(3) Consistency

$$W(y_1, t_1; y_2, t_2; \dots; y_n, t_n) = \int dy_{n+1} W(y_1, t_1; y_2, t_2; \dots; y_n, t_n; y_{n+1}, t_{n+1}).$$

Note that consistency corresponds to conservation of total probability,

$$1 = \int dy_1 \int dy_2 \cdots \int dy_n W(y_1, t_1; y_2, t_2; \dots; y_n, t_n). \quad (2.2)$$

A theorem of Kolomogorov states that if the hierarchy satisfies these conditions, there must exist an underlying classical process [4]. That is, there must exist an ensemble of time tracks  $y(t)$  such that the  $W$ 's are the weighted fraction of time tracks that go through the appropriate intervals.

A natural question is, how is this description changed in the quantum case? The answer will be seen in the following section, but for now one can say that the essential change is that in the quantum case the symmetry condition no longer holds.

### III. QUANTUM DISTRIBUTION FUNCTIONS

The system considered is that of a Brownian particle coupled to a heat bath, a system with an infinite number of degrees of freedom. The quantum mechanical motion of this system is described by a microscopic Hamiltonian  $H$  and corresponds to a unitary transformation of states in Hilbert space,

$$\Psi(t) = U(t)\Psi(0), \quad (3.1)$$

where  $\Psi(t)$  is the state vector at time  $t$  and

$$U(t) = \exp(-iHt/\hbar). \quad (3.2)$$

However, one does not have precise knowledge of the initial state. Instead, there is an initial density matrix. The density

matrix is defined as an operator  $\rho(t)$  in Hilbert space such that  $\langle \Phi, \rho(t)\Phi \rangle / \langle \Phi, \Phi \rangle$  is the relative probability at time  $t$  that the system is in any given state  $\Phi$  [5,6]. Note that consistent with this definition  $\rho$  must be a positive-definite Hermitian operator. Its time development follows from Eq. (3.1),

$$\rho(t) = U(t)\rho(0)U^\dagger(t). \quad (3.3)$$

Before introducing the distribution functions, it is necessary to make some general remarks about measurement in quantum mechanics. By "measurement" here is meant "measurement with selection" (what Pauli in his famous Ziffer 9 called "measurement of the second kind" [7]) so that measurement irreversibly changes the state of the system. When discussed in general terms in textbooks, the accepted description of this change of state is framed in terms of the measurement of a discrete variable (Hermitian operator with a pure point spectrum). Let  $B$  be such a variable, with  $b$  an eigenvalue and  $P_b$  its associated projection operator, so that

$$B = \sum_b b P_b. \quad (3.4)$$

Then the effect of a measurement at time  $t_1$  whose result is that the eigenvalue  $b$  is in the interval  $M$  is to instantaneously transform the density matrix,

$$\rho(t_1) \rightarrow P_M \rho(t_1) P_M, \quad (3.5)$$

where  $P_M$  is the projection operator associated with the interval,

$$P_M = \sum_{b \in M} P_b. \quad (3.6)$$

Here instantaneous means that the duration of the measurement is short compared with the natural periods of the system. The prescription (3.5) may be obtained from various assumptions about the optimal character of the measurement (such that the disturbance of the state is somehow minimal). See, e.g., Lüders [8], Goldberger and Watson [9], Furry [10], or Davies and Lewis [11].

But the prescription (3.5) is too restricted for our purpose; it represents too limited a class of measurements. We must consider measurements of limited precision and involving operators with a continuous spectrum, such as the position of a Brownian particle. For guidance as to how to generalize the prescription, we look to such practical fields as the theory of angular correlations (see, e.g., the article by Frauenfelder and Steffan in Ref. [12], Sec. 3) or the theory of polarization in multiple scattering (see, e.g., Wolfenstein [13], Sec. 4). There one associates a transition operator  $T$  (in the scattering case this would be the Wigner  $T$  matrix) with a measurement with a given result (e.g., the observation of an emitted  $\gamma$  ray in a given direction) and represents the transformation of  $\rho$  brought about by the measurement at time  $t_1$  by

$$\rho(t_1) \rightarrow T\rho(t_1)T^\dagger. \quad (3.7)$$

Note that this is the most general transformation that preserves the positivity of the density matrix. In the special case where the transition operator is a projection operator, one recovers the prescription (3.5). To be consistent with the

probabilistic interpretation of the density matrix, the transition operator must satisfy the general requirement

$$\|T\|^2 \equiv \max_{\Phi} \frac{\langle T\Phi, T\Phi \rangle}{\langle \Phi, \Phi \rangle} \leq 1. \quad (3.8)$$

The diagonal matrix elements of  $\rho$  formed with respect to a complete set of states are interpreted as the probabilities of finding the system in the corresponding states. The sum of these probabilities over all states is the trace, and as a consequence of the requirement (3.8), this is reduced by measurement. Thus,

$$\text{Tr}(T\rho T^\dagger) = \text{Tr}(\rho T^\dagger T) \leq \text{Tr}(\rho) \|T\|^2 \leq \text{Tr}(\rho). \quad (3.9)$$

In fact, the ratio  $\text{Tr}(\rho T^\dagger T)/\text{Tr}(\rho)$  can be interpreted as the probability that measurement will produce the given result.

This reduction of the sum over all states of the probability of finding the system in each state is not a unique feature of quantum probability. In classical probability, where the probabilities after measurement would be interpreted as joint probabilities of the result of the measurement *and* of finding the system in the state, the sum of probabilities is also reduced. The difference is that in the classical case none of the individual probabilities will be increased, while in the quantum case some may increase.

As a simple example illustrating all this, consider a spin-1/2 system initially polarized in the +z direction. This can be represented by the  $2 \times 2$  density matrix

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.10)$$

Note that the diagonal elements, which are, respectively, the probability that the spin is in the +z and -z directions, add up to 1. This is in accordance with the *convention* that the trace of the density matrix is normalized to 1 prior to the first measurement. Suppose that a measurement is made, for example, by a Stern-Gerlach apparatus, whose result is that the spin is in the +x direction. The transition matrix corresponding to this result is

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (3.11)$$

in this case a projection operator. The density matrix after the measurement is

$$\rho_{+x} = T\rho_0 T^\dagger = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \quad (3.12)$$

The probability that the spin is in the +z direction has decreased from 1 to  $\frac{1}{4}$  while the probability that the spin is in the -z direction has *increased* from 0 to  $\frac{1}{4}$ . Nevertheless, the sum of the probabilities is  $\frac{1}{2}$ , less than the sum before the measurement. One might ask, what happened to the probability, and how is it that the probabilities after the measurement do not add up to 1? The answer is that there is another possible result of the measurement: the spin is in the -x direction. Repeating the above argument for this case, we find for the density matrix after the measurement

$$\rho_{-x} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}. \quad (3.13)$$

Again the probability that the spin after the measurement is in the  $\pm z$  direction is  $\frac{1}{4}$ . Thus, in either case the sum of the probabilities is  $\frac{1}{2}$ , the probability that the measurement produces the given result. These probabilities add up to 1, so overall probability is conserved.

With these remarks as a guide, consider the measurement of a dynamical variable  $y$ . Assume that  $y$  is a variable, such as the position or velocity of the Brownian particle, with a continuous spectrum over all real values. In this case one can associate with the measurement a function  $f(y_1)$  such that (here  $y_1$  is a  $c$  number)

$$\frac{\langle f(y-y_1)\Phi, f(y-y_1)\Phi \rangle}{\langle \Phi, \Phi \rangle} dy_1 \quad (3.14)$$

is the conditional probability that if the system is in state  $\Phi$ , the instrument will read in the interval  $dy_1$  about  $y_1$ . Here the choice that only the difference  $y-y_1$  appears is made for convenience, since by choosing  $f(y_1)$  to be peaked the requirement that the measured value be somehow close to the actual value can be satisfied in an obvious way. If this conditional probability is to be normalized, one must require ( $f$  need not be real)

$$\int_{-\infty}^{\infty} dy_1 |f(y_1)|^2 = 1. \quad (3.15)$$

An example to keep in mind is that of a ‘‘Gaussian instrument’’ [14], for which

$$f(y_1) = \frac{1}{(2\pi\sigma_1^2)^{1/4}} \exp\left(-\frac{y_1^2}{4\sigma_1^2}\right), \quad (3.16)$$

where  $\sigma_1$  is the experimental width. The result of a measurement at time  $t_1$  in which the instrument reads in the interval  $I_1$  is therefore to instantaneously transform  $\rho$ ,

$$\rho(t_1) \rightarrow \int_{I_1} dy_1 f(y-y_1)\rho(t_1)f(y-y_1)^\dagger. \quad (3.17)$$

As remarked above, an instantaneous measurement is to be understood as one whose duration is short compared with the natural periods of the motion.

The distribution functions are now constructed as follows. To begin assume that at  $t=0$  (or in the distant past) the system is in equilibrium at temperature  $T$ ,

$$\rho(0) = \rho_0 \equiv \frac{e^{-H/kT}}{\text{Tr}(e^{-H/kT})}, \quad (3.18)$$

so  $\rho$  is initially normalized. If  $y$  is to be measured at a later time  $t_1$ , the system must move in time from 0 to  $t_1$ ,

$$\rho_0 \rightarrow U(t_1)\rho_0 U^\dagger(t_1), \quad (3.19)$$

according to Eq. (3.3). At  $t_1$  a measurement is made,

$$U(t_1)\rho_0U^\dagger(t_1) \rightarrow \int_{I_1} dy_1 f(y-y_1)U(t_1)\rho_0U^\dagger(t_1)f(y-y_1)^\dagger, \quad (3.20)$$

according to Eq. (3.17). Finally, the system moves from  $t_1$  to  $t$ ,

$$\begin{aligned} & \int_{I_1} dy_1 f(y-y_1)U(t_1)\rho_0U^\dagger(t_1)f(y-y_1)^\dagger \\ & \rightarrow \int_{I_1} dy_1 U(t-t_1)f(y-y_1)U(t_1)\rho_0U^\dagger(t_1) \\ & \quad \times f(y-y_1)^\dagger U^\dagger(t-t_1). \end{aligned} \quad (3.21)$$

This last expression can be simplified somewhat by introducing the time-dependent variable (Heisenberg representation)

$$y(t_1) = U^\dagger(t_1)yU(t_1). \quad (3.22)$$

Noting that  $U(t-t_1) = U(t)U(-t_1) = U(t)U^\dagger(t_1)$ , the final density matrix (3.21) can be written

$$\int_{I_1} dy_1 U(t)f(y(t_1)-y_1)\rho_0f(y(t_1)-y_1)^\dagger U^\dagger(t). \quad (3.23)$$

The probability that the measurement of  $y$  is in  $I_1$  is the trace of this final density matrix. This same probability is interpreted as the integral over the one-point distribution  $W(y_1, t_1)$  over the interval  $I_1$ , so that

$$\begin{aligned} \int_{I_1} dy_1 W(y_1, t_1) &= \int_{I_1} dy_1 \text{Tr}[U(t)f(y(t_1)-y_1) \\ & \quad \times \rho_0f(y(t_1)-y_1)^\dagger U^\dagger(t)]. \end{aligned} \quad (3.24)$$

Since  $I_1$  is arbitrary, one can identify

$$W(y_1, t_1) = \text{Tr}[U(t)f(y(t_1)-y_1)\rho_0f(y(t_1)-y_1)^\dagger U^\dagger(t)]. \quad (3.25)$$

Finally, the trace is invariant under cyclic permutation of the factors, so one can write

$$W(y_1, t_1) = \text{Tr}[f(y(t_1)-y_1)\rho_0f(y(t_1)-y_1)^\dagger]. \quad (3.26)$$

In the same way one can show that the two-point distribution is

$$\begin{aligned} W(y_1, t_1; y_2, t_2) &= \text{Tr}[f(y(t_2)-y_2)f(y(t_1)-y_1)\rho_0f(y(t_1)-y_1)^\dagger \\ & \quad \times f(y(t_2)-y_2)^\dagger] \end{aligned} \quad (3.27)$$

and, in general, using an obvious shorthand notation

$$W(1, \dots, n) = \text{Tr}[f(n) \cdots f(1)\rho_0f(1)^\dagger \cdots f(n)^\dagger]. \quad (3.28)$$

Finally, note that under cyclic permutation of the factors in the trace, one can write the expression for the  $n$ -point distribution in the compact form

$$W(1, \dots, n) = \langle f(1)^\dagger \cdots f(n)^\dagger f(n) \cdots f(1) \rangle, \quad (3.29)$$

where the angular brackets indicate the thermal equilibrium expectation. That is, for a given operator  $\mathcal{O}$ ,

$$\langle \mathcal{O} \rangle \equiv \text{Tr}(\mathcal{O}\rho_0). \quad (3.30)$$

Expression (3.29) is the key result of this section.

In connection with expression (3.29) for the joint probability distribution, it should first be emphasized that it is time ordered:  $0 < t_1 < t_2 < \cdots < t_n$ . Also we should point out that in our shorthand notation the label applies to *all* the parameters of the measurement. Thus, for example, the label “ $j$ ” represents not only the value  $y_j$  and the time  $t_j$  but also the instrumental parameters such as the width  $\sigma_j$ . The symmetry property of the classical stochastic process refers to symmetry under permutations of the labels. In other words, the probability distributions of a classical stochastic process are symmetric under the interchange of all the parameters of the measurements. An inspection of expression (3.29) for the quantum probability distribution shows that it does not have that symmetry, because the  $f$ 's at different times do not in general commute. However, the quantum distributions still have the consistency property, providing the integral is over the results of the last measurement. This is sometimes called marginal consistency.

#### IV. CHARACTERISTIC FUNCTIONS

In this section we consider the evaluation of the distribution functions when the dynamical variable  $y(t)$  is taken to be the position operator  $x(t)$  for quantum Brownian motion, introduced in Appendix A. In evaluating these distribution functions, it is convenient in analogy with the classical case to introduce the corresponding characteristic functions, defined by

$$\xi(1, \dots, n) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n W(1, \dots, n) \exp\left(i \sum_{j=1}^n k_j x_j\right). \quad (4.1)$$

An important result is that for quantum Brownian motion the characteristic function can be cast in the form

$$\xi(1, \dots, n) = K(1, \dots, n) \left\langle \exp\left(i \sum_{j=1}^n k_j x(t_j)\right) \right\rangle, \quad (4.2)$$

where the factor  $K(1, \dots, n)$  is given by

$$\begin{aligned} K(1, \dots, n) &= \prod_{j=1}^n \int_{-\infty}^{\infty} dx_j f^* \left( x_j - \sum_{l=j+1}^n k_l \frac{[x(t_j), x(t_l)]}{2i} \right) \\ & \quad \times f \left( x_j + \sum_{l=j+1}^n k_l \frac{[x(t_j), x(t_l)]}{2i} \right) e^{ik_j x_j}. \end{aligned} \quad (4.3)$$

In this expression the sums are to be taken to be zero when  $j=n$ . The importance of this result lies first of all in the fact that the effects of measurement are completely contained in the factor  $K$ , in which the particle dynamics enters only through the commutators. In the classical limit these commutators vanish and  $K$  becomes a simple numerical factor. Indeed in this classical limit one generally considers measurements of perfect precision, for which  $|f(x_j)|^2 \rightarrow \delta(x_j)$  and the factor  $K$  is unity. Expression (4.2) then becomes the familiar

form for classical Brownian motion [15]. On the other hand, in the quantum case, where the commutators do not vanish, it is clear that the particle dynamics is inextricably linked with measurement and, as a consequence, the symmetry property of classical stochastic processes does not hold. A second reason for the importance of this result is that it is convenient for calculation, as we shall illustrate in the example below. Before that, however, we give a brief derivation.

