ANNALS OF PHYSICS 129, 358–377 (1980)

The Post-Post-Newtonian Problem in Classical Electromagnetic Theory

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We find the Lagrangian to order c^{-4} for two charged bodies (with $e_1/m_1 = e_2/m_2$) in electromagnetic theory. This Lagrangian contains acceleration terms in its final form and we show why it is incorrect to eliminate these terms by using the equations of motion in the Lagrangian as was done by Golubenkov and Smorodinskii, and by Landau and Lifshitz. We find the center of inertia and show that the potential energy term does not split equally between particles 1 and 2 as it does in the Darwin Lagrangian (Lagrangian to order c^{-2}). In addition to the infinite self-energy terms in the electromagnetic energy-momentum tensor, which are eliminated using Gupta's method, some new type of divergent terms are found in the moment of electromagnetic field energy and in the electromagnetic field momentum which cancel in the final conservation law for the center of inertia.

I. INTRODUCTION

For certain n-body Lagrangians (to order c^{-2} and in standard coordinates [1]), such as the Darwin (pure electromagnetism) or Einstein–Infeld–Hoffmann (pure gravitation) or Bażański (gravitation and electromagnetism) Lagrangians, it is well known [2, 3, 4, 5] that in finding the center of inertia the potential energy terms $-Gm_im_j/r_{ij}$ and e_ie_j/r_{ij} must be split equally between the particles i and j. We have recently shown [6] that this $\frac{1}{2}$, $\frac{1}{2}$ split, as we shall call it, also holds for the case of the gravitational n-body Lagrangian (to order c^{-2} and in stqndard coordinates [1]) with parameterized post-Newtonian (PPN) parameters γ and β. We have also shown [6] that the $\frac{1}{2}$, $\frac{1}{2}$ split holds only for certain coordinate systems. We have found [6], for the case of the Bazanski Lagrangian, coordinate systems where something other than the $\frac{1}{2}$, $\frac{1}{2}$ split occurs.

In this paper we shall turn our attention to the two-body post-post-Newtonian (i.e., to order c^{-4}) Lagrangian in electromagnetism. In order to postpone dipole radiation from the c^{-3} to the c^{-5} order we must require [7] that $e_1/m_1 = e_2/m_2$. If we were doing the n-body problem we would have to require that the charge to mass ratio for all the particles was the same. As we are dealing with pure electromagnetism

0003-4916/80/120358–20$05.00/0

(i.e., there is no gravitation) we can and shall use only *Cartesian coordinates*. In Section II we shall derive this Lagrangian which contains accelerations terms in its final form. Golubenkov and Smorodinskii [8], and Landau and Lifshitz [9] have given an *incorrect* form of this Lagrangian by using the improper procedure of using the equations of motion in the Lagrangian to eliminate the acceleration terms. *Using the equations of motion in the Lagrangian changes its functional form and, hence, leads in most cases (including the above) to different and, thus, incorrect equations of motion.*

In Section III we discuss Lagrange's equations and the results for the conserved energy, momentum, and angular momentum for our acceleration-dependent Lagrangian. The center of inertia result, $d[\mathscr{E}\mathbf{r}_{\mathrm{CI}}/c^2]/dt =$ momentum, is also given and checked in this section. The results of Sections IV–VII are used to find $\mathscr{E}\mathbf{r}_{\mathrm{CI}}$. This term is not found from the Lagrangian and a guess that it would be given by the $\frac{1}{2}$, $\frac{1}{2}$ split is shown to be incorrect.

