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Is there Unruh radiation?

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Abstract

We present an exact analysis of an oscillator (the detector) moving under a constant force with respect to zero-temperature vacuum and coupled to a one-dimensional scalar field. We show that this system does not radiate despite the fact that it thermalizes at the Unruh temperature. We remark upon a differing opinion expressed regarding a system coupled to the electromagnetic field.

1. Introduction

It is now generally accepted that, as originally pointed out by Davies [1] and Unruh [2], a system undergoing uniform acceleration with respect to zero-temperature vacuum will come to equilibrium at an effective temperature that is proportional to the acceleration. This is the so-called Unruh temperature. What is more controversial is whether or not this implies that the system actually radiates.

Grove [3] was the first to argue that, contrary to the then prevailing opinion, the system does not radiate. This conclusion was supported by Raine et al. [4] who considered a uniformly accelerated oscillator moving under the action of a constant force and analyzed its effect on a detector (represented as an inertial harmonic oscillator). However, Unruh [5] claims that Raine et al. [4] discarded some terms in the autocorrelation function of the field that actually contribute to the excitation of the detector. Also, we note that Ref. [4] uses a Weisskopf–Wigner (or white-noise) approximation. Belief in the reality of the radiation may be gauged by recent suggestions as to how it might be measured [6,7], but there is widespread controversy[1] as to whether the radiation actually exists. Thus, we are motivated to present an exact calculation for the simple model of an oscillator (the detector) coupled to a scalar field (scalar electrodynamics). In particular, we discard no terms and do not introduce the Weisskopf–Wigner approximation. Our approach is also unique in using the oscillator as a detector since it considerably simplifies the analysis in that we only need to treat the motion of one body instead of two. Secondly, making the oscillator massive enough has the merit of ensuring that the back reaction due to the scalar field has no effect on the oscillators dynamics. The methods we use are those of a quantum Langevin approach which we have used for such problems as an oscillator coupled to the radiation field [12].

Since the subject has given rise to so much controversy, our aim will be to present the discussion in a detailed pedagogical manner. We begin in Section 2, where we give a simple description of the real scalar field in one dimension. This field is isomorphic to the case of a stretched string [12] and to make the discussion more intuitive we couch it in terms of the string. Our starting point is the Lagrangian for the field, from which we deduce that the equation of motion of the field and the zero-temperature correlation are invariant under Lorentz transformation. In addition, we obtain explicit expressions for the correlation (which consists of a finite space- and time-dependent part plus a constant divergent term) and the commutator.

In Section 3, we consider the case of a point mass moving under a constant force (hyperbolic motion). There we make

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[1] For example, Barut and Dowling [8] and Narozhny et al. [9] agree with the conclusions of Refs. [3,4], whereas Reznik [10] and Alsing and Milonni [11] disagree.

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use of the convenient parametric representation of the motion sometimes called Rindler coordinates [13]. We find that the zero-temperature correlation of the field evaluated at a space–time point moving along the hyperbolic path is identical with the correlation evaluated at a point fixed in space, but with the field at the elevated Unruh temperature.

Next, we take into account the coupling of an oscillator to the field. In Section 4, we consider the coupling of a charged oscillator to the field (scalar electrodynamics), such that the oscillator is at rest. This system is similar to the Lamb model of a particle attached to a stretched string [12], in that it leads to the same Langevin equation. The solution of the Langevin equation enables us to calculate the field and with that the flux of energy radiated by the oscillator at rest and in equilibrium with the field at temperature T. We find the net energy flux at any point in the field is identically zero, the result of a detailed balance of a flux of field energy emitted by the oscillator and a flux of field energy supplied to the oscillator. This is entirely what one should expect, since the mean energy of the oscillator in equilibrium is constant. The point of this exercise is seen in the next two sections, where we find the for an oscillator in hyperbolic motion through a zero-temperature field the net flux vanishes in the same way and for the same reasons as for an oscillator at rest.

In Section 5, we extend the model discussed in the previous section to consider an oscillator coupled to a moving point in the field. In contrast to Raine et al., who introduced an inertial detector at a fixed point in space to test for the emission of radiation from the moving oscillator, we simply treat the oscillator as a detector and calculate the flux of field energy. In addition, we do not ignore quantum effects in the expressions for the various field correlation functions. As before, we obtain the Unruh temperature. Then in Section 6 we present an explicit calculation to show that for an oscillator in hyperbolic motion the expectation value of the energy flux vanishes, just as for an oscillator at rest. With some concluding remarks in Section 7, we present our conclusion that, whereas one can speak of an Unruh temperature, there is no corresponding radiation to be detected. In this context, we also analyze Unruh's counterclaim [5] and argue that it is not valid. Finally, we emphasize that our discussion is restricted to the specific model of an oscillator coupled to a one-dimensional scalar field. While this is the model used by most authors, including the original work of Davies [1] and Unruh [2], other models (for example, the more realistic one of a charged particle coupled to the electromagnetic field [14]) can give different results. In our concluding remarks we discuss conflicting opinions concerning the radiation with such models.

2. Real scalar field in one dimension (stretched string)

The Lagrangian for the stretched string is

$$L=\int dy\left\{\frac{\sigma}{2}\left(\frac{\partial u}{\partial t}\right)^2-\frac{\tau}{2}\left(\frac{\partial u}{\partial y}\right)^2\right\}, \tag{2.1}$$

where σ is the mass per unit length, τ is the tension and $u(y,t)$ is the string displacement. The integral is along the length of the string, which is stretched in the y direction. For the real scalar field it is customary to put $\sigma=1/4\pi$ and $\tau=c^2/4\pi$, where c is the velocity of light. In that case u has the dimensions $(\text{mass}\cdot\text{length})^{1/2}$. The equation of motion of the string is the homogeneous wave equation,

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial y^2}=0, \tag{2.2}$$

where c is the speed of waves in the string,

$$c=(\tau/\sigma)^{1/2}. \tag{2.3}$$

There is an energy conservation law,

$$\frac{\partial \mathcal{E}}{\partial t}+\frac{\partial j}{\partial y}=0, \tag{2.4}$$

with energy density

$$\mathcal{E}(y,t)=\frac{\sigma}{2}\left(\frac{\partial u(y,t)}{\partial t}\right)^2+\frac{\tau}{2}\left(\frac{\partial u(y,t)}{\partial y}\right)^2 \tag{2.5}$$

and energy flux

$$j(y,t)=-\frac{\tau}{2}\left(\frac{\partial u(y,t)}{\partial t}\frac{\partial u(y,t)}{\partial y}+\frac{\partial u(y,t)}{\partial y}\frac{\partial u(y,t)}{\partial t}\right). \tag{2.6}$$

Although for the string c is not necessarily the speed of light, the equation of motion is still invariant under the Lorentz transformation with velocity v:

$$\begin{aligned}&y'=\gamma(y-vt),\qquad t'=\gamma\left(t-vy/c^2\right),\\&u'(y',t')=u(y,t),\end{aligned} \tag{2.7}$$

where

$$\gamma=\left(1-v^2/c^2\right)^{-1/2}. \tag{2.8}$$

Of course, for the real scalar field in one dimension this is the usual Lorentz transformation.