Consider first the case  $n=1$ . Using expression (3.29) in the definition (4.1) of the characteristic function, we can write

$$\xi(1) = \left\langle \int_{-\infty}^{\infty} dx_1 f^*(x_1 - x(t_1)) f(x_1 - x(t_1)) e^{ik_1 x_1} \right\rangle. \quad (4.4)$$

Making the change of variable  $x_1 \rightarrow x_1 + x(t_1)$ , we see that

$$\xi(1) = K(1) \langle e^{ik_1 x(t_1)} \rangle, \quad (4.5)$$

where

$$K(1) = \int_{-\infty}^{\infty} dx_1 f^*(x_1) f(x_1) e^{ik_1 x_1}. \quad (4.6)$$

Recall that the sums in expression (4.3) are to be taken to be zero when  $j=n$ .

Next consider

$$\begin{aligned} \xi(1,2) &= \left\langle \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 f(1)^\dagger f(2)^\dagger f(2) f(1) e^{i(k_1 x_1 + k_2 x_2)} \right\rangle \\ &= K(2) \left\langle \int_{-\infty}^{\infty} dx_1 f^*(x_1 - x(t_1)) e^{i[k_1 x_1 + k_2 x(t_2)]} \right. \\ &\quad \left. \times f(x_1 - x(t_1)) \right\rangle. \end{aligned} \quad (4.7)$$

Here  $K(2)$  is exactly of the form (4.6) but with the label ‘‘2’’ in place of ‘‘1.’’ Remember also that with our shorthand notation the label also represents the instrumental parameters, so the measurement function  $f$  changes with the label. Next, we apply the generalized Baker-Campbell-Hausdorff formula (B4) to write

$$\begin{aligned} e^{i[k_1 x_1 + k_2 x(t_2)]} f(x_1 - x(t_1)) &= f(x_1 - x(t_1)) \\ &\quad + ik_2 [x(t_1), x(t_2)] e^{i[k_1 x_1 + k_2 x(t_2)]}. \end{aligned} \quad (4.8)$$

With this and making the change of variable  $x_1 \rightarrow x_1 + x(t_1) + k_2 [x(t_1), x(t_2)]/2i$ , we obtain

$$\xi(1,2) = K(1,2) \langle e^{ik_1 x(t_1)} e^{ik_2 x(t_2)} e^{k_1 k_2 [x(t_1), x(t_2)]} \rangle, \quad (4.9)$$

where

$$\begin{aligned} K(1,2) &= \int_{-\infty}^{\infty} dx_1 f^* \left( x_1 - k_2 \frac{[x(t_1), x(t_2)]}{2i} \right) \\ &\quad \times f \left( x_1 + k_2 \frac{[x(t_1), x(t_2)]}{2i} \right) e^{ik_1 x_1} K(2), \end{aligned} \quad (4.10)$$

which is of the form (4.3) with  $n=2$ . As a final step, we use the Baker-Campbell-Hausdorff formula (B3) to write  $e^{ik_1 x(t_1)} e^{ik_2 x(t_2)} e^{k_1 k_2 [x(t_1), x(t_2)]} = e^{i[k_1 x(t_1) + k_2 x(t_2)]}$  and obtain the form (4.2) with  $n=2$ .

For the general case, the argument goes in the same way. Assuming the form (4.3) for smaller  $n$ , we write the definition (4.1) in the form

$$\begin{aligned} \xi(1, \dots, n) &= K(2, \dots, n) \left\langle \int_{-\infty}^{\infty} dx_1 f^*(x_1 - x(t_1)) \right. \\ &\quad \left. \times \exp \left[ i \left( k_1 x_1 + \sum_{l=2}^n k_l x(t_l) \right) \right] f(x_1 - x(t_1)) \right\rangle. \end{aligned} \quad (4.11)$$

Then we bring the exponential factor to the right, using the theorem (4.9) as in Eq. (4.8). Then, shifting the variable of integration and using the Baker-Campbell-Hausdorff formula in the exponential factor, we get the form (4.3).

#### A. Example: Wave packet spreading

Here we consider the case of two successive measurements, each with a measurement function of the form (3.16), corresponding to a Gaussian slit. First, we consider a single measurement with

$$f(x_1) = \frac{1}{(2\pi\sigma_1^2)^{1/4}} \exp\left(-\frac{x_1^2}{4\sigma_1^2}\right). \quad (4.12)$$

With this we use the standard Gaussian integral (B1), to evaluate the integral expression (4.6). We find

$$K(1) = \exp\left(-\frac{1}{2}\sigma_1^2 k_1^2\right). \quad (4.13)$$

Using the Gaussian property (B6) to evaluate the expectation in Eq. (4.5), we find

$$\langle e^{ik_1 x(t_1)} \rangle = \exp\left(-\frac{1}{2}\langle x^2 \rangle k_1^2\right). \quad (4.14)$$

With this, we find

$$\xi(1) = \exp\left(-\frac{1}{2}\sigma^2 k_1^2\right), \quad (4.15)$$

where we have introduced

$$\sigma^2 = \langle x^2 \rangle + \sigma_1^2. \quad (4.16)$$

Finally, we invert the definition (4.1) of the characteristic function to write

$$W(1) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \xi(1) e^{-ik_1 x_1}. \quad (4.17)$$

This again is a standard Gaussian integral, and we obtain the result

$$W(1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x_1^2}{2\sigma^2}\right). \quad (4.18)$$

Thus the probability distribution associated with a single measurement is a Gaussian whose variance is the sum of that of the instrument and that of the underlying quantum state of the particle. We have presented the steps leading to this result in detail since these are the steps that will be used repeatedly in this and our later examples.

Consider now a pair of successive measurements, the first with measurement function of the form (4.12) and the second of the same form but with the index “1” replaced by “2.” Using the standard Gaussian integral to evaluate the integral in expression (4.10) for  $K(1,2)$ , we find

$$K(1,2) = \exp\left[-\frac{1}{2}\sigma_1^2 k_1^2 - \frac{1}{2}\left(\sigma_2^2 - \frac{[x(t_1),x(t_2)]^2}{4\sigma_1^2}\right)k_2^2\right]. \quad (4.19)$$

Then, using the Gaussian property, we see that

$$\langle e^{i[k_1 x(t_1) + k_2 x(t_2)]} \rangle = \exp\left(-\frac{1}{2}\langle x^2 \rangle (k_1^2 + k_2^2) - c(t_2 - t_1) k_1 k_2\right). \quad (4.20)$$

Here we have introduced the correlation (A9). Putting these together, using the expression (4.2) for the characteristic function, we can write

$$\xi(1,2) = \exp\left(-\frac{1}{2}(\sigma^2 k_1^2 + 2\sigma\tau\rho k_1 k_2 + \tau^2 k_2^2)\right), \quad (4.21)$$

where [note the misprint in the Eq. (7.18) of Ref. [1]]

$$\begin{aligned} \sigma^2 &= \langle x^2 \rangle + \sigma_1^2, \\ \tau^2 &= \langle x^2 \rangle + \sigma_2^2 - \frac{[x(t_1),x(t_2)]^2}{4\sigma_1^2}, \\ \sigma\tau\rho &= c(t_2 - t_1). \end{aligned} \quad (4.22)$$

Note that  $\sigma^2$  is the same quantity (4.16) that appears in the single-measurement function. The two-measurement distribution function is given by

$$W(1,2) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \xi(1,2) e^{-i(k_1 x_1 + k_2 x_2)}. \quad (4.23)$$

With the form (4.21) of  $\xi(1,2)$  we can perform the integration using the multidimensional form (B2) of the standard Gaussian integral. The result is

$$W(1,2) = \frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}} \exp\left(-\frac{\tau^2 x_1^2 - 2\sigma\tau\rho x_1 x_2 + \sigma^2 x_2^2}{2\sigma^2 \tau^2 (1-\rho^2)}\right). \quad (4.24)$$

Here we remark first of all that the lack of symmetry of the quantum distribution is obvious:  $W(1,2) \neq W(2,1)$ . The exception is when the commutator vanishes. Note that the symmetry, or lack of it, is with respect to interchange of the labels; that is, one must interchange not only  $x_1 \leftrightarrow x_2$  and  $t_1 \leftrightarrow t_2$  but also  $\sigma_1 \leftrightarrow \sigma_2$ . Another aspect of this asymmetry is that it is possible to make the last measurement one of perfect precision—that is, put  $\sigma_2=0$ .

A second remark is that the time dependence is only through the time difference  $t_2-t_1$ . This is a general feature, independent of the form of the measurement function. It arises from the time-translation invariance of the equilibrium state.

Finally, we remark that for widely separated times ( $t_2-t_1 \rightarrow \infty$ ) the correlation and the commutator for the oscillator vanish. Then we see that  $W(1,2) \rightarrow W(1)W(2)$ . This is a special case of the *cluster property* of the quantum joint distribution functions, a property they share with the classical functions. Whenever the time between any two successive measurements is large, the quantum joint distribution function factors into a product of distribution functions, the one corresponding to the earlier times, the other to the later times. The exception would be the case of a free particle, since in that case there is no approach to an equilibrium value of  $\langle x^2 \rangle$ .

## V. PROBABILITY DISTRIBUTION

In elementary quantum mechanics one interprets the absolute square of the wave function as the probability distribution of the particle position. That is,  $\mathcal{P}(x,t) = |\psi(x,t)|^2$ , where  $\mathcal{P}dx$  is the probability of finding the particle in the interval  $dx$  at time  $t$ . In our discussion this probability distribution becomes the conditional probability, specialized such that the second measurement is a perfect measurement. That is, the probability distribution is given by

$$\mathcal{P}(x_2, t_2 - t_1) \equiv \left. \frac{W(1,2)}{W(1)} \right|_{\sigma_2=0}. \quad (5.1)$$

Here we exhibit only the dependence on the final particle position and the time difference. (Due to the time-translation invariance of the equilibrium state, only the time difference appears.) The picture we have is that the initial state is prepared by the first measurement, a measurement made on the equilibrium state, and that the second (perfect) measurement samples the state. Of course, in the special case of a particle not interacting with the bath and at temperature zero, this reduces to the elementary quantum mechanics prescription.

If we use expression (3.29) for the joint probability distributions, we see that we can write

$$\mathcal{P}(x_2, t_2 - t_1) = \frac{\langle f(1)^\dagger \delta(x_2 - x(t_2)) f(1) \rangle}{\langle f(1)^\dagger f(1) \rangle}. \quad (5.2)$$

While we can use the method described in Sec. IV to calculate  $W(1)$  and  $W(1,2)$  and then put the results in the definition (5.1), it is just as well to evaluate the expression (5.2) directly. For this purpose we introduce the integral expression for the delta function,

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dP}{2\pi\hbar} e^{ixP/\hbar}, \quad (5.3)$$

with which we can write

$$\mathcal{P}(x_2, t_2 - t_1) = \int_{-\infty}^{\infty} \frac{dP}{2\pi\hbar} \frac{\langle f(1)^\dagger e^{-ix(t_2)P/\hbar} f(1) \rangle}{\langle f(1)^\dagger f(1) \rangle} e^{ix_2 P/\hbar}. \quad (5.4)$$

Now we use the formula (B4) to write

$$f(1)^\dagger e^{-ix(t_2)P/\hbar} f(1) = f^*(x(t_1) - x_1) f(x(t_1) - x_1 - GP) e^{-ix(t_2)P/\hbar}, \quad (5.5)$$

where we have used relation (A7) to write the commutator in terms of the Green function,  $G = G(t_2 - t_1)$ . Next we again use the integral expression (5.3) for the delta function to write

$$\begin{aligned} & f^*(x(t_1) - x_1) f(x(t_1) - x_1 - GP) \\ &= \int_{-\infty}^{\infty} dx'_1 f^* \left( x'_1 + \frac{GP}{2} \right) f \left( x'_1 - \frac{GP}{2} \right) \\ & \quad \times \int_{-\infty}^{\infty} \frac{dP'}{2\pi\hbar} e^{i[x_1 + x'_1 - x(t_1) + GP/2]P'/\hbar}. \end{aligned} \quad (5.6)$$

Introducing this in Eq. (5.5) and using the Baker-Campbell-Hausdorff formula (B3) we can write

$$\begin{aligned} \langle f(1)^\dagger e^{-ix(t_2)P/\hbar} f(1) \rangle &= \int_{-\infty}^{\infty} dx'_1 f^* \left( x'_1 + \frac{GP}{2} \right) f \left( x'_1 - \frac{GP}{2} \right) \\ & \quad \times \int_{-\infty}^{\infty} \frac{dP'}{2\pi\hbar} e^{i(x_1 + x'_1)P'/\hbar} \\ & \quad \times \langle e^{-i[x(t_1)P' + x(t_2)P]/\hbar} \rangle. \end{aligned} \quad (5.7)$$

The Gaussian property (B6) allows us to write

$$\langle e^{-i[x(t_1)P' + x(t_2)P]/\hbar} \rangle = \exp \left( - \frac{\langle x^2 \rangle (P'^2 + P^2) + 2cPP'}{2\hbar^2} \right), \quad (5.8)$$

where  $c = c(t_2 - t_1)$  is the correlation (A9). Finally, with this result, the integral over  $P'$  in Eq. (5.7) is a standard Gaussian integral (B1) and we obtain

$$\begin{aligned} & \langle f(1)^\dagger e^{-ix(t_2)P/\hbar} f(1) \rangle \\ &= \exp \left( - \frac{(\langle x^2 \rangle^2 - c^2)P^2}{2\langle x^2 \rangle \hbar^2} \right) \\ & \quad \times \int_{-\infty}^{\infty} dx'_1 f^* \left( x'_1 + \frac{GP}{2} \right) f \left( x'_1 - \frac{GP}{2} \right) \\ & \quad \times \frac{\exp[-(x_1 + x'_1)^2/2\langle x^2 \rangle - i(c/\langle x^2 \rangle)(x_1 + x'_1)P/\hbar]}{\sqrt{2\pi\langle x^2 \rangle}}. \end{aligned} \quad (5.9)$$

Here we should recall that  $c = c(t_2 - t_1)$  is the correlation (A9) and  $G = G(t_2 - t_1)$  is the Green function (A5). This is the key result of this section, valid for any form of the measurement

function  $f$ . Dividing this result by its value for  $P=0$  we get the integrand in the expression (5.4) for the probability distribution. We next consider some examples corresponding to different choices of the measurement function.