In Section IV we find the electromagnetic field energy and in Section V we find the moment of electromagnetic field energy. The infinite self-energy terms in the electromagnetic energy–momentum tensor are eliminated by the method of Gupta [10], which is consistent with Dirac's [11] equations of motion. It was expected that $\mathscr{E}\mathbf{r}_{\mathrm{CI}}$ would be equal to the moment of particle energy + the moment of electromagnetic field energy. However, a new type of divergent term was found in the moment of electromagnetic field energy and its removal had to be justified before the finite $\mathscr{E}\mathbf{r}_{\mathrm{CI}}$ could be obtained. In Section VI conservation laws are given and in Section VII these new type of divergent terms are shown to cancel out in the conservation laws. We give our conclusions in Section VIII. Non-standard forms of the Darwin Lagrangian are included in Appendix A. In Appendix B, using a simple Lagrangian as an example, we demonstrate that using the equations of motion in the Lagrangian to eliminate the higher-order acceleration terms leads to different equations of motion and is thus an incorrect procedure.

II. Lagrangian to Order c^{-4}

The exact Lagrangian for particle 1 in an electromagnetic field (we are using Gaussian units) is

$$\mathscr{L}_1 = -m_1c^2(1 - v_1^2/c^2)^{1/2} + e_1\mathbf{v}_1 \cdot \mathbf{A}_{\mathrm{T}}/c - e_1\phi_{\mathrm{T}}, \tag{1}$$

where $\mathbf{A}_{\mathrm{T}}(\mathbf{r}_1, t)$ and $\phi_{\mathrm{T}}(\mathbf{r}_1, t)$ are the total potentials. Since we are only considering the two-body problem, the total potentials consist of the retarded potentials due to particle 2 and the self potentials [10] due to particle 1. We thus have

$$\phi_{\mathrm{T}} = \phi_2 + \phi_{\mathrm{S}}, \qquad \mathbf{A}_{\mathrm{T}} = \mathbf{A}_2 + \mathbf{A}_{\mathrm{S}}, \tag{2}$$

where

$$\phi_2 = \phi_{2\mathrm{ret}}, \qquad \phi_{\mathrm{S}} = \tfrac{1}{2}(\phi_{1\mathrm{ret}} - \phi_{1\mathrm{adv}}), \tag{3}$$

$$\mathbf{A}_2 = \mathbf{A}_{2\mathrm{ret}}, \qquad \mathbf{A}_{\mathrm{S}} = \tfrac{1}{2}(\mathbf{A}_{1\mathrm{ret}} - \mathbf{A}_{1\mathrm{adv}}). \tag{4}$$

A. *Expansion of the Potentials*

The retarded and advanced potentials in the Lorentz gauge at the field point $\mathbf{r}_1$ due to a particle i at position $\mathbf{r}_i$ are

$$\phi(\mathbf{r}_1, t) = \int \frac{\rho_i[\mathbf{x}', (t \mp |\mathbf{r}_1 - \mathbf{x}'|/c)]}{|\mathbf{r}_1 - \mathbf{x}'|} dV', \tag{5}$$

$$\mathbf{A}(\mathbf{r}_1, t) = \frac{1}{c} \int \frac{\mathbf{j}_i[\mathbf{x}', (t \mp |\mathbf{r}_1 - \mathbf{x}'|/c)]}{|\mathbf{r}_1 - \mathbf{x}'|} dV', \tag{6}$$

where the top (bottom) signs corresponds to the ret (adv) solutions and

$$\rho_i(\mathbf{x}', t) = e_i \delta(\mathbf{r}_i - \mathbf{x}'), \tag{7}$$

$$\mathbf{j}_i(\mathbf{x}', t) = e_i \mathbf{v}_i \delta(\mathbf{r}_i - \mathbf{x}'), \tag{8}$$

where ρ_i, $\mathbf{j}_i$ and e_i are the charge density, current density, and charge of particle i. Expanding Eqs. (5) and (6) in powers of c^{-1} (to order c^{-4} in ϕ and $\mathbf{A}/c$) we obtain