The normal mode expansion of the displacement operator may be written

$$u(y,t)=\sum_k\sqrt{\frac{\hbar}{2\sigma L\omega}}\left(a_k e^{i(ky-\omega t)}+a_k^\dagger e^{-i(ky-\omega t)}\right), \tag{2.9}$$

where L is the length of the string and for periodic boundary conditions the sum is over positive and negative integer multiples of $2\pi/L$. The frequency is given by the dispersion relation,

$$\omega=c|k|. \tag{2.10}$$

The string is quantized when we assume the canonical commutation relations for the dimensionless normal mode amplitudes,

$$\left[a_k,a_{k'}^\dagger\right]=\delta_{k',k},\qquad [a_k,a_{k'}]=0. \tag{2.11}$$

When the string is in equilibrium at temperature T, we have the expectation values

$$\begin{aligned}&\left\langle a_k a_{k'}^\dagger+a_{k'}^\dagger a_k\right\rangle=\coth\frac{\hbar\omega}{2kT}\delta_{k',k},\\&\langle a_k a_{k'}+a_{k'}a_k\rangle=0.\end{aligned} \tag{2.12}$$

The correlation function for the string is

$$C(\Delta y,\Delta t)\equiv\frac{1}{2}\left\langle u(y_1,t_1)u(y_2,t_2)+u(y_2,t_2)u(y_1,t_1)\right\rangle, \tag{2.13}$$

where

$$\Delta y = y_1 - y_2, \qquad \Delta t = t_1 - t_2. \tag{2.14}$$

In the limit of an infinite string ($L \to \infty$) we evaluate this using the above relations together with the prescription

$$\sum_k \to \frac{L}{2\pi} \int_{-\infty}^{\infty} dk. \tag{2.15}$$

The result is

$$C(\Delta y, \Delta t) = \frac{\hbar}{4\pi\sigma} \int_{-\infty}^{\infty} dk \frac{1}{\omega} \coth\frac{\hbar\omega}{2kT} \cos(k\Delta y - \omega\Delta t). \tag{2.16}$$

This integral is divergent at long wavelength ($k = 0$). This is to be expected since the Lagrangian (2.1) is invariant under uniform displacement of the string. Nevertheless, we can still obtain a useful result. To do so note that the derivative is a conditionally convergent integral [15],

$$\begin{aligned}\frac{\partial C(\Delta y, \Delta t)}{\partial \Delta t} &= -\frac{\hbar}{4\pi\sigma c} \int_0^{\infty} d\omega \coth\frac{\hbar\omega}{2kT} \\ &\quad \times \left[\sin\omega\left(\Delta t - \frac{\Delta y}{c}\right) + \sin\omega\left(\Delta t + \frac{\Delta y}{c}\right)\right] \\ &= -\frac{kT}{4\sigma c}\left(\coth\frac{\pi kT(\Delta t - \Delta y/c)}{\hbar} \right. \\ &\quad \left. + \coth\frac{\pi kT(\Delta t + \Delta y/c)}{\hbar}\right).\end{aligned} \tag{2.17}$$

From this we conclude that

$$\begin{aligned}C(\Delta y, \Delta t) = -\frac{\hbar}{4\pi\sigma c}&\left(\log\sinh\frac{\pi kT(\Delta t - \Delta y/c)}{\hbar}\right. \\ &\left. + \log\sinh\frac{\pi kT(\Delta t + \Delta y/c)}{\hbar}\right) + \text{const},\end{aligned} \tag{2.18}$$

where the constant, while infinite, is independent of Δy and Δt. The case of the correlation at a fixed point on the string ($\Delta y = 0$) is of special interest,

$$C(0, \Delta t) = -\frac{\hbar}{2\pi\sigma c} \log\sinh\frac{\pi kT\Delta t}{\hbar} + \text{const}. \tag{2.19}$$

This finite-temperature correlation function in Minkowski space–time will in the next section be compared to the zero-temperature correlation function in Rindler coordinates in order to relate the constant acceleration to the Unruh temperature.

It is of interest to consider the zero-temperature correlation function in Minkowski space,

$$C_0(\Delta y, \Delta t) \equiv \frac{\hbar}{4\pi\sigma} \int_{-\infty}^{\infty} \frac{dk}{\omega} \cos(k\Delta y - \omega\Delta t). \tag{2.20}$$

Introduce in the integral a Lorentz transformation of the wave vector and frequency,

$$k' = \gamma\left(k - v\omega/c^2\right), \qquad \omega' = \gamma(\omega - vk). \tag{2.21}$$

It is a simple matter to show that the dispersion relation (2.10) is preserved under this transformation and that

$$dk'/\omega' = dk/\omega. \tag{2.22}$$

To obtain an explicit expression for the zero-temperature correlation, put $v = -\Delta y/\Delta t$ if $|\Delta y/\Delta t| < c$ and $v = -c^2\Delta t/\Delta y$ if $|\Delta y/\Delta t| > c$. Then, using the dispersion relation (2.10), we obtain the expression

$$C_0(\Delta y, \Delta t) = \frac{\hbar}{2\pi\sigma c} \int_0^{\infty} d\omega \frac{\cos(\omega|\Delta t^2 - \Delta y^2/c^2|^{1/2})}{\omega}. \tag{2.23}$$

This integral is again divergent at long wavelength. Note that the divergence comes from the behavior at $\omega = 0$ and that we can write

$$C_0(\Delta y, \Delta t) = \lim_{\epsilon\to 0^+} \frac{\hbar}{2\pi\sigma c} \int_{\epsilon(\Delta t^2 - \Delta y^2/c^2)^{1/2}}^{\infty} dt \frac{\cos(t)}{t}. \tag{2.24}$$

Here, the integral is logarithmic at $t = 0$, so we see that

$$C_0(\Delta y, \Delta t) = -\frac{\hbar}{4\pi\sigma c} \log\left|\Delta t^2 - \frac{\Delta y^2}{c^2}\right| + \text{const}, \tag{2.25}$$

where the constant is logarithmically divergent as $\epsilon \to 0$, but independent of Δt and Δy. Note that, since it depends only upon the invariant interval, this zero-temperature correlation is invariant under Lorentz transformation.