### A. Example: Wave packet spreading

Again, we consider wave packet spreading with an initial measurement corresponding to a single Gaussian slit, the measurement function being that given in Eq. (4.12). There is no need to evaluate the general expression (5.9) since we already have the expressions (4.18) for  $W(1)$  and (4.24) for  $W(1, 2)$ . Putting these in Eq. (5.1) we can write

$$\mathcal{P}(x, t) = \frac{1}{\sqrt{2\pi w^2(t)}} \exp \left( - \frac{[x - \bar{x}(t)]^2}{2w^2(t)} \right). \quad (5.10)$$

This is a Gaussian distribution with center  $\bar{x}(t)$  and variance  $w^2(t)$ , where

$$\begin{aligned} \bar{x}(t) &= \frac{\tau\rho}{\sigma} x_1 = \frac{c(t)}{\langle x^2 \rangle + \sigma_1^2} x_1, \\ w^2(t) &= \tau^2(1 - \rho^2) = \langle x^2 \rangle - \frac{[x(0), x(t)]^2}{4\sigma_1^2} - \frac{c^2(t)}{\langle x^2 \rangle + \sigma_1^2}. \end{aligned} \quad (5.11)$$

Here we should again recall that  $c(t)$  is the correlation (A9).

As a first consideration, we note that  $\mathcal{P}(x, 0)$  is the probability distribution for the particle distribution immediately after the first measurement. Since  $c(0) = \langle x^2 \rangle$  and the commutator vanishes at  $t=0$ , we see that the center and variance of the initial distribution are

$$\begin{aligned} \bar{x}(0) &= \frac{\langle x^2 \rangle}{\langle x^2 \rangle + \sigma_1^2} x_1, \\ w^2(0) &= \frac{\langle x^2 \rangle \sigma_1^2}{\langle x^2 \rangle + \sigma_1^2}. \end{aligned} \quad (5.12)$$

As one can easily verify, this initial distribution corresponds to the product of a Gaussian distribution of variance  $\langle x^2 \rangle$  centered at the origin with one of variance  $\sigma_1^2$  centered at  $x_1$ . That is, the initial distribution corresponds to the wave packet formed when the equilibrium state of the oscillator is passed through a Gaussian slit of width  $\sigma_1$  centered at  $x_1$ .

### 1. Free particle

The free particle coupled to the bath in the absence of the oscillator potential corresponds to the limit  $\langle x^2 \rangle \rightarrow \infty$ . The point here is simple: the oscillator force can be neglected near the center, and the motion will be that of a free particle. Noting that  $c(t) = \langle x^2 \rangle - s(t)/2$ , where  $s(t)$  is the mean-square displacement and remains finite in the limit, we find that the center and variance (5.11) of the probability distribution become

$$\bar{x}(t) = x_1,$$

$$w^2(t) = \sigma_1^2 + s(t) - \frac{[x(0),x(t)]^2}{4\sigma_1^2}. \quad (5.13)$$

With the commutator expressed in terms of the Green function, this expression for free particle wave packet spreading corresponds to that obtained using path integral methods by Hakim and Ambegoakar [16]. For a free particle not interacting with the bath and at temperature zero, in which case  $s(t)=0$  and  $[x(0),x(t)]=i\hbar t/m$ , this reduces to the well-known expression for wave packet spreading found in elementary quantum textbooks.

**2. Displaced ground-state distribution**

Another limit of interest is that in which  $\sigma_1^2 \rightarrow \infty$ , while at the same time  $x_1 \rightarrow \infty$  such that  $x_0 = \langle x^2 \rangle_{x_1} / \sigma_1^2$  is fixed. In this limit, the center and variance (5.11) become

$$\bar{x}(t) = \frac{c(t)}{\langle x^2 \rangle} x_0, \quad (5.14)$$

$$w^2(t) = \langle x^2 \rangle.$$

The probability distribution (5.12) therefore is that of a displaced equilibrium state. That is,

$$\mathcal{P}(x,t) = \mathcal{P}_{\text{eq}}(x - \bar{x}(t)), \quad (5.15)$$

where

$$\mathcal{P}_{\text{eq}}(x) = \frac{1}{\sqrt{2\pi\langle x^2 \rangle}} \exp\left(-\frac{x^2}{2\langle x^2 \rangle}\right) \quad (5.16)$$

is the probability distribution of the equilibrium state. Thus, the wave packet moves without spreading, the variance being that of the equilibrium state of the oscillator. The center is initially at  $x_0$  and asymptotically approaches the origin as the equilibrium state is reached. We shall come back to discuss this result further in Sec. V when we discuss spreading of an initial coherent state.

**B. Example: Schrödinger cat state**

Here we consider the case where the initial measurement forms two separated wave packets. The first measurement function then has the form

$$f(1) = \frac{\exp[-(x_1 - d/2)^2/4\sigma_1^2] + \exp[-(x_1 + d/2)^2/4\sigma_1^2]}{[8\pi\sigma_1^2(1 + e^{-d^2/8\sigma_1^2})^2]^{1/4}}. \quad (5.17)$$

With this form of the measurement function the integration in Eq. (5.9) involves only the standard Gaussian integral (B1). Note that  $x_1$  is the position of the center of the instrument, which should be chosen to be zero if we wish the wave packet pair to be symmetrically placed about the origin. We then find

$$\begin{aligned} & \left( \frac{\langle f(1)^\dagger e^{i[x_2-x(t)]P/\hbar} f(1) \rangle}{\langle f(1)^\dagger f(1) \rangle} \right)_{x_1=0} \\ &= \frac{\exp(-w^2 P^2/2\hbar^2 + ix_2 P/\hbar)}{1 + \exp\left[-\frac{\langle x^2 \rangle d^2}{8\sigma_1^2(\langle x^2 \rangle + \sigma_1^2)}\right]} \\ & \times \left[ \cos \frac{P\bar{d}}{2\hbar} + \exp\left(-\frac{\langle x^2 \rangle d^2}{8\sigma_1^2(\langle x^2 \rangle + \sigma_1^2)}\right) \right] \\ & \times \cosh \frac{GPd}{4\sigma_1^2}, \end{aligned} \quad (5.18)$$

where  $w^2 = w^2(t_2 - t_1)$  is given in Eqs. (5.11) and  $\bar{d} = \bar{d}(t_2 - t_1)$  with

$$\bar{d}(t) = \frac{c(t)}{\langle x^2 \rangle + \sigma_1^2} d. \quad (5.19)$$

While there is no difficulty evaluating the integral expression (5.4) with this expression for the integrand, our interest will be in the limits of a free particle or, for the oscillator, a displaced ground-state pair, in which case it is simpler to first evaluate the limits of the above expression and then evaluate the integral.

**1. Free particle**

As in the above example of wave packet spreading, we obtain the case of a free particle coupled to the bath in the absence of the oscillator potential by forming the limit  $\langle x^2 \rangle \rightarrow \infty$ . Forming this limit of expression (5.18) and then putting the result into the expression (5.4) for the probability distribution and performing the integral with the standard Gaussian formula (B1) we find

$$\begin{aligned} \mathcal{P}(x,t) &= \frac{1}{2(1 + e^{-d^2/8\sigma_1^2})} \left( \frac{\exp[-(x - d/2)^2/2w^2]}{\sqrt{2\pi w^2}} \right. \\ &+ \frac{\exp[-(x + d/2)^2/2w^2]}{\sqrt{2\pi w^2}} \\ &+ 2a \frac{\exp[-(x^2 + d^2/4)/2w^2]}{\sqrt{2\pi w^2}} \cos \frac{[x(0),x(t)]xd}{4i\sigma_1^2 w^2} \Big), \end{aligned} \quad (5.20)$$

where now  $w^2(t)$  is given by the free particle form (5.13) and we have used relation (A7) to reintroduce the commutator. In this expression  $a(t)$  is the attenuation coefficient, given by

$$a(t) = \exp\left(-\frac{s(t)d^2}{8\sigma_1^2 w^2(t)}\right). \quad (5.21)$$

This expression for the probability distribution is the same as that derived in an earlier brief communication [17].

This probability distribution is the sum of three contributions, corresponding to the three terms within the braces. The first two are probability distributions of the form (5.12) corresponding to a pair of single slits positioned at  $\pm d/2$ , while the third term (that involving the cosine) is an interference



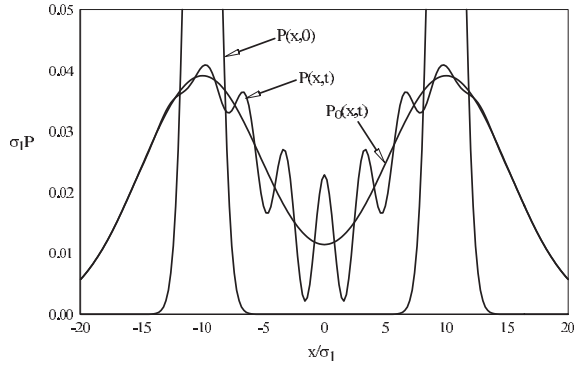


FIG. 1. Probability distribution for a free particle without dissipation in the Schrödinger cat state.  $P(x,0)$  is the initial distribution.  $P(x,t)$  is the distribution at time  $t$ , while  $P_0(x,t)$  is the distribution obtained at time  $t$  by artificially setting the attenuation coefficient equal to zero.

term. The attenuation coefficient is a measure of the size of the interference term and is defined as the ratio of the amplitude of the interference term to twice the geometric mean of the other two terms. The point here is perhaps best seen if we look first at the case of a particle without dissipation and at zero temperature. Then  $s(t)=0$  and  $[x(0),x(t)]=i\hbar t/m$ . The resulting probability distributions are shown in Fig. 1. There  $\mathcal{P}(x,0)$  is the initial probability distribution, which depends only on the initial measurement function and is therefore the same whether or not there is dissipation. In this initial distribution the interference term corresponds to a miniscule peak at the origin, so small that it does not show in the plot. In the same figure  $\mathcal{P}(x,t)$  is the distribution at a time  $t$  such that the width of the individual wave packets has increased by a factor of roughly 3, while  $\mathcal{P}_0(x,t)$  is the same distribution but with the attenuation factor  $a(t)$  set equal to zero. We emphasize that at this later time the amplitude of the interference term is of the order of that of the other terms, despite the fact that initially it is negligibly small. The difference between  $\mathcal{P}(x,t)$ , where the attenuation factor is unity and the interference term is present, and  $\mathcal{P}_0(x,t)$ , where the interference term is absent, is what is called decoherence. Thus, the attenuation coefficient corresponds to the traditional measure of coherence in terms of the relative amplitude of an observed interference term [18].

In the presence of dissipation the attenuation factor decays rapidly when the separation  $d$  of the wave packets is large. Here by “large” we mean not only large compared with the slit width  $\sigma_1$  but also large compared with the mean de Broglie wavelength  $\bar{\lambda}=\hbar/m\sqrt{\langle x^2 \rangle}$ . To obtain the short-time behavior in this case, we note that, as we have seen in Eq. (A13), for short times the mean-square displacement  $s(t)\cong\langle x^2 \rangle t^2$ . Then, since for short times  $w^2(t)\cong\sigma_1^2$ , we see that, for short times,

$$a(t)\cong e^{-t^2/2\tau_d^2}, \quad (5.22)$$

where  $\tau_d$ , the decoherence time, is given by

$$\tau_d = \frac{2\sigma_1^2}{\sqrt{\langle x^2 \rangle}d}. \quad (5.23)$$

In the high-temperature case, where  $\langle x^2 \rangle=kT/m$ , this is the result for the decoherence time obtained previously [17], but the result holds equally well at zero temperature, where  $\langle x^2 \rangle$  is given in Eq. (A29). In either case, the decoherence time is very short when the separation  $d$  of the pair of wave packets is large.

## 2. Displaced ground-state pair

As in the above example of wave packet spreading, we obtain a relatively simple expression for the oscillator case in the limit  $\sigma_1^2 \rightarrow \infty$  and  $d \rightarrow \infty$  such that  $d_0 = \langle x^2 \rangle d / \sigma_1^2$  is fixed. Forming this limit of the result (5.18) and then evaluating the expression ((5.4)) for the probability distribution, we find

$$\begin{aligned} \mathcal{P}(x,t) = \frac{1}{2(1+e^{-d_0^2/8\langle x^2 \rangle})} & \left[ \mathcal{P}_{\text{eq}}\left(x-\frac{\bar{d}}{2}\right) + \mathcal{P}_{\text{eq}}\left(x+\frac{\bar{d}}{2}\right) \right. \\ & \left. + 2a(t)e^{-\bar{d}^2(t)/8\langle x^2 \rangle} \mathcal{P}_{\text{eq}}(x) \cos \frac{[x(0),x(t)]xd_0}{4i\langle x^2 \rangle^2} \right], \end{aligned} \quad (5.24)$$

where  $\mathcal{P}_{\text{eq}}$  is the equilibrium distribution (5.16) and now

$$\bar{d}(t) = \frac{c(t)}{\langle x^2 \rangle} d_0, \quad (5.25)$$

while here the attenuation coefficient is given by

$$a(t) = \exp \left[ -\frac{d_0^2}{8\langle x^2 \rangle} \left( 1 - \frac{c^2}{\langle x^2 \rangle^2} + \frac{[x(0),x(t)]^2}{4\langle x^2 \rangle^2} \right) \right]. \quad (5.26)$$

In connection with this result we remark first that initially the distribution is of the same form as in the free particle case, with two Gaussian peaks at  $\pm d_0/2$  and a miniscule central peak. The difference is in the motion: the two peaks drift back and forth against each other without spreading, eventually arriving at the origin. The interference term therefore has a different effect (sometimes called a “quantum carpet” [19]) but we have nevertheless introduced the attenuation coefficient in the same way as the ratio of the coefficient of the cosine to twice the geometric mean of the first two terms. For very short times,  $c(t)\cong\langle x^2 \rangle - \frac{1}{2}\langle x^2 \rangle t^2$  and  $[x(0),x(t)]\cong i\hbar t/m$ , so  $a(t)$  is of the same form (5.22) but now with

$$\tau_d = \frac{2\langle x^2 \rangle}{\sqrt{\langle x^2 \rangle - \hbar^2/4m^2\langle x^2 \rangle}d_0}. \quad (5.27)$$

Note that the uncertainty principle tells us that  $m^2\langle x^2 \rangle\langle x^2 \rangle \geq \hbar^2/4$ , so the argument of the square root is always positive.