$$\begin{aligned}\phi(\mathbf{r}_1, t) = &\int \frac{\rho_i(\mathbf{x}', t)}{|\mathbf{r}_1 - \mathbf{x}'|} dV' \mp \frac{1}{c}\frac{\partial}{\partial t} \int \rho_i(\mathbf{x}', t)\, dV' \\ &+ \frac{1}{2c^2}\frac{\partial^2}{\partial t^2} \int |\mathbf{r}_1 - \mathbf{x}'|\, \rho_i(\mathbf{x}', t)\, dV' \mp \frac{1}{6c^3}\frac{\partial^3}{\partial t^3} \int (\mathbf{r}_1 - \mathbf{x}')^2 \rho_i(\mathbf{x}', t)\, dV' \\ &+ \frac{1}{24c^4}\frac{\partial^4}{\partial t^4} \int |\mathbf{r}_1 - \mathbf{x}'|^3 \rho_i(\mathbf{x}', t)\, dV', \end{aligned} \tag{9}$$

$$\begin{aligned}\mathbf{A}(\mathbf{r}_1, t) = &\frac{1}{c} \int \frac{\mathbf{j}_i(\mathbf{x}', t)}{|\mathbf{r}_1 - \mathbf{x}'|} dV' \mp \frac{1}{c^2}\frac{\partial}{\partial t} \int \mathbf{j}_i(\mathbf{x}', t)\, dV' \\ &+ \frac{1}{2c^3}\frac{\partial^2}{\partial t^2} \int |\mathbf{r}_1 - \mathbf{x}'|\, \mathbf{j}_i(\mathbf{x}', t)\, dV'. \end{aligned} \tag{10}$$

Next, using Eq. (7) in Eq. (9) and Eq. (8) in Eq. (10) we get

$$\phi(\mathbf{r}_1, t) = \frac{e_i}{|\mathbf{r}_1 - \mathbf{r}_i|} + \frac{e_i}{2c^2}\frac{\partial^2 |\mathbf{r}_1 - \mathbf{r}_i|}{\partial t^2} \mp \frac{e_i}{6c^3}\frac{\partial^3 (\mathbf{r}_1 - \mathbf{r}_i)^2}{\partial t^3} + \frac{e_i}{24c^4}\frac{\partial^4 |\mathbf{r}_1 - \mathbf{r}_i|^3}{\partial t^4}, \tag{11}$$

$$\mathbf{A}(\mathbf{r}_1, t) = \frac{e_i \mathbf{v}_i}{c|\mathbf{r}_1 - \mathbf{r}_i|} \mp \frac{e_i}{c^2}\frac{\partial \mathbf{v}_i}{\partial t} + \frac{e_i}{2c^3}\frac{\partial^2 (\mathbf{v}_i |\mathbf{r}_1 - \mathbf{r}_i|)}{\partial t^2}, \tag{12}$$

where $\partial \mathbf{r}_1/\partial t = 0$ and $\partial \mathbf{r}_i/\partial t = \mathbf{v}_i$ since in the expansion $\mathbf{r}_1$ is held fixed while $\mathbf{r}_i$ is not [12]. We have a notational problem for the self-potentials when $i = 1$. What we shall do is write $\mathbf{r}_1^*$, $\mathbf{v}_1^*$, $\mathbf{a}_1^*$, for $\mathbf{r}_i$, $\mathbf{v}_i$, $\mathbf{a}_i$ when $i = 1$. The operator $\text{grad}_{(1)}$ will operate on $\mathbf{r}_1$ but not on $\mathbf{r}_1^*$. Let us also define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$. We then have

$$\phi_T(\mathbf{r}_1, t) = \frac{e_2}{r} + \frac{e_2}{2c^2}\frac{\partial^2 r}{\partial t^2} - \frac{e_2}{6c^3}\frac{\partial^3 r^2}{\partial t^3} + \frac{e_2}{24c^4}\frac{\partial^4 r^3}{\partial t^4} - \frac{e_1}{6c^3}\frac{\partial^3 (\mathbf{r}_1 - \mathbf{r}_1^*)^2}{\partial t^3}, \tag{13}$$

$$\mathbf{A}_T(\mathbf{r}_1, t) = \frac{e_1 \mathbf{v}_2}{cr} - \frac{e_2 \mathbf{a}_2}{c^2} + \frac{e_2}{2c^3}\frac{\partial^2 (r\mathbf{v}_2)}{\partial t^2} - \frac{e_1 a_1^*}{c^2}. \tag{14}$$