Next, consider the commutator of the field, which can be written

$$\left[u(y_1, t_1), u(y_2, t_2)\right] = \frac{i\hbar}{2\pi\sigma} \int_{-\infty}^{\infty} dk \frac{\sin(k\Delta y - \omega\Delta t)}{\omega}. \tag{2.26}$$

If $|\Delta y/\Delta t| < c$, we introduce a Lorentz transformation corresponding to $v = -\Delta y/\Delta t$ to obtain the expression

$$\begin{aligned}\left[u(y_1, t_1), u(y_2, t_2)\right] &= \frac{-i\hbar}{2\pi\sigma} \int_{-\infty}^{\infty} dk \frac{\sin[\omega\Delta t(1 - \Delta y^2/c^2\Delta t^2)^{1/2}]}{\omega} \\ &= \frac{-i\hbar}{2\sigma c} \operatorname{sgn}(\Delta t).\end{aligned} \tag{2.27}$$

On the other hand, if $|\Delta y/\Delta t| > c$, we introduce a Lorentz transformation corresponding to $v = -c^2\Delta t/\Delta y$ to obtain the expression

$$\left[u(y_1, t_1), u(y_2, t_2)\right] = \frac{-i\hbar}{2\pi\sigma} \int_{-\infty}^{\infty} dk \frac{\sin[k\Delta y(1 - c^2\Delta t^2/\Delta y^2)^{1/2}]}{\omega} = 0. \tag{2.28}$$

Therefore, we have the general result

$$\left[u(y_1, t_1), u(y_2, t_2)\right] = \frac{\hbar}{2i\sigma c} \operatorname{sgn}(\Delta t)\theta\left(\Delta t^2 - \frac{\Delta y^2}{c^2}\right), \tag{2.29}$$

in which θ is the Heaviside function. Note that the commutator is invariant under proper Lorentz transformation, with an extra sign change under time reversal.

3. Hyperbolic motion

A point mass m moving under a constant force F in relativity moves according to the equation of motion

$$\frac{d}{dt}\frac{mv}{(1-v^2/c^2)^{1/2}} = F, \tag{3.1}$$

where $v = dy/dt$ is the velocity. The solution can most simply be written in parametric form,

$$y = \frac{mc^2}{F}\cosh\frac{F\tau}{mc}, \qquad t = \frac{mc}{F}\sinh\frac{F\tau}{mc}, \\ -\infty < \tau < \infty, \tag{3.2}$$

where the parameter τ is the proper time,

$$d\tau = \left(1 - v^2/c^2\right)^{1/2} dt. \tag{3.3}$$

This solution is called hyperbolic motion. It is also called uniformly accelerated motion, since in the instantaneous rest frame the acceleration is a constant equal to F/m. The motion corresponds to a point mass coming at $t = -\infty$ from $y = +\infty$ with velocity $v = -c$, decelerating with a constant force F until at $t = 0$ it comes to rest at $y = mc^2/F$. The mass then accelerates back to $y = +\infty$ at $t = \infty$.

For this hyperbolic motion, taking the two points to be on the same world line (3.2), we see that

$$\left(\Delta t^2 - \Delta y^2/c^2\right)^{1/2} = \frac{2mc}{F}\sinh\frac{F\Delta\tau}{2mc}, \tag{3.4}$$

where $\Delta\tau = \tau_1 - \tau_2$. The zero-temperature correlation function (2.25) for the scalar field in one dimension, when evaluated on the world line, therefore takes the form

$$C_0(\Delta y, \Delta t) = -\frac{\hbar}{2\pi\sigma c}\log\sinh\frac{F\Delta\tau}{2mc} + \text{const.} \tag{3.5}$$

Therefore, for hyperbolic motion the correlation is a function of $\Delta\tau$ alone.

Recall that the proper time τ is the time as measured on a moving clock. Therefore, comparing (3.5) and (2.19), we see that the zero-temperature correlation evaluated along the hyperbolic path is identical to the finite-temperature correlation evaluated at a fixed point if we make the identification

$$kT = \frac{\hbar F}{2\pi mc}. \tag{3.6}$$

This is the Unruh temperature [2].

4. Oscillator coupled to a one-dimensional scalar field

We consider a coupling of the oscillator to the field through the velocity. The Lagrangian is

$$\mathrm{L} = \frac{1}{2}mv^2 - \frac{1}{2}Kx^2 - 2\sigma cv\phi(0,t) \\ + \int dy\left\{\frac{\sigma}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\sigma c^2}{2}\left(\frac{\partial\phi}{\partial y}\right)^2\right\}. \tag{4.1}$$

This is sometimes called scalar electrodynamics. Note that the particle displacement is in the x direction, while the field extends in the y direction. The oscillator interacts with the field at the origin ($y = 0$). Thus, the system is very like the Lamb model (in which the particle is attached to the center of an infinite stretched string [12,16]) and we shall see that it leads to the same quantum Langevin equation. However, if the system is to be invariant under time reversal then ϕ must be odd under time reversal. In this sense, the field ϕ is different from the displacement u of a string. Otherwise, the discussion of the previous sections applies to the free field ϕ.

The equation of particle motion is that of a driven oscillator,

$$m\frac{d^2x}{dt^2} + Kx = 2\sigma c\frac{\partial\phi(0,t)}{\partial t}. \tag{4.2}$$

The field equation of motion is the inhomogeneous wave equation,

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial y^2} = -2c\frac{dx(t)}{dt}\delta(y). \tag{4.3}$$

We now eliminate the field variable between these two equations. Treating the right-hand side as known, the solution of the field equation is

$$\phi(y,t) = \phi^h(y,t) - x\left(t - \frac{|y|}{c}\right), \tag{4.4}$$

where $\phi^h(y,t)$ is the general solution of the homogeneous wave equation (2.2). Putting this solution in the particle equation of motion, we get the Langevin equation

$$m\ddot{x} + \zeta\dot{x} + Kx = F(t), \tag{4.5}$$

where

$$\zeta = 2\sqrt{\sigma\tau} \tag{4.6}$$

is the friction constant and

$$F(t) = \zeta\frac{\partial\phi^h(0,t)}{\partial t} \tag{4.7}$$

is a fluctuating force operator.