## VI. WIGNER FUNCTION

The Wigner function is the analog of the probability distribution in which the second measurement is a perfect mea-

surement of position *and* momentum. Of course, a quantum particle cannot have simultaneously a precise position and momentum, so this last cannot be a proper quantum measurement of the general form (3.7). The Wigner function is therefore not a probability distribution but rather what is called a “quasiprobability distribution function.” To get the Wigner function corresponding to the density matrix at time  $t_2$ , which we denote by  $\mathcal{W}(q, p; t_2 - t_1)$ , we make the replacement  $f^\dagger(2)f(2) \rightarrow “\delta(q - x(t_2))\delta(p - m\dot{x}(t_2))” = F(q - x(t_2), p - m\dot{x}(t_2))$ , where

$$F(q, p) = \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dP e^{i(Pq + Qp)/\hbar}. \quad (6.1)$$

That is, in place of the general formula (5.2) for the probability distribution we have its generalization to quantum phase space given by the general formula

$$\begin{aligned} \mathcal{W}(q, p; t_2 - t_1) &= \frac{\langle f^\dagger(1)F(q - x(t_2), p - m\dot{x}(t_2))f(1) \rangle}{\langle f^\dagger(1)f(1) \rangle} \\ &= \int_{-\infty}^{\infty} \frac{dQ}{2\pi\hbar} \int_{-\infty}^{\infty} \frac{dP}{2\pi\hbar} e^{i(Pq + Qp)/\hbar} \\ &\quad \times \frac{\langle f^\dagger(1)e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar}f(1) \rangle}{\langle f^\dagger(1)f(1) \rangle}. \end{aligned} \quad (6.2)$$

Note first of all that as a classical function,  $F(q, p) = \delta(q)\delta(p)$ . This expression would be unsatisfactory for our purposes, since in the general formula (6.2) the arguments of the delta functions would not commute. The integral expression (6.1) corresponds to the Fourier–von Neumann representation of the classical operator. Second, we note that the momentum operator is interpreted as the *mechanical* momentum  $m\dot{x}$ . This, as we have noted above, is in accordance with the macroscopic description of a dissipative system. The point here is that the canonical momentum is an operator of the microscopic description, which for the same macroscopic description may or may not be equal to the mechanical momentum [20]. Finally, we should emphasize that the Wigner function is not a probability distribution as discussed in Sec. III. For this reason we use a calligraphic  $\mathcal{W}$  to help keep this in mind. This formula for the Wigner function is unique in the sense that it satisfies certain general requirements such as that it be a real function, that the integral over  $p$  or  $q$  must give the corresponding probability distribution in position or momentum, etc. For a thorough discussion of the Wigner function as it has appeared in the literature, see the review article of Hillery *et al.* [21] [see especially their Eq. (2.45)].

It is convenient to introduce the Fourier transform of the Wigner function,

$$\tilde{\mathcal{W}}(Q, P; t) = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp e^{-i(Pq + Qp)/\hbar} \mathcal{W}(q, p; t). \quad (6.3)$$

This Fourier transform is what in the literature is called the “characteristic function” [21]. We adopt this convenient terminology, but warn that this Wigner characteristic function should not be confused with the quantum analog of the char-

acteristic functions of classical probability introduced in Sec. IV. The inverse Fourier transform is

$$\mathcal{W}(q, p; t) = \int_{-\infty}^{\infty} \frac{dQ}{2\pi\hbar} \int_{-\infty}^{\infty} \frac{dP}{2\pi\hbar} \tilde{\mathcal{W}}(Q, P; t) e^{i(Pq + Qp)/\hbar}. \quad (6.4)$$

Comparing this with the general formula (6.2), we obtain by inspection a simple formula for the Wigner characteristic function:

$$\tilde{\mathcal{W}}(Q, P; t_2 - t_1) = \frac{\langle f^\dagger(1)e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar}f(1) \rangle}{\langle f^\dagger(1)f(1) \rangle}. \quad (6.5)$$

This is the key result of this section, valid for any form of the measurement function  $f$ . As we next shall show, it allows us to readily calculate the Wigner function for a variety of examples.

### A. Example: Equilibrium Wigner function

As a first simple example we consider the equilibrium Wigner function, which we denote by  $\mathcal{W}_{\text{eq}}(q, p)$  and which we get when we make no initial measurement. This corresponds to  $f(1) \rightarrow 1$ , and for this case the formula (6.5) for the Wigner characteristic function becomes

$$\tilde{\mathcal{W}}_{\text{eq}}(Q, P) = \langle e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} \rangle. \quad (6.6)$$

Using the Gaussian formula (B6) we find

$$\tilde{\mathcal{W}}_{\text{eq}}(Q, P) = \exp\left(-\frac{1}{2\hbar^2}(\langle x^2 \rangle P^2 + m^2 \langle \dot{x}^2 \rangle Q^2)\right), \quad (6.7)$$

where we have used the fact that  $\langle x\dot{x} + \dot{x}x \rangle = 0$ . With this in Eq. (6.4), the integrals are standard Gaussian integrals and we find, for the equilibrium Wigner function,

$$\mathcal{W}_{\text{eq}}(q, p) = \frac{1}{2\pi m \sqrt{\langle x^2 \rangle \langle \dot{x}^2 \rangle}} \exp\left(-\frac{q^2}{2\langle x^2 \rangle} - \frac{p^2}{2m^2 \langle \dot{x}^2 \rangle}\right). \quad (6.8)$$

Here  $\langle x^2 \rangle$  and  $\langle \dot{x}^2 \rangle$  are given in expressions (A30) and (A29). The familiar weak-coupling form, well known as the equilibrium solution of the master equation, results if we recall the relations for weak coupling given in Eq. (A39).

### B. Example: Motion of a coherent state

Coherent states are generally defined for the free oscillator by operating on the oscillator ground state with the general displacement operator  $\exp[i(mv_0x - x_0p)/\hbar]$ . The resulting coherent state corresponds to a displaced ground state, centered at  $x_0$  and moving with velocity  $v_0$ . Here we define a generalized coherent state for an oscillator interacting with a linear passive heat bath. The corresponding density matrix is obtained by acting on the equilibrium density matrix with the measurement function

$$f(1) = \exp\{im[v_0x(t_1) - x_0\dot{x}(t_1)]/\hbar\}. \quad (6.9)$$

With this, the expression (6.5) for the Wigner characteristic function becomes

$$\begin{aligned} \tilde{\mathcal{W}}(Q, P; t_2 - t_1) \\ = \langle e^{-im[v_0x(t_1) - x_0\dot{x}(t_1)]/\hbar} e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} e^{im[v_0x(t_1) - x_0\dot{x}(t_1)]/\hbar} \rangle. \end{aligned} \quad (6.10)$$

Use the Baker-Campbell-Hausdorff formula (B3) to write

$$\begin{aligned} e^{-im[v_0x(t_1) - x_0\dot{x}(t_1)]/\hbar} e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} e^{im[v_0x(t_1) - x_0\dot{x}(t_1)]/\hbar} \\ = e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} e^{m[x(t_2)P + m\dot{x}(t_2)Q, v_0x(t_1) - x_0\dot{x}(t_1)]/\hbar^2}. \end{aligned} \quad (6.11)$$

With this, we find

$$\tilde{\mathcal{W}}(Q, P; t) = \exp\left(-i\frac{1}{\hbar}[\bar{x}(t)P + m\bar{v}(t)Q]\right) \tilde{\mathcal{W}}_{\text{eq}}(Q, P), \quad (6.12)$$

where, using Eq. (A7) to express the commutator in terms of the Green function,

$$\bar{x}(t) = m\dot{G}(t)x_0 + mG(t)v_0,$$

$$\bar{v}(t) = \frac{d\bar{x}(t)}{dt} = m\ddot{G}(t)x_0 + m\dot{G}(t)v_0. \quad (6.13)$$

The Wigner function is given by the inverse Fourier transform (6.4). We find

$$\mathcal{W}(q, p; t) = \mathcal{W}_{\text{eq}}(q - \bar{x}(t), p - m\bar{v}(t)). \quad (6.14)$$

Thus the Wigner function corresponding to an initial coherent state has the form of an equilibrium Wigner function whose center moves according to Eqs. (6.13).

We should point out that the motion (6.13) of this center is not the solution of the mean of the quantum Langevin equation (A1). Rather, it is the solution of the mean of the initial value Langevin equation [22]

$$m\frac{d^2\bar{x}}{dt^2} + \int_0^t dt' \mu(t-t') \frac{d\bar{x}(t')}{dt'} + K\bar{x} = -\mu(t)\bar{x}(0), \quad (6.15)$$

with initial data  $\bar{x}(0) = x_0$  and  $\bar{v}(t) = v_0$ . The effect of the term on the right-hand side can be seen clearly in the Ohmic case, where the Green function has the form (A20). With that form we see that  $\bar{x}(t+0^+) = x_0$ , while  $\bar{v}(t+0^+) = v_0 - \gamma x_0$ . Thus the center of initial distribution in the  $qp$  plane makes a jump in the  $p$  direction, down (up) if  $x_0$  is positive (negative), after which the motion of the center is that of a damped harmonic oscillator. At all times the shape of the distribution is that of a displaced thermal equilibrium state of the oscillator.

The probability distribution is obtained by integrating over  $p$ ,

$$\mathcal{P}(x, t) = \int_{-\infty}^{\infty} dp \mathcal{W}(x, p; t). \quad (6.16)$$

With the expression (6.14) for the Wigner function in which  $\mathcal{W}_{\text{eq}}$  is of the form (6.8) this becomes

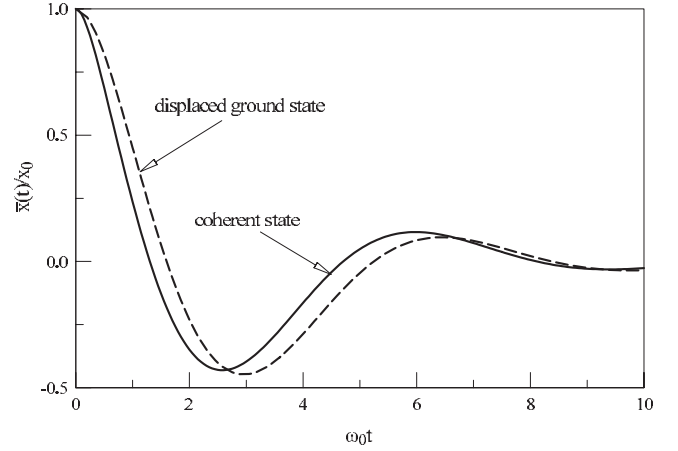


FIG. 2. The motion of the wave packet center for the displaced ground state and for the coherent state, both for initial velocity zero. The displaced ground-state motion is computed at zero temperature. The parameters chosen are  $\gamma/\omega_0 = 10/13$  and  $\Omega/\omega_0 = 5$ .

$$\mathcal{P}(x, ; t) = \mathcal{P}_{\text{eq}}(x - \bar{x}(t)), \quad (6.17)$$

where  $\mathcal{P}_{\text{eq}}$  is the equilibrium distribution (5.16). This is exactly the form we encountered above in the example of wave packet spreading, where the parameters for the displaced ground state are given in Eq. (5.14). In either case the probability distribution is that of a displaced ground state. The difference lies in the motion of the center, which is temperature independent in the case of the coherent state but has a temperature-dependent form for the displaced ground state. In either case, of course, the distribution approaches that of equilibrium for long times. The lesson we learn here is that the time dependence of the approach to equilibrium depends on how the initial state is formed. In Fig. 2 we plot  $\bar{x}(t)$  for the two cases, the displaced ground-state motion being calculated at zero temperature. The parameters chosen were  $\gamma/\omega_0 = 10/13$  and  $\Omega/\omega_0 = 5$ , but despite this rather strong coupling, there is not much difference between the two curves.

### C. Example: Coherent-state pair

The idea here is to form an initial state like the Schrödinger cat state discussed in Sec. V B. There the initial state was prepared with a pair of Gaussian slits. Here we consider instead an initial state which is a superposition of two separated coherent states. This is accomplished with a measurement function of the simple form

$$f(1) = \cos \frac{m d \dot{x}(t_1)}{2\hbar}, \quad (6.18)$$

which results in a superposed pair of generalized coherent states, centered at  $x = \pm d/2$  and each with zero velocity. Expression (6.5) for the Fourier transform of the normalized Wigner function at time  $t_2$  therefore becomes

$$\begin{aligned} \tilde{\mathcal{W}}(Q, P; t_2 - t_1) &= \frac{\langle \cos[md\dot{x}(t_1)/2\hbar] e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} \cos[md\dot{x}(t_1)/2\hbar] \rangle}{\langle \cos^2[md\dot{x}(t_1)/2\hbar] \rangle}. \end{aligned} \quad (6.19)$$

Now, using again the Baker-Campbell-Hausdorff formula (B3), we see that

$$\begin{aligned} &\cos \frac{md\dot{x}(t_1)}{2\hbar} e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} \cos \frac{md\dot{x}(t_1)}{2\hbar} \\ &= \frac{1}{4} \left( 2e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} \cos \frac{md(\dot{G}P + m\ddot{G}Q)}{2\hbar} \right. \\ &\quad \left. + e^{-i[x(t_2)P + m\dot{x}(t_2)Q - md\dot{x}(t_1)]/\hbar} + e^{-i[x(t_2)P + m\dot{x}(t_2)Q + md\dot{x}(t_1)]/\hbar} \right), \end{aligned} \quad (6.20)$$

where, to shorten the expression, we have used expression (A7) to express the commutator in terms of the Green function. Putting this into Eq. (6.19) and using the Gaussian property (B6) we find

$$\begin{aligned} \tilde{\mathcal{W}}(Q, P; t) &= \frac{\exp[-(\langle x^2 \rangle P^2 + m^2 \langle \dot{x}^2 \rangle Q^2)/2\hbar^2]}{1 + \exp(-m^2 \langle \dot{x}^2 \rangle d^2/2\hbar^2)} \\ &\times \left[ \cos \frac{md(\dot{G}P + m\ddot{G}Q)}{2\hbar} + \exp\left(-\frac{m^2 d^2 \langle \dot{x}^2 \rangle}{2\hbar^2}\right) \right. \\ &\quad \left. \times \cosh \frac{md(\dot{s}P + m\ddot{s}Q)}{2\hbar^2} \right], \end{aligned} \quad (6.21)$$

where  $G=G(t)$  is the Green function (A5) and  $s=s(t)$  is the mean-square displacement (A8). Putting this into Eq. (6.4), the integral is a two-dimensional standard Gaussian (B2) and we find

$$\begin{aligned} \mathcal{W}(q, p; t) &= \frac{1}{2[1 + \exp(-m^2 \langle \dot{x}^2 \rangle d^2/2\hbar^2)]} \left[ \mathcal{W}_{\text{eq}}\left(q - \frac{m\dot{G}d}{2}, \right. \right. \\ &\quad \left. \left. p - \frac{m^2\ddot{G}d}{2}\right) + \mathcal{W}_{\text{eq}}\left(q + \frac{m\dot{G}d}{2}, p + \frac{m^2\ddot{G}d}{2}\right) \right. \\ &\quad \left. + 2e^{-A(t)} \mathcal{W}_{\text{eq}}(q, p) \cos \Phi(q, p; t) \right], \end{aligned} \quad (6.22)$$

where  $\mathcal{W}_{\text{eq}}(q, p)$  is the equilibrium Wigner function (6.8) and we have introduced

$$\begin{aligned} A(t) &= \frac{m^2 d^2 \langle \dot{x}^2 \rangle}{2\hbar^2} \left( 1 - \frac{\dot{s}^2(t)}{4\langle x^2 \rangle \langle \dot{x}^2 \rangle} - \frac{\ddot{s}^2(t)}{4\langle \dot{x}^2 \rangle^2} \right), \\ \Phi(q, p; t) &= \left( \frac{m\dot{s}q}{\langle x^2 \rangle} + \frac{\ddot{s}p}{\langle \dot{x}^2 \rangle} \right) \frac{d}{2\hbar}. \end{aligned} \quad (6.23)$$

This expression for the Wigner function is identical with that obtained by Romero and Paz using path integral methods [23].

