Generalization of the Schott energy in electrodynamic radiation theory

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We discuss the origin of the Schott energy in the Abraham-Lorentz version of electrodynamic radiation theory and how it can be used to explain some apparent paradoxes. We also derive the generalization of this quantity for the Ford-O'Connell equation, which has the merit of being derived exactly from a microscopic Hamiltonian for an electron with structure and has been shown to be free of the problems associated with the Abraham-Lorentz theory. We emphasize that the instantaneous power supplied by the applied force not only gives rise to radiation (acceleration fields), but it can change the kinetic energy of the electron and change the Schott energy of the velocity fields. The important role played by boundary conditions is noted. © *2006 American Association of Physics Teachers.*

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I. INTRODUCTION

Since its introduction in 1912,[1] there has been much confusion concerning the nature of the Schott energy, despite the fact that some authors[2–4] have presented a clear explanation of its origin. The origin of the Schott energy arises from the fact that the power supplied by an external force to a charged particle not only contributes to the energy radiated (acceleration fields) but also to the velocity fields. This feature is not connected with the well-known deficiencies of the Abraham-Lorentz theory (for example, runaway solutions). Previous discussions of the Schott energy arose in the context of the Abraham-Lorentz equation of motion for a radiating electron.

In this paper we define a (generalized) Schott energy that is applicable not only to the Abraham-Lorentz theory but to all theories of a radiating electron. To do so, we start by recalling the classical Newtonian equation of motion for a particle of mass M under the action of an external field $\boldsymbol{f}(t)$:

$$M\boldsymbol{a}=\boldsymbol{f}, \tag{1}$$

where $\boldsymbol{a}$ is the acceleration. The only effect of $f(t)$ is to change the kinetic energy T of the particle, where

$$T=\tfrac{1}{2}Mv^2, \tag{2}$$

and v is the velocity. We stress these elementary facts because they are often overlooked in the development of radiation theories for a charged particle because for $\boldsymbol{f}(t)=0$, the equation of motion should reduce to $M\boldsymbol{a}=0$. In particular, this requirement is not obeyed by the Abraham-Lorentz equation (so that runaway solutions emerge) whereas it is obeyed by the Ford-O'Connell theory as we will discuss.

We consider an electron of charge e subject to an external force $\boldsymbol{f}(t)$. The total work done by $\boldsymbol{f}(t)$ during an arbitrary time interval consists of three parts (1) the change in kinetic energy ΔT (which is independent of e), (2) the radiated energy, which is the energy in the acceleration or far fields,[5] and (3) the change in energy in the velocity or near fields,[5] which does not give rise to radiation. This change can be positive or negative.

The energy in the velocity fields is the Schott energy. Thus, at any time t, the instantaneous power, $P(t)$, supplied by $\boldsymbol{f}(t)$ does not contribute just to the radiated energy. A feature of the Schott energy is that its time derivative appears in the expression for $P(t)$ so that "… if we… consider only intervals over which the system returns to its initial state, then the energy in the velocity fields is the same at both ends, and the only net loss is in the form of radiation … ."[4] That is, the Schott energy is the energy contributed to the velocity fields by the external field and does not contribute to the radiated energy (which is due to acceleration fields).

A related question is whether or not radiation can occur for constant acceleration (because the Larmor result for the radiated energy depends only on the acceleration squared whereas the radiation reaction term in the Abraham-Lorentz equation of motion depends on the rate of change of the acceleration). The solution to this apparent paradox is summarized succintly in Ref. 2 where it is noted that "… the radiated energy and the work of the radiation friction are not equal to each other in the nonstationary state," due again to the existence of the Schott energy. For that reason, it is desirable to consider energy exchange between the particle and the field at each instant of time, rather than using conservation laws integrated over time.