The free field has the normal mode expansion (2.9),

$$\phi^h(y,t) = \sum_k \sqrt{\frac{\hbar c}{\zeta L\omega}}\left(a_k e^{i(ky-\omega t)} + a_k^\dagger e^{-i(ky-\omega t)}\right), \tag{4.8}$$

in which we combined (2.3) and (4.6) to write $\zeta = 2\sigma c$. From this we see that the fluctuating force can be expanded,

$$F(t) = \sum_k \sqrt{\frac{\hbar c\zeta\omega}{L}}\left(-ia_k e^{-i\omega t} + ia_k^\dagger e^{i\omega t}\right). \tag{4.9}$$

From this, using the canonical commutation rules (2.11) and the expectation values (2.12), we can obtain the correlation and commutator for the fluctuating force. If we then form the limit $L \to \infty$, using the prescription (2.15) and the dispersion relation (2.10), we get

$$\frac{1}{2}\left\langle F(t_1)F(t_2) + F(t_2)F(t_1)\right\rangle \\ = \frac{\hbar}{\pi}\int_0^\infty d\omega\,\zeta\omega\coth\frac{\hbar\omega}{2kT}\cos\left[\omega(t_1 - t_2)\right],$$

$$\big[F(t_1), F(t_2)\big] = -i\frac{2\hbar}{\pi}\int_0^\infty d\omega\,\zeta\omega\sin\big[\omega(t_1-t_2)\big]. \tag{4.10}$$

Using the expansion (4.9) of the fluctuating force in the right-hand side of the Langevin equation (4.5), we see that the solution has the expansion

$$x(t) = \sum_k \sqrt{\frac{\hbar c\zeta\omega}{L}}\big[-i\alpha(\omega)a_k e^{-i\omega t} + i\alpha(\omega)^* a_k^\dagger e^{i\omega t}\big], \tag{4.11}$$

where $\alpha(\omega)$ is the oscillator susceptibility,

$$\alpha(\omega) = \big(-m\omega^2 - i\omega\zeta + K\big)^{-1}. \tag{4.12}$$

It is of interest to form the position correlation for the oscillator. Using the expectation values (2.12) and forming the limit $L\to\infty$, we find

$$\frac{1}{2}\big\langle x(t_1)x(t_2) + x(t_2)x(t_1)\big\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega\,\mathrm{Im}\big\{\alpha(\omega)\big\}\coth\frac{\hbar\omega}{2kT}\cos\big[\omega(t_1-t_2)\big], \tag{4.13}$$

in which we have used the fact that $\mathrm{Im}\{\alpha(\omega)\} = \zeta\omega|\alpha(\omega)|^2$.

Consider now the flux of field energy radiated by the oscillator as measured at some point y away from the origin. The energy flux operator is given by (2.6) with $u(y,t)\to\phi(y,t)$, the total field given by the solution (4.4) of the field equation. With this, forming the expectation, we can write

$$\big\langle j(y,t)\big\rangle = \big\langle j_0(y,t)\big\rangle + \big\langle j_{\mathrm{dir}}(y,t)\big\rangle + \big\langle j_{\mathrm{int}}(y,t)\big\rangle, \tag{4.14}$$

where $\langle j_0(y,t)\rangle$ is the energy flux in the absence of the oscillator,

$$\big\langle j_0(y,t)\big\rangle = -\frac{1}{2}\zeta c\,\mathrm{Re}\left\langle\frac{\partial\phi^h(y,t)}{\partial t}\frac{\partial\phi^h(y,t)}{\partial y}\right\rangle, \tag{4.15}$$

$\langle j_{\mathrm{dir}}(y,t)\rangle$ is the energy flux arising from source alone,

$$\big\langle j_{\mathrm{dir}}(y,t)\big\rangle = \frac{1}{2}\frac{y}{|y|}\zeta\left\langle\dot{x}^2\left(t-\frac{|y|}{c}\right)\right\rangle, \tag{4.16}$$

and $\langle j_{\mathrm{int}}(y,t)\rangle$ is the interference term

$$\big\langle j_{\mathrm{int}}(y,t)\big\rangle = \frac{1}{2}\zeta\,\mathrm{Re}\left\langle\dot{x}\left(t-\frac{|y|}{c}\right)\left(c\frac{\partial\phi^h(y,t)}{\partial y} - \frac{y}{|y|}\frac{\partial\phi^h(y,t)}{\partial t}\right)\right\rangle. \tag{4.17}$$

To get some insight into the significance of these terms, we multiply both sides of the Langevin equation (4.5) by dx/dt, symmetrize the factors in each term and form the expectation of the resulting equation to get the oscillator energy balance equation:

$$\frac{d}{dt}\left\langle\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2\right\rangle + \zeta\big\langle\dot{x}^2\big\rangle = \frac{1}{2}\langle\dot{x}F + F\dot{x}\rangle. \tag{4.18}$$

Here the first term on the left is clearly the rate of change of the mean oscillator energy. The second term is interpreted as the mean rate of radiation of field energy by the oscillator, while the right-hand side is interpreted as the mean rate at which work is done on the oscillator by the fluctuating force. Of course, in equilibrium the mean oscillator energy is constant and the remaining two terms must balance. Now, the direct term (4.16) is clearly the energy flux corresponding to the radiated energy, directed away from the oscillator with half to the left and half to the right. Note, incidentally, that the radiated power $\zeta\langle\dot{x}^2\rangle$ is the analog of the well-known Larmor formula for the power radiated by an oscillating electric dipole (proportional to the square of the velocity rather than the square of the acceleration since the coupling is to a scalar rather than a vector field).

We now evaluate these fluxes using the expansion (4.8) for the free field and the expansion (4.11) for the oscillator displacement. Consider first $\langle j_0(y,t)\rangle$, the current in the absence of the oscillator. This, of course, must vanish on very general grounds. In this case we see that when we insert the expansion for the free field the result is a sum over k of an odd function of k, which vanishes. Next, consider the direct flux (4.16). Using the expansion (4.11) and the expectation values (2.12), we find after a little rearrangement

$$\big\langle j_{\mathrm{dir}}(y,t)\big\rangle = \frac{y}{|y|}\frac{\hbar c}{2L}\sum_k\omega^3\zeta^2\big|\alpha(\omega)\big|^2\coth\frac{\hbar\omega}{2kT}. \tag{4.19}$$

If we use the prescription (2.15) to form the limit $L\to\infty$, we can write

$$\big\langle j_{\mathrm{dir}}(y,t)\big\rangle = \frac{1}{2}\frac{y}{|y|}\frac{\hbar}{\pi}\int_0^\infty d\omega\,\omega^3\zeta^2\big|\alpha(\omega)\big|^2\coth\frac{\hbar\omega}{2kT}. \tag{4.20}$$

In the same way the interference term (4.17) becomes

$$\big\langle j_{\mathrm{int}}(y,t)\big\rangle = \frac{y}{|y|}\frac{\hbar c^2}{2L}\times\sum_k\mathrm{Re}\left\{i\left(\frac{ky}{|y|}+|k|\right)\omega\zeta\alpha(\omega)e^{-i(ky-\omega|y|/c)}\right\}\times\coth\frac{\hbar\omega}{2kT}. \tag{4.21}$$