B. *Gauge Transformation*

We shall now go to the Coulomb gauge by making the gauge transformation

$$\phi_T' = \phi_T - \frac{1}{c}\frac{\partial \Lambda_T}{\partial t}, \quad \mathbf{A}_T' = \mathbf{A}_T + \mathrm{grad}_{(1)} \Lambda_T, \tag{15}$$

$$\Lambda_T = \frac{e_2}{2c}\frac{\partial r}{\partial t} - \frac{e_2}{6c^2}\frac{\partial^2 r^2}{\partial t^2} + \frac{e_2}{24c^3}\frac{\partial^3 r^3}{\partial t^3} - \frac{e_1}{6c^2}\frac{\partial^2(\mathbf{r}_1 - \mathbf{r}_1^*)^2}{\partial t^2}. \tag{16}$$

We then have

$$\phi_T'(\mathbf{r}_1, t) = e_2/r, \tag{17}$$

$$\begin{aligned} \mathbf{A}_T'(\mathbf{r}_1, t) = {} & \frac{e_2\mathbf{v}_2}{cr} + \frac{e_2}{2c}\frac{\partial}{\partial t}[\mathrm{grad}_{(1)} r] \\ & - \frac{e_2\mathbf{a}_2}{c^2} - \frac{e_2}{6c^2}\frac{\partial^2}{\partial t^2}[\mathrm{grad}_{(1)} r^2] \\ & - \frac{e_1\mathbf{a}_1^*}{c^2} - \frac{e_1}{6c^2}\frac{\partial^2}{\partial t^2}[\mathrm{grad}_{(1)}(\mathbf{r}_1 - \mathbf{r}_1^*)^2] \\ & + \frac{e_2}{2c^3}\frac{\partial^2(r\mathbf{v}_2)}{\partial t^2} + \frac{e_2}{24c^3}\frac{\partial^3}{\partial t^3}[\mathrm{grad}_{(1)} r^3]. \end{aligned} \tag{18}$$

After evaluating the terms involving $\mathrm{grad}_{(1)}$, Eq. (18) can be written as

$$\begin{aligned} \frac{1}{c}\mathbf{A}_T'(\mathbf{r}_1, t) = {} & \frac{e_2\mathbf{v}_2}{2c^2 r} + \frac{e_2(\mathbf{v}_2\cdot\mathbf{r})\,\mathbf{r}}{2c^2r^3} - \frac{2}{3}\frac{e_2\mathbf{a}_2}{c^3} - \frac{2}{3}\frac{e_1\mathbf{a}_1^*}{c^3} \\ & + \frac{e_2}{2c^4}\left[\frac{\partial^2(r\mathbf{v}_2)}{\partial t^2} + \frac{1}{4}\frac{\partial^3(r\mathbf{r})}{\partial t^3}\right]. \end{aligned} \tag{19}$$

The new Lagrangian

$$\mathscr{L}_1' = -m_1c^2(1 - v_1^2/c^2)^{1/2} + e_1\mathbf{v}_1\cdot\mathbf{A}_T'/c - e_1\phi_T', \tag{20}$$

differs from the old Lagrangian of Eq. (1) by a total time derivative as

$$\mathscr{L}_1' = \mathscr{L}_1 + d(e_1\Lambda_T/c)/dt. \tag{21}$$

The equations of motion for particle 1 are

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}_1'}{\partial\mathbf{v}_1}\right) - \frac{\partial\mathscr{L}_1'}{\partial\mathbf{r}_1} = 0, \tag{22}$$

since the potentials are regarded as functions of $\mathbf{r}_1$ and t only [i.e., we do *not* have a term $-d^2(\partial\mathscr{L}_1'/\partial\mathbf{a}_1^*)/dt^2$ added to the left-hand side of Eq. (22)]. Using