There has also been a long-standing recognition that the Abraham-Lorentz analysis has a fundamental flaw related to the existence of runaway solutions, which are a manifestation of the fact that causality is violated.[5] A solution to the latter problem was presented by Ford and O'Connell,[6–9] who pointed out the necessity of ascribing structure to the electron. Their solution led to a second-order equation of motion that is simple and well-behaved and incorporates quantum and fluctuation effects and the presence of a potential V. Here we confine ourselves to the nonrelativistic classical case with $V=0$, which is the case most often considered in the literature.[4,5]

In Sec. II we consider the generalization of the Schott energy for the radiation reaction force F_d (the subscript d indicates its dissipative nature) without specifying its specific form. A key feature of our analysis is that because the electron motion and the rate of radiation are continually chang-

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ing in time, we consider conservation of power [the power $P(t)$ supplied by the external force to the particle is equal to the rate of change of the particle's kinetic energy plus the rate of change of the velocity fields and the acceleration fields], as distinct from energy (which is integrated power). The latter gives less information and obscures the nonstationary aspect of the problem. As a result, we find that although the radiated power depends on $d\boldsymbol{f}/dt$, the integrated radiated power, that is, the radiated energy, does not. In Secs. III and IV we apply our general analysis to the Abraham-Lorentz and Ford-O'Connell theories and describe the physical nature of the generalized Schott terms. We also emphasize why the Ford-O'Connell theory is superior to the Abraham-Lorentz theory. Our conclusions are presented in Sec. V.

II. GENERAL EQUATION OF MOTION FOR A RADIATING ELECTRON

The equation of a radiating electron may be written in the form

$$M\boldsymbol{a} = \boldsymbol{f} + \boldsymbol{F}_d, \tag{3}$$

where $\boldsymbol{a}$ is the acceleration, $\boldsymbol{f}$ is the applied force, and $\boldsymbol{F}_d$ is a dissipative force arising from the back reaction due to the emitted radiation. All of these quantities are functions of time. The instantaneous power supplied by the external force $\boldsymbol{f}(t)$ to the electron is

$$P(t) = \boldsymbol{f}(t) \cdot \boldsymbol{v}(t), \tag{4}$$

where $\boldsymbol{v}(t)$ is the velocity. From Eq. (3), we obtain

$$P(t) = M\boldsymbol{a} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{F}_d \tag{5a}$$

$$= \frac{dT}{dt} - \boldsymbol{v} \cdot \boldsymbol{F}_d \equiv P_N(t) + P_d(t), \tag{5b}$$

where the kinetic energy T is given by Eq. (2). The rate of change of the kinetic energy of the electron, $P_N(t)$, arises from the application of Newton's law when $F_d=0$. Our main interest is in the $P_d(t)$ term and, as we will discuss, this term contributes not only to the radiated electromagnetic energy but also to the energy in the velocity fields.

The total work done by the external force during the time interval t_2-t_1 is

$$W \equiv W(t_1,t_2) = \int_{t_1}^{t_2} P(t')dt' \tag{6a}$$

$$= \int_{t_1}^{t_2} P_d(t')dt' + \Delta T \equiv W_d + \Delta T, \tag{6b}$$

where $\Delta T = T(t_2) - T(t_1)$ is the change in the kinetic energy. Thus $P_d(t)$ is the instantaneous power delivered to the fields by the external force (only part of which goes into radiated energy); when $P_d(t)=0$ there is no radiated energy. Note that W_d is the total integrated energy transmitted to the fields.

It is useful to write

$$W_d = W_v + W_R, \tag{7}$$

where from Eqs. (5) and (6),

$$W_d = \int_{t_1}^{t_2} P_d(t')dt' = -\int_{t_1}^{t_2} \boldsymbol{v}(t') \cdot \boldsymbol{F}_d(t')dt', \tag{8}$$

W_v is the work done on the velocity fields, and W_R is the radiated energy (associated with the acceleration fields). In other words, the total work done by the external field on the electron at time t consists of three parts. The total work changes the kinetic energy of the electron and concomitantly contributes both to the acceleration fields (which give rise to radiation) and the velocity fields (which do not give rise to radiation).