Recalling that the sum is over positive and negative k, we discard terms that are odd in k. The result, again after forming the limit $L\to\infty$, can be written

$$\big\langle j_{\mathrm{int}}(y,t)\big\rangle = -\frac{1}{2}\frac{y}{|y|}\frac{\hbar}{\pi}\int_0^\infty d\omega\,\omega^2\zeta\,\mathrm{Im}\big\{\alpha(\omega)\big\}\coth\frac{\hbar\omega}{2kT}. \tag{4.22}$$

But, as we see from (4.12), $\mathrm{Im}\{\alpha(\omega)\} = \omega\zeta|\alpha(\omega)|^2$. Therefore, comparing the expressions (4.20) and (4.22), we see that $\langle j_{\mathrm{int}}(y,t)\rangle = -\langle j_{\mathrm{dir}}(y,t)\rangle$ and

$$\big\langle j(y,t)\big\rangle = 0. \tag{4.23}$$

This result, which took some doing to obtain, should have been expected from the beginning. After all, in equilibrium the mean energy of the oscillator is constant, so the mean energy flux radiated into the field by the oscillator must be balanced by a mean energy flux from the field into the oscillator. This is just the result (4.23). Put another way, we can now interpret

$\langle j_{\rm int}(y,t)\rangle$ as the inward flux of field energy to balance the radiated power.

Finally, we remark on the situation when the oscillator is excited, say, by an impulse applied at $t=0$. In this case there will be a mean motion superposed on the random thermal motion of the oscillator. One can then calculate the net radiated flux of energy using only the expression (4.16) for $\langle j_{\rm dir}(y,t)\rangle$, evaluated for the mean motion. There will be no interference term since the mean motion will be uncorrelated with the random motion of the field.

5. Oscillator moving in the field

Consider now an oscillator undergoing a given motion in the field direction [4]. The idea is that in addition to the x-motion the oscillator has a given y-motion,

$$y=y(\tau),\qquad t=t(\tau), \tag{5.1}$$

where τ is the proper time,

$$d\tau=\left(dt^2-dy^2/c^2\right)^{1/2}. \tag{5.2}$$

We shall later take this to be the hyperbolic motion described in Section 3, but for now we assume only that the motion is mechanically allowed, so $|dy/dt|<c$. The Lagrangian (4.1) must be modified to take the motion into account. First of all, consider the kinetic energy, which must be replaced by the relativistic free particle Lagrangian [13],

$$\mathrm{L}_{\rm free}=-mc^2\sqrt{1-\frac{1}{c^2}\left(\frac{dy}{dt}\right)^2-\frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}. \tag{5.3}$$

But the x-motion is nonrelativistic while the y-motion is arbitrary, so we expand

$$\mathrm{L}_{\rm free}\cong-mc^2\sqrt{1-\frac{1}{c^2}\left(\frac{dy}{dt}\right)^2}-\frac{1}{\sqrt{1-\frac{1}{c^2}(\frac{dy}{dt})^2}}\frac{1}{2}m\left(\frac{dx}{dt}\right)^2. \tag{5.4}$$

We drop the first term, since the y-motion is given, and replace the kinetic energy in the Lagrangian (4.1) with the second term. Next, consider the potential energy, which must be multiplied by the time-dilation factor $\sqrt{1-\frac{1}{c^2}(\frac{dy}{dt})^2}$. Finally, the interaction must involve the field at the instantaneous position of the particle. The resulting Lagrangian can be written

$$\mathrm{L}=\frac{dt}{d\tau}\frac{1}{2}m\left(\frac{dx}{dt}\right)^2-\frac{d\tau}{dt}\frac{1}{2}Kx^2-2\sigma c\frac{dx}{dt}\phi\big[y(\tau),t(\tau)\big] \\ +\int dy\frac{\sigma}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2-c^2\left(\frac{\partial\phi}{\partial y}\right)^2\right], \tag{5.5}$$

where we have used the definition (5.2) of the proper time.

With this Lagrangian, the oscillator equation of motion is

$$m\frac{d^2x}{d\tau^2}+Kx=2\sigma c\frac{d\phi[y(\tau),t(\tau)]}{d\tau}, \tag{5.6}$$

while that for the field is the inhomogeneous wave equation,

$$\frac{\partial^2\phi}{\partial t^2}-c^2\frac{\partial^2\phi}{\partial y^2}=-2c\frac{dx}{dt}\delta\big[y-y(\tau)\big]. \tag{5.7}$$

Treating the right-hand side as known, the solution of this wave equation is

$$\phi(y,t)=\phi^h(y,t)-x\left(\tau^{\rm ret}\right), \tag{5.8}$$

where $\phi^h(y,t)$ is the general solution (4.8) of the free wave equation and $\tau^{\rm ret}$ is the retarded time. The retarded time is defined implicitly as a function of the field point (y,t) by the relation

$$t-t\left(\tau^{\rm ret}\right)=\left|y-y\left(\tau^{\rm ret}\right)\right|/c, \tag{5.9}$$

and corresponds to the point on the mechanical path (5.1) where it pierces the backward light cone centered at the field point. Note in particular that when the field point is on the mechanical path, then $\tau^{\rm ret}=\tau$. Thus, the solution (5.8) of the inhomogeneous wave equation can be written $\phi[y(\tau),t(\tau)]=\phi^h[y(\tau),t(\tau)]-x(\tau)$. Putting this in the right-hand side of the particle equation of motion (5.6), we obtain a quantum Langevin equation,

$$m\frac{d^2x}{d\tau^2}+\zeta\frac{dx}{d\tau}+Kx=F(\tau), \tag{5.10}$$

where ζ is the friction constant, given by the same expression (4.6) obtained for the oscillator at a fixed point. In this Langevin equation, the fluctuating operator force $F(\tau)$ is given by

$$F(\tau)=\zeta\frac{d\phi^h[y(\tau),t(\tau)]}{d\tau}. \tag{5.11}$$

We note that this Langevin equation has the same form as the Langevin equation (4.5) corresponding to the oscillator at a fixed point. Indeed, it reduces to that equation for the special motion $y(\tau)=0$, $t(\tau)=t$.