$$\mathbf{E}_T = -\mathrm{grad}_{(1)}\phi_T' - (\partial\mathbf{A}_T'/\partial t)/c, \tag{23}$$

$$\mathbf{H}_T = \mathrm{curl}_{(1)}\,\mathbf{A}_T', \tag{24}$$

together with Eq. (22) we obtain

$$\frac{d}{dt}\left[\frac{m_1\mathbf{v}_1}{(1-v_1{}^2/c^2)^{1/2}}\right] = e_1\mathbf{E}_T + e_1\mathbf{v}_1 \times \mathbf{H}_T. \tag{25}$$

However, if we use

$$\mathbf{E}_2 = -\mathrm{grad}_{(1)}\phi_2' - (\partial\mathbf{A}_2'/\partial t)/c, \tag{26}$$

$$\mathbf{H}_2 = \mathrm{curl}_{(1)}\,\mathbf{A}_2', \tag{27}$$

where

$$\phi_T' = \phi_2' + \phi_S', \qquad \mathbf{A}_T' = \mathbf{A}_2' + \mathbf{A}_S', \tag{28}$$

together with Eq. (22) we obtain

$$\frac{d}{dt}\left[\frac{m_1\mathbf{v}_1}{(1-v_1{}^2/c^2)^{1/2}}\right] = e_1\mathbf{E}_2 + e_1\mathbf{v}_1 \times \mathbf{H}_2 + \frac{2}{3}\frac{e_1{}^2\dot{\mathbf{a}}_1^*}{c^3}, \tag{29}$$

which is in the form of Dirac's equation [11] (to the order that we have expanded the potentials). At this stage the asterisk can be removed from $\dot{\mathbf{a}}_1^*$ in Eq. (29).

We now require that $e_1/m_1 = e_2/m_2$. Then using the equations of motion where $m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = \mathcal{O}(c^{-2})$ we find that the c^{-3} term in Eq. (19) is actually of order c^{-5} and can be neglected. We can now, using Eqs. (17) and (19), split our Lagrangian of Eq. (20) into two parts (to order c^{-4}) as

$$\mathscr{L}_1' = \mathscr{L}_{1D}' + \mathscr{L}_{1(4)}', \tag{30}$$

where

$$\mathscr{L}_{1D}' = -m_1c^2 + \frac{1}{2}m_1v_1{}^2 + \frac{1}{8}m_1v_1{}^4/c^2 - \frac{e_1e_2}{r}\left[1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{2c^2} - \frac{(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})}{2c^2r^2}\right], \tag{31}$$

$$\begin{aligned}\mathscr{L}_{1(4)}' &= \frac{1}{16}m_1v_1{}^6/c^4 + \frac{e_1e_2}{2c^4}\left[\frac{\partial^2(r\mathbf{v}_2)}{\partial t^2} + \frac{1}{4}\frac{\partial^3(r\mathbf{r})}{\partial t^3}\right]\cdot\mathbf{v}_1 \\ &= \frac{1}{16}m_1v_1{}^6/c^4 + \frac{e_1e_2}{8c^4}\frac{\partial\mathbf{F}_2}{\partial t}\cdot\mathbf{v}_1,\end{aligned} \tag{32}$$

and $\mathbf{F}_2$ is defined as

$$\mathbf{F}_2 \equiv \frac{\partial}{\partial t}\left[3r\mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{r}}{r}\mathbf{r}\right]. \tag{33}$$

C. *Addition of Total Time Derivative*

Let us now define

$$\mathscr{L}''_{1(4)} \equiv \mathscr{L}'_{1(4)} - \frac{d}{dt}\left[\frac{e_1 e_2}{8c^4}(\mathbf{v}_1 \cdot \mathbf{F}_2)\right], \tag{34}$$

as well as

$$\mathscr{L}''_1 \equiv \mathscr{L}'_{1\mathrm{D}} + \mathscr{L}''_{1(4)} . \tag{35}$$