Viewed in the  $qp$  plane, the expression (6.22) for the Wigner function shows three peaks: an outlying pair in the form of single coherent states, centered initially at  $q = \pm \frac{d}{2}$ ,  $p=0$ , and an interference peak, centered at the origin and modulated by the factor  $\cos \Phi$ . Initially, since  $A(0)=0$ , the amplitude of the interference peak is twice that of either of the two outlying peaks. In general, the interest is in the case of a widely separated coherent-state pair—that is,  $d \gg \sqrt{\langle x^2 \rangle}$ . In that case for very short times  $A(t)$  becomes large and the interference peak is practically zero. This disappearance of the interference peak is the phenomenon of decoherence as seen with the Wigner function. To be more explicit, with the expansion (A26) in the expression (A24) for  $s(t)$ , we can readily evaluate this expression for short times. In the limit of small bath relaxation time this takes the simple form

$$A(t) \cong -\frac{t^2}{2\tau_d^2} \ln \frac{t}{\tau}, \quad (6.24)$$

where

$$\tau_d = \tau \sqrt{\frac{\pi\hbar}{\zeta d^2}}. \quad (6.25)$$

Here  $\zeta$  is the friction constant and  $\tau$  is the bath relaxation time, the parameters in the single relaxation time model. We can interpret  $\tau_d$ , the time of the order of which the central peak in the Wigner distribution vanishes, as a decoherence time. Note that this expression for the decoherence time is very different from that for the displaced ground-state pair given in Eq. (5.27), despite the similarity of the initial states. The qualitative nature of the phenomenon is the same: the rapid disappearance of an interference term.

Since the Wigner function is not directly observable, we should consider the probability distribution. That is, we put the expression (6.22) for the Wigner function into the integral (6.16) for the probability distribution to obtain

$$\begin{aligned} \mathcal{P}(x, t) &= \frac{1}{2[1 + \exp(-m^2 \langle \dot{x}^2 \rangle d^2/2\hbar^2)]} \left[ \mathcal{P}_{\text{eq}}\left(x - m\dot{G}\frac{d}{2}\right) \right. \\ &\quad \left. + \mathcal{P}_{\text{eq}}\left(x + m\dot{G}\frac{d}{2}\right) \right. \\ &\quad \left. + 2a \exp\left(-\frac{m^2\ddot{G}^2 d^2}{8\langle x^2 \rangle}\right) \mathcal{P}_{\text{eq}}(x) \cos \frac{m\dot{s}dx}{2\hbar\langle x^2 \rangle} \right], \end{aligned} \quad (6.26)$$

where  $a=a(t)$  is the attenuation coefficient, now given by

$$a(t) = \exp\left[-\left(\frac{4m^2\langle \dot{x}^2 \rangle \langle x^2 \rangle}{\hbar^2} - \frac{m^2\dot{s}^2(t)}{\hbar^2} - m^2\ddot{G}^2\right) \frac{d^2}{8\langle x^2 \rangle}\right]. \quad (6.27)$$

As in our discussion of the Schrödinger cat state in Sec. V, the attenuation coefficient is defined as the ratio of the coefficient of the cosine term to the geometric mean of the first two terms and corresponds to the traditional measure of coherence. Its disappearance is the phenomenon of decoher-

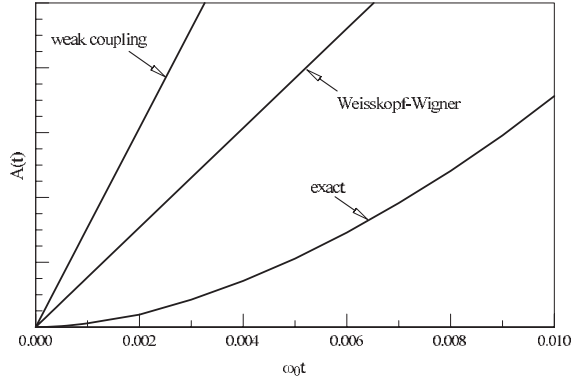


FIG. 3. The function  $A(t)$  for a coherent state pair. The parameters chosen are  $\gamma/\omega_0=10/13$  and  $\Omega/\omega_0=5$ .

ence as seen in the probability distribution. But here the initial value of the attenuation coefficient,

$$a(0) = \exp\left[-\left(\frac{4m^2\langle\dot{x}^2\rangle\langle x^2\rangle}{\hbar^2} - 1\right)\frac{d^2}{8\langle x^2\rangle}\right], \quad (6.28)$$

is already vanishingly small for large separations (the uncertainty principle tells us that the factor in the exponent is necessarily positive). Indeed, for the Ohmic model, the mean-square velocity is logarithmically divergent, so the attenuation coefficient is identically zero for all times. In any event, we would say that as seen in the probability distribution, the decoherence for a coherent state pair occurs initially and there is no notion of a decoherence time.

The earliest discussion of this problem of a displaced pair of coherent states at zero temperature was that of Walls and Milburn [24], who based their discussion on the master equation. As we note in the last paragraph of Appendix A, this would correspond to the Weisskopf-Wigner approximation. In this approximation

$$A(t) = \frac{m\omega_0 d^2}{4\hbar} (1 - e^{-\gamma t}) \quad (6.29)$$

and

$$a(t) = \exp[-A(t)]. \quad (6.30)$$

Note first of all that in this Weisskopf-Wigner approximation the decay of coherence is identical whether viewed in the Wigner function or in the probability distribution. However, the short-time behavior is very different from that of the exact result. As an illustration, in Fig. 3 we compare the short-time behavior of the exact result with that of the Weisskopf-Wigner approximation as well as that of the weak-coupling approximation. In this range the weak-coupling expression for  $A(t)$  is just twice that from the Weisskopf-Wigner approximation, while both are much larger than the exact result and would give a correspondingly much shorter estimate of the decoherence time. The conclusion to be drawn is that the master equation can give misleading results at short times.

#### D. Example: Squeezed state

The squeezed state that appears in the quantum optics literature is obtained by operating on the ground state of the free oscillator with the squeeze operator [25],

$$S = \exp\left(\frac{r}{2}(e^{-i\theta}a^2 - e^{i\theta}a^{\dagger 2})\right), \quad (6.31)$$

where  $r$  and  $\theta$  are real parameters and  $a=(m\omega_0x + ip)/\sqrt{2m\hbar\omega_0}$  is the usual annihilation operator for the free oscillator. Here we consider the so-called ideal squeeze operator, corresponding to  $\theta=0$ , and again replace the canonical momentum with the mechanical momentum. The ideal squeeze operation would therefore correspond to an initial measurement operator of the form

$$f(1) = \exp\left(i\frac{mr}{2\hbar}[x(t_1)\dot{x}(t_1) + \dot{x}(t_1)x(t_1)]\right). \quad (6.32)$$

Since this is a unitary operator, we see that

$$\langle f^\dagger(1)e^{-i[x(t_2)P+m\dot{x}(t_2)Q]/\hbar}f(1)\rangle = \langle e^{-i[X(r,t_2)P+m\dot{X}(r,t_2)Q]/\hbar}\rangle, \quad (6.33)$$

where we have introduced the operator

$$X(r;t_2) = f^\dagger(1)x(t_2)f(1). \quad (6.34)$$

To evaluate this operator, form the derivative with respect to  $r$ ,

$$\begin{aligned} \frac{\partial X(r;t_2)}{\partial r} &= f^\dagger(1)\frac{im}{2\hbar}[x(t_2),x(t_1)\dot{x}(t_1) + \dot{x}(t_1)x(t_1)]f(1) \\ &= -m\dot{G}(t_2-t_1)X(r;t_1) + mG(t_2-t_1)\dot{X}(r;t_1), \end{aligned} \quad (6.35)$$

where we have used relation (A7) to express the commutator in terms of the Green function. Also, we have

$$\frac{\partial \dot{X}(r;t_2)}{\partial r} = -m\ddot{G}(t_2-t_1)X(r;t_1) + m\dot{G}(t_2-t_1)\dot{X}(r;t_1). \quad (6.36)$$

If we set  $t_2=t_1$  and use the fact that  $G(0)=0$  and  $\dot{G}(0)=1/m$ , we find from Eq. (6.35) that

$$X(r;t_1) = x(t_1)e^{-r}, \quad (6.37)$$

while from Eq. (6.36) we find that

$$\dot{X}(r;t_1) = \dot{x}(t_1)e^r. \quad (6.38)$$

Here we should emphasize that we use the single relaxation time model, for which  $\ddot{G}(0)=0$ . For the Ohmic model there would be an extra term. Putting these results into Eqs. (6.35) and (6.36) and integrating, we find

$$X(r;t_2) = x(t_2) - (1 - e^{-r})m\dot{G}x(t_1) + (e^r - 1)mG\dot{x}(t_1),$$

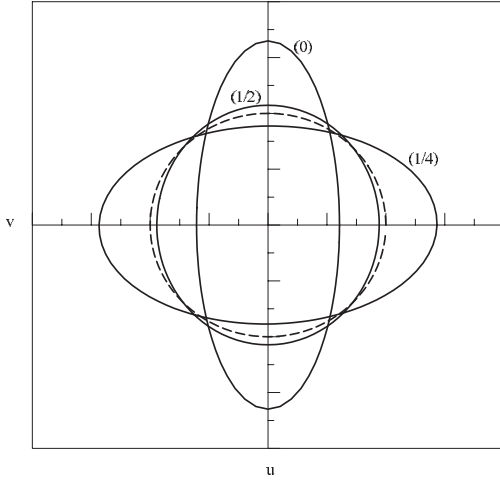


FIG. 4. Constant density contours of the Wigner function for a squeezed state, shown in the plane of the dimensionless variables  $u = q/\sqrt{\langle x^2 \rangle}$  and  $v = p/m\sqrt{\langle \dot{x}^2 \rangle}$ . The dashed circle corresponds to the equilibrium state, the state just before the initial squeeze as well as the state at long times. The contour marked (0) corresponds to the initial squeezed state. The contour marked (1/4) is that corresponding to a quarter period, while that marked (1/2) is that corresponding to a half period. The parameters chosen are  $\gamma/\omega_0 = 10/13$ ,  $\Omega/\omega_0 = 5$ .

$$\dot{X}(r; t_2) = \dot{x}(t_2) - (1 - e^{-r})m\ddot{G}\dot{x}(t_1) + (e^r - 1)m\dot{G}\dot{x}(t_1), \quad (6.39)$$

where  $G = G(t_2 - t_1)$ .

Since  $X(r; t_2)$  and  $\dot{X}(r; t_2)$  are linear in the operators  $x(t_1)$  and  $\dot{x}(t_1)$ , they have the Gaussian property and we can use the identity (B6) to evaluate expression (6.33). With this result the Wigner characteristic function (6.5) can be written in the form

$$\tilde{W}(Q, P; t_2) = \exp\left(-\frac{A_{11}P^2 + 2A_{12}QP + A_{22}Q^2}{2\hbar^2}\right), \quad (6.40)$$

where, for this present example,

$$A_{11} = \langle X^2(r; t_2) \rangle = [1 - (1 - e^{-r})m\dot{G}]^2 \langle x^2 \rangle + (e^r - 1)^2 m^2 G^2 \langle \dot{x}^2 \rangle \\ + (1 - e^{-r})m\dot{G}s + (e^r - 1)mG\dot{s}, \quad (6.41a)$$

$$A_{12} = \frac{m}{2} \langle X(r; t_2) \dot{X}(r; t_2) + \dot{X}(r; t_2) X(r; t_2) \rangle \\ = -[1 - (1 - e^{-r})m\dot{G}](1 - e^{-r})m^2 \dot{G} \langle \dot{x}^2 \rangle \\ + (e^r - 1)^2 m^3 G \dot{G} \langle \dot{x}^2 \rangle + \frac{1}{2}(1 - e^{-r})m^2 (\ddot{G}s + \dot{G}\dot{s}) \\ + \frac{1}{2}(e^r - 1)m^2 (\dot{G}\dot{s} + G\ddot{s}), \quad (6.41b)$$

$$A_{22} = m^2 \langle \dot{X}^2(r; t_2) \rangle \\ = [1 + (e^r - 1)^2 m^2 \dot{G}^2] m^2 \langle \dot{x}^2 \rangle + (1 - e^{-r})^2 m^4 \ddot{G}^2 \langle \dot{x}^2 \rangle \\ + (e^r - 1)m^3 \dot{G}\ddot{s} + (1 - e^{-r})m^3 \ddot{G}\dot{s}. \quad (6.41c)$$

Here we should again recall that  $s = s(t_2 - t_1)$  is the mean-square displacement (A8) and  $G = G(t_2 - t_1)$  is the Green function (A5).