Next, consider the correlation of the fluctuating force. Using the above definition, we see that

$$\frac{1}{2}\big\langle F(\tau_1)F(\tau_2)+F(\tau_2)F(\tau_1)\big\rangle \\ =\zeta^2\frac{d^2}{d\tau_1\,d\tau_2}C\big[y(\tau_1)-y(\tau_2),t(\tau_1)-t(\tau_2)\big], \tag{5.12}$$

where $C(\Delta y,\Delta t)$ is the correlation function (2.13) for the real scalar field. At $T=0$, this correlation is given by the explicit expression (2.25), so we can write

$$\frac{1}{2}\big\langle F(\tau_1)F(\tau_2)+F(\tau_2)F(\tau_1)\big\rangle \\ =\frac{\hbar\zeta}{2\pi}\frac{d^2}{d\tau_1\,d\tau_2}\log\left(\Delta t^2-\Delta y^2/c^2\right)\quad(T=0). \tag{5.13}$$

The commutator of the fluctuating force is given by

$$\big[F(\tau_1),F(\tau_2)\big]=\zeta^2\frac{d^2}{d\tau_1\,d\tau_2}\big[\phi(y_1,t_1),\phi(y_2,t_2)\big]. \tag{5.14}$$

Using the explicit expression (2.29), we can write

$$\big[F(\tau_1),F(\tau_2)\big]=-i\hbar\zeta\frac{d^2}{d\tau_1\,d\tau_2}\operatorname{sgn}(\Delta\tau) \\ =2i\hbar\zeta\delta'(\tau_1-\tau_2). \tag{5.15}$$

Note that the form of the Langevin equation and the commutator of the fluctuating force operator are independent of the motion. However, the correlation of the force is explicitly dependent upon the motion.

Now we consider the special case of hyperbolic motion. We could use the explicit expression (5.13) for the force correlation, but it will be useful in the later discussion to obtain the expression in a different form. We begin with the normal mode expansion (4.8) of the free field, which with the expression (5.11) for the fluctuating force results in the expansion

$$F(\tau)=\frac{d}{d\tau}\sum_k\sqrt{\frac{\zeta\hbar c}{L\omega}}\times\left(a_k e^{i[ky(\tau)-\omega t(\tau)]}+a_k^\dagger e^{-i[ky(\tau)-\omega t(\tau)]}\right). \tag{5.16}$$

We next introduce the Fourier expansion

$$e^{i[ky(\tau)-\omega t(\tau)]}=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega'\,c(k;\omega')e^{-i\omega'\tau}, \tag{5.17}$$

to write

$$F(\tau)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\omega'\,\omega'\sum_k\sqrt{\frac{\zeta\hbar c}{L\omega}}\times\left(a_k c(k;\omega')e^{-i\omega'\tau}-a_k^\dagger c(k;\omega')^* e^{i\omega'\tau}\right). \tag{5.18}$$

Forming the zero-temperature correlation, using the expectation values (2.12) we can write

$$\frac{1}{2}\langle F(\tau_1)F(\tau_2)+F(\tau_2)F(\tau_1)\rangle =\frac{\zeta}{4\pi^2}\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2\,\omega_1\omega_2\times\mathrm{Re}\left\{e^{-i(\omega_1\tau_1-\omega_2\tau_2)}\sum_k\frac{\hbar c}{L\omega}c(k;\omega_1)c(k;\omega_2)^*\right\}. \tag{5.19}$$

To evaluate this expression, we first consider the Fourier transform,

$$c(k;\omega')=\int_{-\infty}^{\infty}d\tau\,e^{i[\omega'\tau+ky(\tau)-\omega t(\tau)]}. \tag{5.20}$$

Note first that for hyperbolic motion $y(\tau)$ is even and $t(\tau)$ is odd as a function of τ, so

$$c(-k;\omega')=c(k;\omega')^*. \tag{5.21}$$

It is therefore sufficient to consider positive $k=\omega/c$. For this case, using Eqs. (4.8) of hyperbolic motion, we make in the integral (5.17) the substitution $z=e^{-F\tau/mc}$ to get

$$c\left(\frac{\omega}{c};\omega'\right)=\frac{mc}{F}\int_0^{\infty}dz\,z^{-1-imc\omega'/F}e^{-mc\omega z/F}. \tag{5.22}$$

Finally, we rotate the path of integration into the positive imaginary axis, and use the well-known integral representation of the gamma function [15], to obtain the result

$$c\left(\frac{\omega}{c};\omega'\right)=\frac{mc}{F}\left(\frac{mc\omega}{F}\right)^{imc\omega'/F}e^{\pi mc\omega'/2F}\Gamma\left(-i\frac{mc\omega'}{F}\right). \tag{5.23}$$

Next, consider

$$\sum_k\frac{\hbar c}{L\omega}c(k;\omega_1)c(k;\omega_2)^* =\frac{\hbar m^2c^2}{\pi F^2}e^{\pi mc(\omega_1+\omega_2)/2F}\Gamma\left(-i\frac{mc\omega_1}{F}\right)\Gamma\left(i\frac{mc\omega_2}{F}\right)\times\mathrm{Re}\left\{\int_0^{\infty}d\omega\,\frac{1}{\omega}\left(\frac{mc\omega}{F}\right)^{imc(\omega_1-\omega_2)/F}\right\}, \tag{5.24}$$

where we have used the prescription (2.15) for the limit $L\to\infty$, then the condition (5.21) and the dispersion relation (2.10) to write the integral as over positive frequencies. With the substitution $v=\log(mc\omega/F)$ the integral here becomes the familiar integral for the Dirac delta-function and is therefore given by $(2\pi F/mc)\delta(\omega_1-\omega_2)$. It follows that

$$\sum_k\frac{\hbar c}{L\omega}c(k;\omega_1)c(k;\omega_2)^* =2\pi\hbar\frac{e^{\pi mc\omega_1/F}}{\omega_1\sinh\frac{\pi mc\omega_1}{F}}\delta(\omega_1-\omega_2), \tag{5.25}$$

where we have used the identity $|\Gamma(ix)|^2=(\pi/x)\sinh\pi x$. Using this result in Eq. (5.19), we find that the zero-temperature correlation can be expressed in the form

$$\frac{1}{2}\langle F(\tau_1)F(\tau_2)+F(\tau_2)F(\tau_1)\rangle =\frac{\hbar}{\pi}\int_0^{\infty}d\omega\,\zeta\omega\coth\frac{\pi mc\omega}{F}\cos[\omega(\tau_1-\tau_2)]. \tag{5.26}$$

Again, we see the Unruh temperature (3.6). That is, this force autocorrelation seen by the oscillator in hyperbolic motion through a zero-temperature field is identical with that (4.10) seen by an oscillator at rest in a field at the Unruh temperature. (Recall that the proper time τ is the time as measured on a clock moving with the oscillator.) We emphasize that this means that the moving oscillator is itself at the Unruh temperature.

6. Energy radiated by an oscillator undergoing hyperbolic motion

We now calculate the flux of energy radiated by the oscillator undergoing hyperbolic motion in a zero-temperature field. As we have seen, the moving oscillator is at the elevated Unruh temperature. This picture of a hot oscillator moving through a zero-temperature background leads one to expect that there should be radiation. After all, doesn't heat always flow from a hot body to a cold body? But in this section we shall show by explicit calculation that the net energy flux is zero.