III. ABRAHAM-LORENTZ THEORY

The Abraham-Lorentz equation of motion[5,6] gives

$$\boldsymbol{F}_d = M\tau \frac{d\boldsymbol{a}}{dt}, \tag{9}$$

so that Eq. (3) reduces to

$$M\boldsymbol{a} = \boldsymbol{f} + M\tau \frac{d\boldsymbol{a}}{dt}, \tag{10}$$

where $\tau = 2e^2/(3Mc^3) = 6\times 10^{-24}$ s is proportional to the time it takes light to travel the classical radius of the electron. We see that when the acceleration is constant in the Abraham-Lorentz theory, $\boldsymbol{F}_d$ is zero and thus from Eq. (5) there is no radiated energy. More generally, from Eqs. (5) and (9) we obtain

$$P_d(t) = -M\tau\boldsymbol{v} \cdot \frac{d\boldsymbol{a}}{dt} \tag{11a}$$

$$= -M\tau\left[\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{a}) - a^2\right] = P_L - \frac{d}{dt}E_s, \tag{11b}$$

where

$$P_L = M\tau a^2 \tag{12}$$

is the familiar Larmor rate of radiation,[5] and

$$E_s = M\tau(\boldsymbol{v}\cdot\boldsymbol{a}) \tag{13}$$

is the Schott energy. Note that the total time derivative of E_s appears in the expression for $P_d(t)$. It follows that

$$W_d = \int_{t_1}^{t_2} P_L dt - \{E_s(t_2) - E_s(t_1)\} \tag{14a}$$

$$= \int_{t_1}^{t_2} P_L dt - M\tau\{\boldsymbol{v}(t_2)\cdot\boldsymbol{a}(t_2) - \boldsymbol{v}(t_1)\cdot\boldsymbol{a}(t_1)\}. \tag{14b}$$

Thus, if the accelerations are equal at times t_2 and t_1 then $W_v=0$ and $W_d=W_R$, the usual result for the radiated energy. Because the initial and find velocities are generally different, we see from Eq. (6) that $W=W_R+\Delta T$. The same scenario approximately occurs when t_1 and t_2 correspond to the times at which the applied force is zero and thus from Eqs. (3) and (9) the acceleration is of order τ and hence very small.

In addition, the Abraham-Lorentz equation (10) has serious problems. In particular, when $\boldsymbol{f}=0$, it is clear that Eq. (10) does not reduce to Newton's equation as it should, and consequently the well-known runaway solutions emerge. We now turn to the Ford-O'Connell theory which does not manifest this problem.

IV. FORD-O'CONNELL THEORY

The Ford-O'Connell theory[6–9] is based on a rigorous microscopic approach whose starting point is the universally accepted Hamiltonian of nonrelativistic quantum electrodynamics generalized to allow for electron structure.[10,11] The use of Heisenberg's equation of motion (or the corresponding Poisson equations of motion in the classical case) leads to an equation of motion that incorporates electron structure and quantum effects and an arbitrary potential V.[6] In the classical limit and for $V=0$, the Ford-O'Connell equation of motion reduces to the Abraham-Lorentz equation in the limit of a point particle (and thus, as a bonus, we have the first Hamiltonian derivation of the Abraham-Lorentz equation). More generally, electron structure is taken into account by incorporating a form factor[6,10] (the Fourier transform of the charge distribution), which is written in terms of a large cutoff frequency Ω. The point electron limit corresponds to $\Omega \to \infty$. More generally, small Ω implies an extended electron structure. As shown in Ref. 6, values of Ω larger than τ_e^{-1} lead to violation of causality. This violation shows that the problem with the Abraham-Lorentz theory arises from the assumption of a point electron. In addition, choosing $\Omega = \tau_e^{-1}$ (corresponding to the maximum allowed value of Ω and hence to the smallest electron structure consistent with causality), leads in the classical limit and for $V=0$ to