Forming the corresponding Wigner function, we find, for the squeezed state,

$$\mathcal{W}(q, p; t) = \frac{1}{2\pi\sqrt{A_{11}A_{22} - A_{12}^2}} \\ \times \exp\left(-\frac{A_{11}p^2 - 2A_{12}pq + A_{22}q^2}{2(A_{11}A_{22} - A_{12}^2)}\right). \quad (6.42)$$

In Fig. 4 we plot a constant density contour for this function in the plane of the dimensionless variables  $u = q/\sqrt{\langle x^2 \rangle}$  and  $v = p/m\sqrt{\langle \dot{x}^2 \rangle}$ . The dashed circle corresponds to the equilibrium state, the state just before the initial squeeze as well as the state at long times. The contour marked (0) corresponds to the initial squeezed state. In the course of time this contour rotates clockwise. The contour marked (1/4) is that corresponding to a quarter period, while that marked (1/2) is that corresponding to a half period when the squeezing is much reduced. The relatively strong coupling chosen,  $\gamma/\omega_0 = 10/13$ , emphasizes the effect: dissipation leads to a loss of squeezing.

### E. Example: Schrödinger cat state

Here we consider the Wigner function when the initial measurement corresponds to the measurement function (5.17). The calculation goes exactly the same as that beginning with Eq. (5.5) in the previous section, so we shall simply quote the results. For general  $f$  we find

$$\langle f^\dagger(1) e^{-i[x(t_2)P + m\dot{x}(t_2)Q]/\hbar} f(1) \rangle = \exp\left(-\frac{\langle x^2 \rangle P^2 + m^2 \langle \dot{x}^2 \rangle Q^2}{2\hbar^2} + \frac{(cP + m\dot{c}Q)^2}{2\langle x^2 \rangle \hbar^2}\right) \int_{-\infty}^{\infty} dx'_1 f^\dagger\left(x'_1 + \frac{GP + m\dot{G}Q}{2}\right) f\left(x'_1 - \frac{GP + m\dot{G}Q}{2}\right) \\ \times \frac{1}{\sqrt{2\pi\langle x^2 \rangle}} \exp\left(-\frac{(x_1 + x'_1)^2}{2\langle x^2 \rangle} - i(x_1 + x'_1) \frac{cP + m\dot{c}Q}{\langle x^2 \rangle \hbar}\right). \quad (6.43)$$

Here  $c=c(t_2-t_1)$  is the correlation (A9) and  $G=G(t_2-t_1)$  is the Green function (A5). With the form (5.17) for the measurement function, this becomes (remember we must choose  $x_1=0$  if we wish the initial wave packet pair to be placed symmetrically about the origin)

$$\begin{aligned} & \langle f^\dagger(1)e^{-i[x(t_2)P+m\dot{x}(t_2)Q]/\hbar}f(1) \rangle_{x_1=0} \\ &= \frac{\exp[-(A_{11}P^2+2A_{12}PQ+A_{22}Q^2)/2\hbar^2]}{(1+e^{-d^2/8\sigma_1^2})\sqrt{2\pi(\langle x^2 \rangle + \sigma_1^2)}} \\ & \times \left( e^{-d^2/8(\langle x^2 \rangle + \sigma_1^2)} \cos \frac{(cP+m\dot{c}Q)d}{2(\langle x^2 \rangle + \sigma_1^2)\hbar} + e^{-d^2/8\sigma_1^2} \right. \\ & \left. \times \cosh \frac{(GP+m\dot{G}Q)d}{4\sigma_1^2} \right), \end{aligned} \quad (6.44)$$

where in this present example

$$\begin{aligned} A_{11} &= \langle x^2 \rangle + \frac{\hbar^2 G^2}{4\sigma_1^2} - \frac{c^2}{\langle x^2 \rangle + \sigma_1^2}, \\ A_{12} &= m \left( \frac{\hbar^2 G\dot{G}}{4\sigma_1^2} - \frac{c\dot{c}}{\langle x^2 \rangle + \sigma_1^2} \right), \\ A_{22} &= m^2 \left( \langle \dot{x}^2 \rangle + \frac{\hbar^2 \dot{G}^2}{4\sigma_1^2} - \frac{\dot{c}^2}{\langle x^2 \rangle + \sigma_1^2} \right). \end{aligned} \quad (6.45)$$

The Wigner characteristic function is therefore

$$\begin{aligned} \tilde{W}(Q,P;t) &= \frac{\exp[-(A_{11}P^2+2A_{12}PQ+A_{22}Q^2)/2\hbar^2]}{1+\exp[-\langle x^2 \rangle d^2/8\sigma_1^2(\langle x^2 \rangle + \sigma_1^2)]} \\ & \times \left[ \cos \frac{(cP+m\dot{c}Q)d}{2(\langle x^2 \rangle + \sigma_1^2)\hbar} + \exp\left(-\frac{\langle x^2 \rangle d^2}{8\sigma_1^2(\langle x^2 \rangle + \sigma_1^2)}\right) \right. \\ & \left. \times \cosh \frac{(GP+m\dot{G}Q)d}{4\sigma_1^2} \right]. \end{aligned} \quad (6.46)$$

With this form of the Wigner characteristic function there is no difficulty evaluating the inverse transform (6.4) to obtain the corresponding Wigner function. However, the interest will be in the limits of a free particle or, for the oscillator, a displaced ground-state pair, in which case it is simpler to first evaluate the limits of the above expression and then evaluate the inverse transform.

### 1. Free particle

We obtain the case of a free particle coupled to the bath in the absence of an oscillator potential by forming the limit

$\langle x^2 \rangle \rightarrow \infty$ . In forming this limit we should recall that  $c(t) = \langle x^2 \rangle - s(t)/2$ , where the mean-square displacement  $s(t)$  is finite in the free particle limit. We find

$$\begin{aligned} \tilde{W}(Q,P;t) &= \frac{\exp[-(A_{11}P^2+2A_{12}PQ+A_{22}Q^2)/2\hbar^2]}{1+\exp\left(-\frac{d^2}{8\sigma_1^2}\right)} \left[ \cos \frac{Pd}{2\hbar} \right. \\ & \left. + \exp\left(-\frac{d^2}{8\sigma_1^2}\right) \cosh \frac{(GP+m\dot{G}Q)d}{4\sigma_1^2} \right]. \end{aligned} \quad (6.47)$$

In this free particle case, expressions (6.45) become

$$\begin{aligned} A_{11} &= \sigma_1^2 + s + \frac{\hbar^2 G^2}{4\sigma_1^2}, \\ A_{12} &= m \left( \frac{\dot{s}}{2} + \frac{\hbar^2 G\dot{G}}{4\sigma_1^2} \right), \\ A_{22} &= m^2 \left( \langle \dot{x}^2 \rangle + \frac{\hbar^2 \dot{G}^2}{4\sigma_1^2} \right). \end{aligned} \quad (6.48)$$

Forming the inverse Fourier transform (6.4) we obtain

$$\begin{aligned} \mathcal{W}(q,p;t) &= \frac{1}{2(1+e^{-d^2/8\sigma_1^2})} \left[ \mathcal{W}_0\left(q-\frac{d}{2},p;t\right) \right. \\ & \left. + \mathcal{W}_0\left(q+\frac{d}{2},p;t\right) \right. \\ & \left. + 2 \exp[-A(t)] \mathcal{W}_0(q,p;t) \cos \Phi(q,p;t) \right], \end{aligned} \quad (6.49)$$

where  $\mathcal{W}_0$  is the Wigner function corresponding an initial measurement forming a single wave packet at the origin,

$$\begin{aligned} \mathcal{W}_0(q,p;t_2) &= \frac{\exp[-(A_{22}q^2-2A_{12}qp+A_{11}p^2)/2(A_{11}A_{22}-A_{12}^2)]}{2\pi\sqrt{A_{11}A_{22}-A_{12}^2}}. \end{aligned} \quad (6.50)$$

In this example the phase  $\Phi$  is given by

$$\Phi(q,p;t) = \frac{(GA_{22}-m\dot{G}A_{12})q + (m\dot{G}A_{11}-GA_{12})p}{A_{11}A_{22}-A_{12}^2} \frac{\hbar d}{4\sigma_1^2} \quad (6.51)$$

and the quantity  $A$  by

$$A(t) = \frac{(A_{11}-\hbar^2 G^2/4\sigma_1^2)(A_{22}-\hbar^2 m^2 \dot{G}^2/4\sigma_1^2) - (A_{12}-\hbar^2 m G \dot{G}/4\sigma_1^2)^2}{A_{11}A_{22}-A_{12}^2} \frac{d^2}{8\sigma_1^2}. \quad (6.52)$$

As in the case of the coherent state pair, The Wigner function for the free particle Schrödinger cat state shows three peaks, an outlying pair centered at  $q = \pm d/2, p = 0$  and an interference peak centered at the origin. However, in this free particle case,

$$A(0) = \frac{\sigma_1^2 \langle \dot{x}^2 \rangle}{\sigma_1^2 \langle \dot{x}^2 \rangle + \hbar^2/4m^2} \frac{d^2}{8\sigma_1^2} = \frac{d^2}{8\sigma_1^2 + 2\bar{\lambda}^2} > 0, \quad (6.53)$$

where  $\bar{\lambda}$  is the mean de Broglie wavelength in equilibrium,

$$\bar{\lambda} = \frac{\hbar}{m\sqrt{\langle \dot{x}^2 \rangle}}. \quad (6.54)$$

Therefore, the amplitude of the interference peak will initially be vanishingly small whenever the separation  $d$  of the wave packets is large compared with both the slit width  $\sigma_1$  and the mean de Broglie wavelength  $\bar{\lambda}$ .

## 2. Displaced ground-state pair

Again we form the limit  $\sigma_1^2 \rightarrow \infty$  and  $d \rightarrow \infty$  such that  $d_0 = \langle \dot{x}^2 \rangle d / \sigma_1^2$  is fixed. The coefficients (6.45) become

$$A_{11} = \langle \dot{x}^2 \rangle, \quad A_{12} = 0, \quad A_{22} = m^2 \langle \dot{x}^2 \rangle. \quad (6.55)$$

The Wigner characteristic function (6.46) then becomes

$$\begin{aligned} \tilde{W}(Q, P; t) &= \frac{\exp[-(\langle \dot{x}^2 \rangle P^2 + m^2 \langle \dot{x}^2 \rangle Q^2) / 2\hbar^2]}{1 + \exp(-d_0^2 / 8\langle \dot{x}^2 \rangle)} \\ &\times \left( \cos \frac{(cP + m\dot{G}Q)d_0}{2\langle \dot{x}^2 \rangle \hbar} + \exp(-d_0^2 / 8\langle \dot{x}^2 \rangle) \right. \\ &\left. \times \cosh \frac{(GP + m\dot{G}Q)d_0}{4\langle \dot{x}^2 \rangle} \right). \end{aligned} \quad (6.56)$$

With this, the Wigner function is

$$\begin{aligned} \mathcal{W}(q, p; t) &= \frac{1}{2[1 + \exp(-d_0^2 / 8\langle \dot{x}^2 \rangle)]} \left[ \mathcal{W}_{\text{eq}} \left( q - \frac{cd_0}{2\langle \dot{x}^2 \rangle}, \right. \right. \\ &\quad \left. \left. p - \frac{m\dot{G}d_0}{2\langle \dot{x}^2 \rangle} \right) + \mathcal{W}_{\text{eq}} \left( q + \frac{cd_0}{2\langle \dot{x}^2 \rangle}, p + \frac{m\dot{G}d_0}{2\langle \dot{x}^2 \rangle} \right) \right. \\ &\quad \left. + 2e^{-A} \mathcal{W}_{\text{eq}}(q, p) \cos \Phi \right], \end{aligned} \quad (6.57)$$

where  $\mathcal{W}_{\text{eq}}(q, p)$  is the equilibrium Wigner function (6.8) and

$$\begin{aligned} \Phi(q, p; t) &= \left( \frac{\hbar G}{\langle \dot{x}^2 \rangle} q + \frac{\hbar \dot{G}}{m\langle \dot{x}^2 \rangle} p \right) \frac{d_0}{4\langle \dot{x}^2 \rangle}, \\ A(t) &= \frac{d_0^2}{8\langle \dot{x}^2 \rangle} \left( 1 - \frac{\hbar^2 G^2}{4\langle \dot{x}^2 \rangle^2} - \frac{\hbar^2 \dot{G}^2}{4\langle \dot{x}^2 \rangle \langle \dot{x}^2 \rangle} \right). \end{aligned} \quad (6.58)$$

Again, as in the free particle case the initial value of  $A$  is not zero,

$$A(0) = \frac{d_0^2}{8\langle \dot{x}^2 \rangle} \left( 1 - \frac{\hbar^2}{4m^2 \langle \dot{x}^2 \rangle \langle \dot{x}^2 \rangle} \right) > 0, \quad (6.59)$$

and the interference term is vanishingly small.

This situation, in which the initial state is a Schrödinger cat state formed by passing the particle through a pair of Gaussian slits, is to be contrasted with that described in Sec. VI C, in which the initial state is prepared by displacing the equilibrium state to form a coherent state pair. The Wigner functions, given by Eqs. (6.57) and (6.22), respectively, are identical in form, but in the Schrödinger cat case the interference peak is initially vanishingly small and remains so for all time, while in that of the coherent state pair the interference peak is initially twice as high as the outlying peaks, becoming vanishingly small only after a short relaxation time. The reverse is true for the probability distributions, given in Eqs. (5.24) and (6.26), respectively. That is, the interference term in the probability distribution is vanishingly small at all times for the coherent state pair, while for the Schrödinger cat state the attenuation coefficient multiplying the interference term vanishes only after a short decoherence time. We must conclude that the notion of decoherence time is arbitrary, depending on the situation and how one chooses to view it.

## VII. CONCLUDING REMARKS

The quantum probability distributions are *measured* distributions. That is, they depend explicitly on the parameters of the measurements. This is seen clearly in the general formula (3.29) for the  $n$ -point distribution, where  $f(j)$  is the measurement function for the  $j$ th measurement. This is also seen in the case of quantum Brownian motion in expression (4.2) for the characteristic function. There it is seen that the characteristic function can be factored, with a factor  $K(1, \dots, n)$  multiplying a quantum expectation independent of measurement. The factor  $K(1, \dots, n)$  contains the measurement parameters and depends upon the dynamics through the non-equal-time commutator. In the classical limit, where this commutator vanishes, this factor becomes a numerical factor independent of the dynamics that can in practice be taken to be unity. The result is the familiar expression for the classical characteristic function [15].

An important part of this work has been to demonstrate that the general expression (4.2) for the characteristic function can be very useful for practical calculations. The use of this formula together with its specializations (5.4) to the probability distribution and Eq. (6.5) to the Wigner characteristic function has been illustrated with a number of examples, each of which is an important application. Among the results we point out expression (5.13) for wave packet spreading in the presence of dissipation, a generalization of the well-known expression found in elementary textbooks. Another result is illustrated in Fig. 4 where the disappearance of squeezing in the presence of dissipation is illustrated. Finally, a comparison of a ‘‘Schrödinger cat’’ state formed by passing the particle through a pair of Gaussian slits with the nearly identical state formed with a coherent state pair shows that a quantitative measure of decoherence, depends on how the state is formed.