Consider now the flux of field energy radiated by the oscillator as measured at some point to the left of the point of closest approach in hyperbolic motion. The energy flux operator is given by (2.6) with $u(y,t) \to \phi(y,t)$, the total field given by the solution (5.8) of the field equations for the oscillator in hyperbolic motion. Forming the expectation, we can write just as in Section 4,

$$\langle j(y,t)\rangle = \langle j_0(y,t)\rangle + \langle j_{\rm dir}(y,t)\rangle + \langle j_{\rm int}(y,t)\rangle, \tag{6.1}$$

where $\langle j_0(y,t)\rangle$ is the energy flux (4.15) in the absence of the oscillator, while now the direct flux is given by

$$\langle j_{\rm dir}(y,t)\rangle = -\frac{\zeta c}{2}\,{\rm Re}\left\langle \frac{\partial x(\tau^{\rm ret})}{\partial t}\frac{\partial x(\tau^{\rm ret})}{\partial y}\right\rangle, \tag{6.2}$$

and the interference term is given by

$$\langle j_{\rm int}(y,t)\rangle = \frac{\zeta c}{2}\,{\rm Re}\left\langle \frac{\partial \phi^h(y,t)}{\partial t}\frac{\partial x(\tau^{\rm ret})}{\partial y} + \frac{\partial x(\tau^{\rm ret})}{\partial t}\frac{\partial \phi^h(y,t)}{\partial y}\right\rangle. \tag{6.3}$$

In these expressions, the retarded time is determined by the condition (5.9). For a field point to the left of the point of closest approach, $y < mc^2/F$ and, using Eqs. (3.2) of hyperbolic motion, we find

$$\tau^{\rm ret} = \frac{mc}{F}\log\frac{F(t+y/c)}{mc}. \tag{6.4}$$

Thus, $c\partial\tau^{\rm ret}/\partial y = \partial\tau^{\rm ret}/\partial t$ and using the chain rule we can write

$$\langle j_{\rm dir}(y,t)\rangle = -\frac{1}{2}\left(\frac{\partial\tau^{\rm ret}}{\partial t}\right)^2 \zeta\left\langle\left(\frac{dx(\tau^{\rm ret})}{d\tau^{\rm ret}}\right)^2\right\rangle \tag{6.5}$$

and

$$\langle j_{\rm int}(y,t)\rangle = \frac{\zeta}{2}\frac{\partial\tau^{\rm ret}}{\partial t}\,{\rm Re}\left\langle\frac{dx(\tau^{\rm ret})}{d\tau^{\rm ret}}\left(\frac{\partial\phi^h(y,t)}{\partial t} + c\frac{\partial\phi^h(y,t)}{\partial y}\right)\right\rangle. \tag{6.6}$$

As in Section 4, we get some insight into the significance of these terms if we consider the energy balance equation for the moving oscillator,

$$\frac{d}{d\tau}\left\langle\frac{1}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + \frac{1}{2}Kx^2(\tau)\right\rangle + \zeta\left\langle\left(\frac{dx(\tau)}{d\tau}\right)^2\right\rangle = \frac{1}{2}\left\langle\frac{dx(\tau)}{d\tau}F(\tau) + F(\tau)\frac{dx(\tau)}{d\tau}\right\rangle. \tag{6.7}$$

This is identical with the corresponding energy balance equation for the oscillator at rest, the only difference being that here τ is the time as measured on a clock moving with the oscillator. Thus, in a frame moving with the oscillator, the energy balance is identical with that for an oscillator at rest. In particular, we can interpret the second term on the left as the mean rate of radiation of field energy by the oscillator, while the right-hand side is the rate at which energy is supplied to the oscillator by the fluctuating force, all as seen in the moving frame. The direct flux (6.5) is half the rate of radiation of energy multiplied by the time dilation factor $(\partial\tau^{\rm ret}/\partial t)^2$. The factor of two is accounted for by the fact that half the radiation is to the left, half to the right. The time dilation factor corresponds to the transformation from the proper time kept on a clock moving with the oscillator, where the rate of radiation is uniform, to a stationary clock at the field point. Using the expression (6.4) for the retarded time, we see that

$$\frac{\partial\tau^{\rm ret}}{\partial t} = \frac{mc}{F(t+y/c)}, \quad 0 < t + y/c < \infty. \tag{6.8}$$

Thus, $\langle j_{\rm dir}(y,t)\rangle$ corresponds to a flux that is zero for $t < -y/c$, then suddenly infinite and decaying to zero for long times. Finally, since $\langle j_{\rm dir}(y,t)\rangle$ corresponds to the energy lost by the oscillator through radiation, the interference term $\langle j_{\rm int}(y,t)\rangle$ must be the inward flux of field energy absorbed by the oscillator. Since the oscillator is in a stationary equilibrium state corresponding to the Unruh temperature, one should expect that these two fluxes should balance, just as they do for an oscillator at rest. We next show by explicit calculation that these two fluxes do indeed cancel to give a net flux of zero.

We begin using the expression (5.18) for the fluctuating force, to write the solution of the quantum Langevin equation (5.10) in the form

$$x(\tau) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega'\,\omega'\sum_k\sqrt{\frac{\zeta\hbar c}{L\omega}} \times \left(a_k\alpha(\omega')c(k;\omega')e^{-i\omega'\tau} - a_k^\dagger\alpha(\omega')^*c(k;\omega')^*e^{i\omega'\tau}\right), \tag{6.9}$$

where $\alpha(\omega)$ is the oscillator susceptibility (4.12). With this, forming the expectation using the expectation values (2.12) with $T = 0$, then using the result (5.25) we find

$$\zeta\left\langle\left(\frac{dx(\tau)}{d\tau}\right)^2\right\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega\,\omega^3\zeta^2\left|\alpha(\omega)\right|^2\coth\frac{\pi mc\omega}{F}. \tag{6.10}$$

As we have seen, this is the rate at which the oscillator loses energy through radiation. It is independent of time as measured in the moving frame and identical with the same quantity for a stationary oscillator. With this, we see that the direct flux can be written

$$\langle j_{\rm dir}(y,t)\rangle = -\left(\frac{\partial\tau^{\rm ret}}{\partial t}\right)^2\frac{\hbar}{2\pi} \times \int_0^\infty d\omega\,\omega^3\zeta^2\left|\alpha(\omega)\right|^2\coth\frac{\pi mc\omega}{F}. \tag{6.11}$$

Next, consider the interference term (6.6). Using the expansion (4.8) of the free field and the expression (6.9) for the oscillator displacement, we form the expectation using the expectation values (2.12) with $T = 0$ to get

$$\langle j_{\rm int}(y,t)\rangle = \frac{\zeta}{2}\frac{\partial\tau^{\rm ret}}{\partial t}\,{\rm Im}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega'\,\omega'^2\alpha(\omega')e^{-i\omega'\tau^{\rm ret}} \times \frac{\hbar c}{L}\sum_k\frac{\omega - ck}{\omega}c(k;\omega')e^{-i(ky-\omega t)}. \tag{6.12}$$