$$\boldsymbol{F}_d = \tau \frac{d\boldsymbol{f}}{dt}, \tag{15}$$

so that Eq. (1) becomes

$$M\boldsymbol{a} = \boldsymbol{f} + \tau \frac{d\boldsymbol{f}}{dt}. \tag{16}$$

Note that $\boldsymbol{F}_d$ depends on both the electron (through the factor τ) and the external field. This dependence contrasts with the corresponding result given by Eq. (9) for the Abraham-Lorentz theory, where the external force does not appear explicitly. Also in the Ford-O'Connell theory, when the applied force $\boldsymbol{f}(t)$ is constant, $\boldsymbol{F}_d$ is zero and thus from Eq. (3) we see that there is no radiation. It also follows that we can write

$$P_d(t) = -\tau\left(\boldsymbol{v}\cdot\frac{d\boldsymbol{f}}{dt}\right) = -\tau\left[\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{f}) - \boldsymbol{f}\cdot\boldsymbol{a}\right] \tag{17a}$$

$$= -\tau\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{f}) + \frac{\tau}{M}\left[f^2 + \frac{\tau}{2}\frac{d}{dt}f^2\right] \tag{17b}$$

$$= P_{\mathrm{FO}} - \frac{d}{dt}E_{\mathrm{FO}}, \tag{17c}$$

where

$$P_{\mathrm{FO}} = \frac{\tau}{M}f^2 \tag{18}$$

is the result obtained in Refs. 7 and 8 for the rate of radiation. In fact, Ford-O'Connell used two different derivations in obtaining Eq. (18), one based on energy conservation[7] and another based on a generalization of Larmor's derivation to include electron structure.[8] Also

$$E_{\mathrm{FO}} = \tau(\boldsymbol{v}\cdot\boldsymbol{f}) - \frac{\tau^2}{2M}f^2 \tag{19}$$

is the generalization of the Schott energy. It follows that the negative of the time derivative of the Schott energy is the power fed into the velocity fields by the external force. It immediately follows that the integrated power is given by

$$W_d = \int_{t_1}^{t_2} P_{\mathrm{FO}}dt - [E_{\mathrm{FO}}(t_2) - E_{\mathrm{FO}}(t_1)]. \tag{20}$$

We note that E_{FO} differs from E_s by terms of order τ^2 and that E_{FO} also appears as a total time derivative in the expression for the instantaneous power radiated. In contrast to E_s, E_{FO} vanishes exactly when the applied force is zero (a more physically appealing boundary condition than in the Abraham-Lorentz analysis), in which case $W_v=0$ and W_d is equal to the first term in Eq. (20), which is the result obtained in Refs. 7 and 8. Thus, for a constant external field, P_d is always zero except when the field is turned on and off, and it is then that energy is radiated with an average rate given by Eq. (18). We point out that Eq. (18) was obtained in Ref. 7 by integrating the equation of motion (16) and then using energy conservation. The same result was verified in Ref. 8 by generalizing Larmor's radiation theory to incorporate electron structure.

V. CONCLUSION

The Schott energy and its generalization corresponds to energy given to or taken from the velocity fields and always occurs as a total time derivative in the expression for the instantaneous power supplied by the external force. Thus the total work done by the applied force is only equal to the radiated energy plus the change in kinetic energy when the boundary conditions ensure that the change in the Schott energy (the energy of the velocity fields) is equal to zero during the time interval of interest. These conditions occur naturally in the Ford-O'Connell theory (as distinct from the Abraham-Lorentz theory) because $f(t)$ is zero at the initial and final times. Moreover, it is immediately clear from the Ford-O'Connell equation of motion (16), that when f =constant, Eq. (16) reduces to the Newtonian equation of motion (1). In other words, there is no radiation reaction term in the equation of motion reflecting the fact that there is no radiation when f=constant, a conclusion that also emerges from a relativistic generalization.[12] This result is also consistent with the conclusion[13] that an oscillator moving under a constant force with respect to the zero-temperature vacuum does not radiate despite the fact that it thermalizes at the Unruh temperature. Finally, we remark that when quantum effects are taken into account, there are additional fluctuating force terms in the equation of motion.[14]

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