In Sec. VI we discuss the Wigner function. Some authors prefer instead the density matrix element in the coordinate representation, which is given by a kind of half-Fourier transform



$$\langle x|\rho(t)|x'\rangle = \int_{-\infty}^{\infty} dp \mathcal{W}\left(\frac{x+x'}{2}, p; t\right) e^{i(x-x')p/\hbar}. \quad (7.1)$$

Clearly, the Wigner function and the density matrix element contain the same information. Indeed, it is not difficult to see that the central interference peak in, say, the Wigner function (6.49) corresponding to a coherent-state pair becomes a pair of off-diagonal peaks in the density matrix element. We prefer the language of the Wigner function since it is always a real function that in the classical limit becomes a real probability distribution.

The examples are all in one dimension. The reader should be aware that this is not a necessary restriction, but has been made to keep the discussion within bounds. The generalization to higher dimensions is straightforward. All that one must keep in mind is that the fluctuating force operator is independent in the different directions and that the effects of dissipation (the fluctuating force and the memory force) are independent of the applied force [20]. Finally, we have restricted the discussion to the single relaxation time model of dissipation and its limiting Ohmic case. Again this has been done to keep the discussion within bounds. There is no problem with the discussion for more general models such as the coupling to the blackbody radiation field [20].

## APPENDIX A: QUANTUM BROWNIAN MOTION

### 1. Quantum Langevin equation

Quantum Brownian motion for an oscillator coupled to a heat bath at temperature  $T$  is described by the quantum Langevin equation

$$m\ddot{x} + \int_{-\infty}^t dt' \mu(t-t')\dot{x}(t') + Kx = F(t). \quad (A1)$$

This is a Heisenberg equation for the position operator  $x(t)$ . On the right-hand side  $F(t)$  is a Gaussian random operator force, with mean zero,  $\langle F(t) \rangle = 0$ , and with autocorrelation and commutator given by

$$\begin{aligned} & \frac{1}{2} \langle F(t)F(t') + F(t')F(t) \rangle \\ &= \frac{\hbar}{\pi} \int_0^{\infty} d\omega \operatorname{Re}[\tilde{\mu}(\omega + i0^+)] \omega \coth \frac{\hbar\omega}{2kT} \cos \omega(t-t'), \\ [F(t), F(t')] &= \frac{\hbar}{i\pi} \int_0^{\infty} d\omega \operatorname{Re}[\tilde{\mu}(\omega + i0^+)] \omega \sin \omega(t-t'). \end{aligned} \quad (A2)$$

In these expressions  $\tilde{\mu}$  is the Fourier transform of the memory function,

$$\tilde{\mu}(z) = \int_0^{\infty} dt \mu(t) e^{izt}, \quad \operatorname{Im}\{z\} > 0. \quad (A3)$$

It is a consequence of the second law of thermodynamics that  $\tilde{\mu}(z)$  must be what is called a positive real function:

analytic with real part positive everywhere in the upper half plane.

In our present discussion we take the view that the above is a macroscopic description, which is complete as it stands. For a thorough discussion, including the derivation from a number of microscopic models, we refer to a paper of Ford, Lewis, and O'Connell [20].

The solution of the quantum Langevin equation (A1) can be written

$$x(t) = \int_{-\infty}^t dt' G(t-t') F(t'), \quad (A4)$$

where the Green function  $G(t)$  is given by

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \alpha(\omega + i0^+) e^{-i\omega t}, \quad (A5)$$

in which  $\alpha(z)$ , the response function, is given by

$$\alpha(z) = \frac{1}{-mz^2 - iz\tilde{\mu}(z) + K}. \quad (A6)$$

With this solution, we can obtain the following expressions for the commutator:

$$\begin{aligned} [x(t), x(t')] &= \frac{2\hbar}{i\pi} \int_0^{\infty} d\omega \operatorname{Im}[\alpha(\omega + i0^+)] \sin \omega(t-t') \\ &= i\hbar[G(t'-t) - G(t-t')]. \end{aligned} \quad (A7)$$

The Green function vanishes for negative times, while  $G(0)=0$  and  $\dot{G}(0)=1/m$ . We see, therefore, that the canonical commutator  $[x, p]=i\hbar$  holds with  $p=m\dot{x}$ , the mechanical momentum. (The canonical momentum may or may not be the same as the mechanical momentum, depending on the form of the microscopic Hamiltonian.)

Also of interest is the mean-square displacement

$$s(t-t') \equiv \langle [x(t) - x(t')]^2 \rangle = 2\langle x^2 \rangle - 2c(t-t'). \quad (A8)$$

Here  $c(t)$  is the correlation

$$c(t) = \frac{1}{2} \langle x(t)x(0) + x(0)x(t) \rangle. \quad (A9)$$

Using the solution (A4) or, more directly without recourse to the Langevin equation, the fluctuation-dissipation theorem of Callen and Welton [26], we obtain the following expression for the correlation:

$$c(t) = \frac{\hbar}{\pi} \int_0^{\infty} d\omega \operatorname{Im}[\alpha(\omega + i0^+)] \coth \frac{\hbar\omega}{2kT} \cos \omega t. \quad (A10)$$

With this, we also have

$$s(t) = \frac{2\hbar}{\pi} \int_0^{\infty} d\omega \operatorname{Im}[\alpha(\omega + i0^+)] \coth \frac{\hbar\omega}{2kT} (1 - \cos \omega t). \quad (A11)$$

Here we should note first of all that for very long times the correlation vanishes and

$$s(t) \sim 2\langle x^2 \rangle. \quad (\text{A12})$$

The exception is the free particle case, where  $K=0$  and consequently  $\langle x^2 \rangle = \infty$ . In that case  $s(t)$  grows for long times without limit, with a time dependence that depends on the model as well as the temperature [27,28]. On the other hand, for very short times we can expand the cosine to obtain the general result

$$s(t) \cong \langle \dot{x}^2 \rangle t^2. \quad (\text{A13})$$

## 2. Explicit expressions

Here we obtain explicit, closed-form expressions for the Green function  $G(t)$  and the mean-square displacement  $s(t)$ . For this purpose, the model of choice for most applications, due to its simplicity, is the Ohmic model, for which  $\tilde{\mu}(z) = \zeta$ , the friction constant. While it is adequate for most purposes, this Ohmic model is singular, particularly at short times or high frequencies, so we here consider a more general model in which the higher frequencies are suppressed. The simplest of these is the single relaxation time model, for which

$$\tilde{\mu}(z) = \frac{\zeta}{1 - iz\tau}. \quad (\text{A14})$$

Here  $\tau$  is the relaxation time (at times called the bath correlation time) which we assume is small in the sense that  $\zeta\tau/m \ll 1$ . Putting this form of  $\tilde{\mu}$  in the response function (A6) and replacing the parameters  $\tau$ ,  $\zeta$ , and  $K$  with parameters  $\Omega$ ,  $\gamma$ , and  $\omega_0$  through the relations

$$\tau = \frac{1}{\Omega + \gamma}, \quad \zeta = m\gamma \frac{\Omega(\Omega + \gamma) + \omega_0^2}{(\Omega + \gamma)^2}, \quad K = m\omega_0^2 \frac{\Omega}{\Omega + \gamma}, \quad (\text{A15})$$

we obtain the convenient form

$$\alpha(z) = \frac{z + i(\Omega + \gamma)}{m(z + i\Omega)(-z^2 - i\gamma z + \omega_0^2)}. \quad (\text{A16})$$

Note that  $-i\Omega$  is the pole of the response function far down on the negative imaginary axis. For small relaxation time  $\Omega \sim 1/\tau$  we have the expansion

$$\Omega = \frac{1}{\tau} - \frac{\zeta}{m} - \frac{\zeta^2\tau}{m^2} + \left( \frac{K\zeta}{m^2} - 2\frac{\zeta^3}{m^3} \right) \tau^2 + \dots \quad (\text{A17})$$

The Ohmic model, for which  $\tau \rightarrow 0$ , therefore corresponds to  $\Omega \rightarrow \infty$ , while  $\gamma \rightarrow \zeta/m$  and  $\omega_0^2 \rightarrow K/m$ .

### a. Green function

With the form (A16) of the response function, we evaluate the integral in expression (A5) for the Green function by deforming the path of integration into the lower half plane, picking up the residues at the poles of the response function. The result is

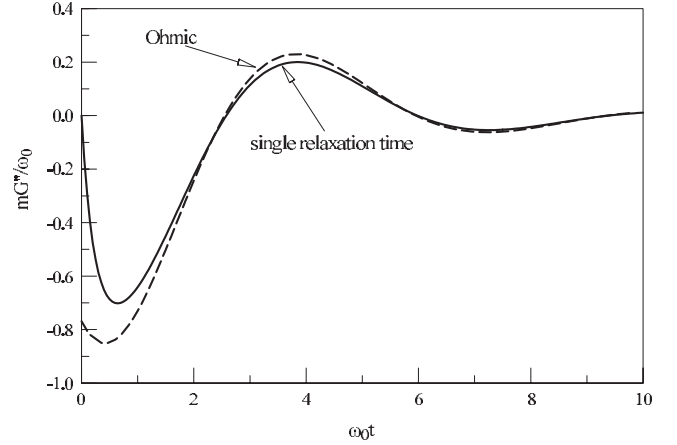


FIG. 5. Second derivative of the Green function for the oscillator. Parameters for the single relaxation time model are  $\gamma/\omega_0 = 10/13$  and  $\Omega/\omega_0 = 5$ .

$$G(t) = \frac{\gamma}{m(\Omega^2 - \gamma\Omega + \omega_0^2)} (e^{-\Omega t} - e^{-\gamma t/2} \cos \omega_1 t) + \frac{\Omega^2 + \omega_0^2 - \gamma^2/2}{\Omega^2 - \gamma\Omega + \omega_0^2} e^{-\gamma t/2} \frac{\sin \omega_1 t}{m\omega_1}, \quad (\text{A18})$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4}. \quad (\text{A19})$$

In the limit,  $\Omega \rightarrow \infty$ , this becomes the familiar Ohmic Green function

$$G_{\text{Ohmic}}(t) = e^{-\gamma t/2} \frac{\sin \omega_1 t}{m\omega_1}. \quad (\text{A20})$$

For reasonable choices of the parameters the Ohmic Green function is very little different from that for the single relaxation time model. The difference between the two models is more apparent in the second derivative of the Green function, which we show in Fig. 5 for the two models. The difference is small except for short times, where  $\ddot{G}(0)$  vanishes, while  $\ddot{G}_{\text{Ohmic}}(0) = -\zeta/m^2$  is finite.

### b. Mean-square displacement at high temperature

First we consider the mean square displacement in the high-temperature limit, in which in expression (A10) we replace the hyperbolic cotangent by the reciprocal of its argument. If we compare the resulting expression with the expression (A5) for the Green function, we see that in this high-temperature limit,

$$s(t) = 2kT \int_0^t dt' G(t'). \quad (\text{A21})$$

The long- and short-time behaviors of the mean-square displacement are of the general form given in Eqs. (A12) and (A13). Note that in this high-temperature limit  $\langle x^2 \rangle$  and  $\langle \dot{x}^2 \rangle$  are given by the classical equipartition formulas,

$$\langle x^2 \rangle = \frac{kT}{K}, \quad \langle \dot{x}^2 \rangle = \frac{kT}{m}. \quad (\text{A22})$$

### c. Mean-square displacement at zero temperature

We consider now the effect of zero-point oscillations on the mean-square displacement. At temperature zero, we replace the hyperbolic cotangent in expression (A10) by unity. Then, if we write

$$\begin{aligned} \text{Im}[\alpha(\omega + i0^+)] = & -\frac{\gamma}{m(\Omega^2 - \gamma\Omega + \omega_0^2)} \frac{\Omega^2}{\omega(\omega^2 + \Omega^2)} \\ & + \text{Im} \left( \frac{\Omega^2 - (\gamma/2 - i\omega_1)^2}{m\omega_1(\Omega^2 - \gamma\Omega + \omega_0^2)} \right. \\ & \left. \times \frac{(\gamma/2 + i\omega_1)^2}{\omega[\omega^2 + (\gamma/2 + i\omega_1)^2]} \right), \end{aligned} \quad (\text{A23})$$

we can write expression (A11) for the mean-square displacement in the form

$$\begin{aligned} s(t) = & \frac{2\hbar}{m\pi} \left\{ -\frac{\gamma}{\Omega^2 - \gamma\Omega + \omega_0^2} V(\Omega t) \right. \\ & \left. + \text{Im} \left[ \frac{\Omega^2 - (\gamma/2 - i\omega_1)^2}{\omega_1(\Omega^2 - \gamma\Omega + \omega_0^2)} V\left(\frac{\gamma t}{2} + i\omega_1 t\right) \right] \right\}. \end{aligned} \quad (\text{A24})$$

Here we have introduced the function

$$V(z) \equiv \int_0^\infty dy \frac{1 - \cos zy}{y(1+y^2)} = \ln z + \gamma_E - \frac{1}{2} [e^{-z} \text{Ei}(z) + e^z \text{Ei}(-z)], \quad (\text{A25})$$

where  $\gamma_E = 0.577\,215\,665$  is Euler's constant and Ei is the exponential integral [29]. Using the expansion of the exponential integral for small argument, we obtain the expansion

$$\begin{aligned} V(z) = & -(\ln z + \gamma_E)(\cosh z - 1) \\ & - \frac{1}{2} \left( e^{-z} \sum_{n=1}^\infty \frac{z^n}{n!n} + e^z \sum_{n=1}^\infty \frac{(-z)^n}{n!n} \right). \end{aligned} \quad (\text{A26})$$

We see from this that, for small  $z$ ,

$$V(z) \equiv -\frac{1}{2} z^2 \left( \ln z + \gamma_E - \frac{3}{2} \right) - \frac{1}{24} z^4 \left( \ln z + \gamma_E - \frac{25}{12} \right) + \dots \quad (\text{A27})$$

Note that  $V(0)=0$ , in agreement with the definition (A25). For large  $z$ , using the asymptotic formulas for the exponential integral, we obtain the asymptotic expansion

$$V(z) \sim \ln z + \gamma_E - \frac{1}{z^2} - \frac{3!}{z^4} - \frac{5!}{z^6} - \dots \quad (\text{A28})$$

With these results, we see that for very short times ( $t \ll \tau$ ) the mean-square displacement again takes the form (A13) but now with the mean-square velocity, given by

$$\begin{aligned} \langle \dot{x}^2 \rangle = & \hbar \frac{[\Omega^2(\omega_0^2 - \gamma^2/2) + \omega_0^4] \arccos(\gamma/2\omega_0) + \gamma\omega_1\Omega^2 \ln(\Omega/\omega_0)}{\pi m \omega_1 (\Omega^2 - \gamma\Omega + \omega_0^2)} \\ \equiv & \frac{\hbar}{\pi m} \left( -\gamma \ln \omega_0 \tau + \frac{\omega_0^2 - \gamma^2/2}{\omega_1} \arccos \frac{\gamma}{2\omega_0} \right), \end{aligned} \quad (\text{A29})$$

where the second form is that for small relaxation time,  $\Omega \rightarrow 1/\tau$ . In the Ohmic limit this is logarithmically divergent, so we have here a case where the single relaxation time makes a difference. On the other hand,

$$\begin{aligned} \langle x^2 \rangle = & \hbar \frac{(\Omega^2 + \omega_0^2 - \gamma^2/2) \arccos(\gamma/2\omega_0) - \gamma\omega_1 \ln(\Omega/\omega_0)}{\pi m \omega_1 (\Omega^2 - \gamma\Omega + \omega_0^2)} \\ \equiv & \frac{\hbar}{\pi m \omega_1} \arccos \frac{\gamma}{2\omega_0}. \end{aligned} \quad (\text{A30})$$

Thus the Ohmic limit of  $\langle x^2 \rangle$  is finite.