Since $\omega = c|k|$, the sum can be restricted to negative k. Then, replacing the sum by an integral, using the prescription (2.15) and using the identity (5.21) and the expression (5.23) for $c(\frac{\omega}{c};\omega')$ we get

$$\frac{\hbar c}{L}\sum_k \frac{\omega - ck}{\omega} c(k;\omega') e^{-i(ky-\omega t)} = e^{\pi mc\omega'/2F}\Gamma\left(i\frac{mc\omega'}{F}\right)\frac{mc\hbar}{F\pi} \times \int_0^\infty d\omega \left(\frac{mc\omega}{F}\right)^{-imc\omega'/F} e^{i\omega(t+y/c)}. \tag{6.13}$$

Next, we rotate the path of integration into the positive real axis and use the integral representation of the gamma function [15] to write

$$\frac{\hbar c}{L}\sum_k \frac{\omega - ck}{\omega} c(k;\omega') e^{-i(ky-\omega t)} = \frac{mc}{F(t+y/c)}\left(\frac{mc}{F(t+y/c)}\right)^{-imc\omega'/F} \times e^{\pi mc\omega'/F}\frac{i\hbar}{\pi}\Gamma\left(i\frac{mc\omega'}{F}\right)\Gamma\left(1-i\frac{mc\omega'}{F}\right). \tag{6.14}$$

We use in the first factor the expression (6.8) for $d\tau^{\text{ret}}/dt$ and in the second factor the expression (6.4) for τ^{ret}. Then, using the identity $i\Gamma(iz)\Gamma(1-ix) = \pi/\sinh \pi x$, we obtain the result

$$\frac{\hbar c}{L}\sum_k \frac{\omega - ck}{\omega} c(k;\omega') e^{-i(ky-\omega t)} = \hbar\frac{\partial\tau^{\text{ret}}}{\partial t}\frac{e^{\pi mc\omega'/F}}{\sinh\frac{mc\omega'}{F}} e^{i\omega'\tau^{\text{ret}}}. \tag{6.15}$$

Putting this in the expression (6.12) we get

$$\langle j_{\text{int}}(y,t)\rangle = \left(\frac{\partial\tau^{\text{ret}}}{\partial t}\right)^2 \frac{\hbar\zeta}{2\pi} \times \int_{-\infty}^{\infty} d\omega\,\omega^2\,\text{Im}\{\alpha(\omega)\}\coth\frac{\pi mc\omega}{F}. \tag{6.16}$$

But, as we see from the expression (4.12) for the oscillator susceptibility, $\text{Im}\{\alpha(\omega)\} = \omega\zeta|\alpha(\omega)|^2$. Therefore, this is just the negative of the expression (6.11) for $\langle j_{\text{dir}}(y,t)\rangle$. That is,

$$\langle j_{\text{dir}}(y,t)\rangle + \langle j_{\text{int}}(y,t)\rangle = 0. \tag{6.17}$$

Thus, the expected energy flux vanishes:

$$\langle j(y,t)\rangle = 0. \tag{6.18}$$

We conclude that a system that undergoes uniform acceleration with respect to the vacuum of flat space–time does not radiate despite the fact that it does in fact thermalize at the Unruh temperature.

7. Concluding remarks

A system undergoing hyperbolic motion through a zero-temperature vacuum experiences a finite temperature, the Unruh temperature (3.6). This was pointed out by Davies [1] and Unruh [2]. Our explicit calculation for the scalar electrodynamics model verifies this for an oscillator in hyperbolic motion. The effect is real: the moving oscillator is in an equilibrium state identical with that of one at rest at the Unruh temperature, with a corresponding distribution over excited states.

This picture of a system at a finite temperature moving through a zero-temperature vacuum might lead one to expect that there would be energy radiated. However Grove [3] and Raine et al. [4] argued that this was not the case, there is no radiation. In agreement with them, our explicit model calculation shows that there is in fact no radiation of energy. The situation is exactly the same as that for a system at rest in a zero-temperature vacuum. The system is driven by the zero-point oscillations of the vacuum field while simultaneously radiating energy into the vacuum. But the driving force and the radiation reaction exactly balance, so the system remains in equilibrium with no net radiation of energy. For our simple model of an oscillator with scalar electrodynamics we show by explicit calculation that the net flux of radiant energy at a point in the field away from the oscillator is zero, for an oscillator at rest in Section 4 and for an oscillator in hyperbolic motion in Section 6. The fact that the argument is identical for an oscillator at rest and one in hyperbolic motion makes it difficult to escape the conclusion that, on very general grounds, there is no radiation in either case. You cannot have the one without the other.

We have seen in Section 4 that when the oscillator is excited by an external agent it will radiate, since the externally excited motion is uncorrelated with the fluctuations of the vacuum field. Unruh [5], in his response to the paper of Raine et al. [4], introduces a heat bath moving with the oscillator. This bath is assumed to be at the Unruh temperature and when it drives the oscillator there will be radiation, the bath acting as an external agent. We have serious reservations about this picture, but whatever its merits, it certainly does not represent the situation envisioned in the many proposals to observe the radiation, all of which involve a single particle or at most a single atomic system in accelerated motion. Moreover, Unruh places emphasis on "[...] the radiation [...] expected from the oscillator/heat bath coming into equilibrium with the thermal radiation in the far past" [5]. Next, Parentani [17] expands on this discussion and shows explicitly by an "[...] analysis of the transients when one switches off the interaction [...]" that such transients lead to radiation. However, during the switching-on and switching-off of the external force, we have a situation which is outside the realm of what is understood to be the basis for Unruh radiation.

Of course, hyperbolic motion is an idealization, with the force F applied over an infinite time. More realistically, one could assume that at some distant but finite time in the past the oscillator is impulsively accelerated into hyperbolic motion and the constant force is switched on. At that time there must be an exchange of energy with the field, but it would not be what one would call Unruh radiation. A description of this exchange is outside the range of the present discussion.

Our conclusion is that a system in hyperbolic motion through a zero-temperature vacuum does not radiate, despite the fact that it is in a state corresponding to the elevated Unruh temperature. We should point out that it has been argued by some authors [18,19] that this is an artifact of the model we have used. In particular, the interaction of a charged oscillator with the electromagnetic field was discussed by Vanzella and Matsas [18] and the authors conclude that there is radiation. However, we are skeptical since, as we have remarked above, the argument is essentially one of detailed balance: for a system in equilibrium the rate of emission of radiation is exactly balanced by a corresponding absorption, there is no net radiation. What we have done here is to demonstrate in detail that detailed balance holds for a system in hyperbolic motion exactly as it does for a system at rest at a finite temperature. It is difficult to believe that this principle is model-dependent.

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