### d. Free particle

The free particle corresponds to the absence of the oscillator potential—that is, to the limit  $K \rightarrow 0$ . For the Green function we can obtain this limit by setting  $\omega_0 = 0$  and  $\omega_1 = i\gamma/2$  in expression (A18). This gives

$$G(t) = \frac{\Omega^2(1 - e^{-\gamma t}) - \gamma^2(1 - e^{-\Omega t})}{\zeta(\Omega^2 - \gamma^2)}. \quad (\text{A31})$$

In this free particle case the parameters  $\Omega$  and  $\gamma$  are given by the relations (A15) with  $K=0$ . These can then be inverted to give

$$\Omega = \frac{1 + \sqrt{1 - 4\zeta\tau/m}}{2\tau}, \quad \gamma = \frac{1 - \sqrt{1 - 4\zeta\tau/m}}{2\tau}. \quad (\text{A32})$$

With this expression for the Green function, the high-temperature form (A21) of the mean-square displacement becomes

$$\begin{aligned} s(t) = & \frac{2kT}{\zeta} \left( t - \frac{\Omega^3(1 - e^{-\gamma t}) - \gamma^3(1 - e^{-\Omega t})}{\gamma\Omega(\Omega^2 - \gamma^2)} \right) \\ \equiv & \frac{2kT}{\zeta} \left( t - \frac{1 - e^{-\gamma t}}{\gamma} \right). \end{aligned} \quad (\text{A33})$$

Note that at long time this increases linearly with time, consistent with Eq. (A12) in the sense that  $\langle x^2 \rangle = \infty$  for the free particle. On the other hand, for short times, we get exactly the short-time result (A13) with  $\langle \dot{x}^2 \rangle$  given by the equipartition form (A22). In other words, the short-time behavior of the oscillator is that of the free particle.

At zero temperature the result (A24) becomes, for the free particle,

$$s(t) = \frac{2\hbar \Omega^2 V(\gamma t) - \gamma^2 V(\Omega t)}{\pi \zeta (\Omega^2 - \gamma^2)}. \quad (\text{A34})$$

At very short times ( $t \ll \tau$ ) this takes the form (A13) with now

$$\langle \dot{x}^2 \rangle = \frac{\hbar \gamma \Omega}{\pi m (\Omega - \gamma)} \ln \frac{\Omega}{\gamma} \cong -\frac{\hbar \gamma}{\pi m} \ln \gamma \tau. \quad (\text{A35})$$

At very long times ( $t \gg \gamma^{-1}$ ), we find

$$\begin{aligned} s(t) &\sim \frac{2\hbar}{\pi \zeta} \left( \frac{\Omega + \gamma}{\Omega} (\ln \gamma t + \gamma_E) - \frac{\gamma^2}{\Omega(\Omega - \gamma)} \ln \frac{\Omega}{\gamma} \right) \\ &\cong \frac{2\hbar}{\pi \zeta} \ln \zeta t / m. \end{aligned} \quad (\text{A36})$$

### 3. Weak coupling

The coupling to the heat bath is measured by the function  $\tilde{\mu}(z)$ . If this is small, the response function will be sharply peaked about  $\omega_0 = \sqrt{K/m}$ , the natural frequency of the oscillator. In the integral expression (A5) for the Green function, one is therefore led to make the replacement  $\tilde{\mu}(\omega) \rightarrow \tilde{\mu}(\omega_0)$ . The result is an expression of the Ohmic form (A20) with

$$\gamma = \frac{1}{m} \text{Re}[\tilde{\mu}(\omega_0)]. \quad (\text{A37})$$

(The imaginary part gives a negligible contribution to  $\omega_0$ .) Next, in the integral expression (A11) for the mean-square displacement, we make the same approximations with, in addition the replacement  $\hbar \omega \coth \frac{\hbar \omega}{2kT} \rightarrow \hbar \omega_0 \coth \frac{\hbar \omega_0}{2kT}$ , to obtain

$$\begin{aligned} s(t) &\cong 2m \langle \dot{x}^2 \rangle \int_0^t dt' G(t') \\ &= 2 \langle x^2 \rangle \left[ 1 - e^{-\gamma t / 2} \left( \cos \omega_1 t + \frac{\gamma}{2\omega_1} \sin \omega_1 t \right) \right], \end{aligned} \quad (\text{A38})$$

where

$$\langle x^2 \rangle \cong \omega_0^2 \langle x^2 \rangle \cong \frac{\hbar \omega_0}{2m} \coth \frac{\hbar \omega_0}{2kT}. \quad (\text{A39})$$

This constitutes the weak-coupling approximation [30]. We should emphasize that this weak-coupling approximation is not valid for the free particle, as should be clear from the above argument. Note, incidentally, that for the Ohmic model this approximation is exact in the high-temperature limit. The usual statement is that it is valid in the limit  $\gamma \ll \omega_0$ , but consideration of the exact results given above tells us that even in this limit the approximation is not correct for very short times ( $\omega_0 t \ll 1$ ) or for very long times ( $\gamma t \gg 1$ ). However, for all other times, the weak-coupling approximation is very good for surprisingly large values of the coupling. As an illustration in Fig. 6 we compare the mean-square displacement at zero temperature as calculated first with the exact formula (A24) and then with the weak-coupling approximation. The parameters  $\Omega/\omega_0=5$  and  $\gamma/\omega_0=10/13$  were chosen to exaggerate the difference. What we see is that the weak-coupling approximation is surprisingly good, even for rather strong coupling.

If one makes the further approximation of neglecting quantities of relative order  $\gamma/\omega_0$ , one gets what at times is

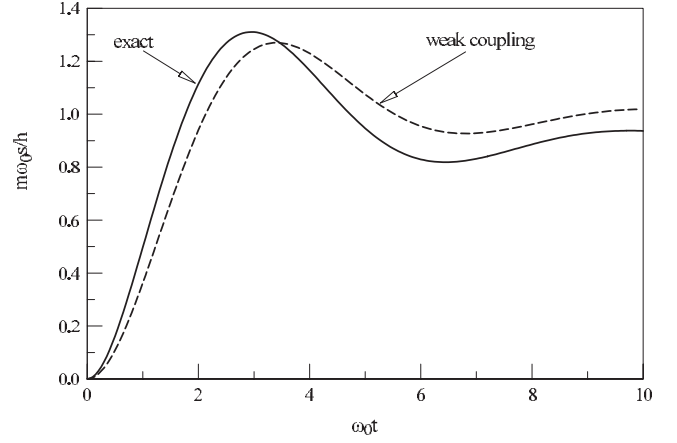


FIG. 6. Comparison of the exact and weak-coupling expressions for the mean-square displacement at zero temperature for the oscillator. The parameters chosen are  $\gamma/\omega_0=10/13$  and  $\Omega/\omega_0=5$ .

called the Weisskopf-Wigner approximation. In Eq. (A38) it would correspond to replacing  $\omega_1 \rightarrow \omega_0$  and dropping the second term after the exponential. This then would correspond exactly to what one obtains by solving the well-known weak-coupling master equation [31]. Because of this, in the literature the Weisskopf-Wigner approximation is often called the weak-coupling approximation. The difference between the weak-coupling approximations we have defined it and the Weisskopf-Wigner approximation is illustrated dramatically in Fig. 3.

## APPENDIX B: MATHEMATICAL FORMULAS

Here we collect some formulas used in the evaluation of the various examples. These formulas are all simple and more or less well known. The first is the standard Gaussian integral

$$\int_{-\infty}^{\infty} du \exp\left(-\frac{1}{2}au^2 + bu\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right). \quad (\text{B1})$$

The generalization to  $d$  dimensions takes the form

$$\begin{aligned} &\int d\mathbf{u} \exp\left(-\frac{1}{2}\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u} + \mathbf{B} \cdot \mathbf{u}\right) \\ &= \frac{(2\pi)^{d/2}}{\sqrt{\det \mathbf{A}}} \exp\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{A}^{-1} \cdot \mathbf{B}\right). \end{aligned} \quad (\text{B2})$$

Here we have used dyadic notation, with  $\mathbf{A}$  a positive-definite symmetric matrix and  $\mathbf{B}$  a vector (not necessarily real) in  $d$  dimensions. This generalization follows from the standard integral, using the fact that a symmetric matrix can be diagonalized by an orthogonal transformation.

The second formula is the Baker-Campbell-Hausdorff formula. If  $A$  and  $B$  are a pair of operators (not necessarily Hermitian) whose commutator is a  $c$  number, then

$$e^A e^B = e^{A+B} e^{[A,B]/2} = e^B e^A e^{[A,B]}. \quad (\text{B3})$$

This formula is easily checked by expanding the exponentials in powers of their argument. A generalization of this theorem is the convenient formula

$$e^A g(B) = g(B + [A,B]) e^A, \quad (\text{B4})$$

which holds for a general function  $g(B)$ . Again, this can be verified by expanding  $g$  in powers of its argument.

Finally, we have a couple of formulas based on the notion of a Gaussian variable. In general a set of operators (each with mean zero) is Gaussian if the expectation of a product of an odd number of the operators is zero while the product of an even number is equal to the sum of products of pair expectations, the sum being over all  $(2n-1)!!$  pairings with the order within the pairs preserved. A Gaussian variable—e.g.,  $x(t)$ —is such a set with the members labeled with the time. Thus, for example, with an obvious shorthand,

$$\langle 1234 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle. \quad (\text{B5})$$

Note that within each pair the order is the same as the original order. A straightforward consequence of this Gaussian property is that for a Gaussian operator  $\mathcal{O}$ , we have the formula

$$\langle e^{i\mathcal{O}} \rangle = e^{-\langle \mathcal{O}^2 \rangle / 2}. \quad (\text{B6})$$

This is easily verified by expanding the exponentials. Another convenient result is

$$\begin{aligned} & \left\langle \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mathcal{O}-a)^2}{2\sigma^2}\right) \right\rangle \\ &= \frac{1}{\sqrt{2\pi(\sigma^2 + \langle \mathcal{O}^2 \rangle)}} \exp\left(-\frac{a^2}{2(\sigma^2 + \langle \mathcal{O}^2 \rangle)}\right). \quad (\text{B7}) \end{aligned}$$

To obtain this result, form the Fourier transform with respect to the parameter  $a$  using the standard Gaussian integral (B1) and then form the expectation using (B6).

- 
- [1] G. W. Ford and J. T. Lewis, *Adv. Math. Suppl. Stud.* **9**, 169 (1986).
- [2] M. C. Wang and G. E. Uhlenbeck, *Rev. Mod. Phys.* **17**, 323 (1945).
- [3] N. Wax, *Selected Papers on Noise and Stochastic Processes* (Dover, New York, 1954).
- [4] A. N. Kolmogorov, *Foundations of the Theory of Probability* (Chelsea, New York, 1956).
- [5] P. A. M. Dirac, *Quantum Mechanics*, 3rd ed. (Oxford University Press, London, 1947).
- [6] U. Fano, *Rev. Mod. Phys.* **29**, 74 (1957).
- [7] W. Pauli, *General Principles of Quantum Mechanics* (Springer-Verlag, New York, 1980).
- [8] G. Lüders, *Ann. Phys.* **443**, 322 (1951).
- [9] M. L. Goldberger and K. M. Watson, *Phys. Rev.* **134**, B919 (1964).
- [10] W. H. Furry, *Lect. Theor. Phys.* **8A**, 1 (1966).
- [11] E. B. Davies and J. T. Lewis, *Commun. Math. Phys.* **17**, 239 (1970).
- [12] *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland, Amsterdam, 1965), Vol. 2.
- [13] L. Wolfenstein, *Annu. Rev. Nucl. Sci.* **6**, 43 (1956).
- [14] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
- [15] J. L. Doob, *Stochastic Processes* (Wiley, New York, 1953).
- [16] V. Hakim and V. Ambegaokar, *Phys. Rev. A* **32**, 423 (1985).
- [17] G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A* **64**, 032101 (2001).
- [18] F. Zernike, *Physica (Amsterdam)* **5**, 785 (1938).
- [19] A. E. Kaplan, P. Stifter, K. A. H. van Leeuwen, W. E. Lamb, and W. P. Schleich, *Phys. Scr.* **T76**, 93 (1998).
- [20] G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A* **37**, 4419 (1988).
- [21] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.* **106**, 121 (1984).
- [22] G. W. Ford and M. Kac, *J. Stat. Phys.* **45**, 803 (1987).
- [23] L. D. Romero and J. P. Paz, *Phys. Rev. A* **55**, 4070 (1997).
- [24] D. F. Walls and G. J. Milburn, *Phys. Rev. A* **31**, 2403 (1985).
- [25] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, UK, 1997).
- [26] H. B. Callen and T. A. Welton, *Phys. Rev.* **83**, 34 (1951).
- [27] H. Grabert, P. Schramm, and G.-L. Ingold, *Phys. Rev. Lett.* **58**, 1285 (1987).
- [28] G. W. Ford and R. F. O'Connell, *Phys. Rev. A* **73**, 032103 (2006).
- [29] *Bateman Manuscript Project, Higher Transcendental Functions*, edited by A. Erdlyi (McGraw-Hill, New York, 1954), Vol. 2.
- [30] G. W. Ford and R. F. O'Connell, *Ann. Phys. (N.Y.)* **276**, 144 (1999).
- [31] W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964).