Noting that

$$\frac{d(\mathbf{v}_1 \cdot \mathbf{F}_2)}{dt} = \mathbf{a}_1 \cdot \mathbf{F}_2 + \mathbf{v}_1 \cdot \frac{\partial \mathbf{F}_2}{\partial t} + \mathbf{v}_1 \cdot \mathrm{grad}_{(1)}(\mathbf{v}_1 \cdot \mathbf{F}_2), \tag{36}$$

we obtain

$$\mathscr{L}''_{1(4)} = \frac{1}{16} m_1 v_1{}^6/c^4 - \frac{e_1 e_2}{8c^4}[\mathbf{v}_1 \cdot \mathrm{grad}_{(1)}(\mathbf{v}_1 \cdot \mathbf{F}_2) + \mathbf{F}_2 \cdot \mathbf{a}_1]. \tag{37}$$

The Lagrangian $\mathscr{L}_1$ contains $\mathbf{a}_2$, $\dot{\mathbf{a}}_2$ and $\ddot{\mathbf{a}}_2$ terms; the Lagrangian $\mathscr{L}'_1$ contains $\mathbf{a}_2$ and $\dot{\mathbf{a}}_2$ terms; and the Lagrangian $\mathscr{L}''_1$ contains $\mathbf{a}_1$ and $\mathbf{a}_2$ terms.

D. *Two-Body Lagrangian*

After a lengthy calculation the [ ] term in Eq. (37) turns out to be symmetrical [13] under interchange of particle indices 1 and 2. Because of this, it is possible to obtain the two-body Lagrangian $\mathscr{L}''$ (good for particle 2 as well as particle 1) by adding $-m_2c^2(1 - v_2{}^2/c^2)^{1/2}$ to $\mathscr{L}''_1$. We have (to order c^{-4})

$$\mathscr{L}'' = \mathscr{L}'_{\mathrm{D}} + \mathscr{L}''_{(4)} , \tag{38}$$

where

$$\begin{aligned} \mathscr{L}'_{\mathrm{D}} = &- m_1c^2 + \frac{1}{2} m_1 v_1{}^2 + \frac{1}{8} m_1 v_1{}^4/c^2 - m_2c^2 + \frac{1}{2} m_2 v_2{}^2 + \frac{1}{8} m_2 v_2{}^4/c^2 \\ &- \frac{e_1 e_2}{r}\left[1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{2c^2} - \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{2c^2 r^2}\right], \end{aligned} \tag{39}$$

$$\begin{aligned} \mathscr{L}''_{(4)} = &\frac{1}{16} m_1 v_1{}^6/c^4 + \frac{1}{16} m_2 v_2{}^6/c^4 + \frac{e_1 e_2}{8c^4}\left[\frac{2(\mathbf{v}_1 \cdot \mathbf{v}_2)^2}{r} - \frac{v_1{}^2 v_2{}^2}{r}\right. \\ &+ \frac{(\mathbf{v}_1 \cdot \mathbf{r})^2\, v_2{}^2}{r^3} + \frac{(\mathbf{v}_2 \cdot \mathbf{r})^2\, v_1{}^2}{r^3} - \frac{3(\mathbf{v}_1 \cdot \mathbf{r})^2\,(\mathbf{v}_2 \cdot \mathbf{r})^2}{r^5} - \frac{2(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_1 \cdot \mathbf{a}_2)}{r} \\ &+ \frac{2(\mathbf{v}_2 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{a}_1)}{r} + \frac{v_1{}^2(\mathbf{a}_2 \cdot \mathbf{r})}{r} - \frac{v_2{}^2(\mathbf{a}_1 \cdot \mathbf{r})}{r} - \frac{(\mathbf{v}_1 \cdot \mathbf{r})^2\,(\mathbf{a}_2 \cdot \mathbf{r})}{r^3} \\ &\left.+ \frac{(\mathbf{v}_2 \cdot \mathbf{r})^2\,(\mathbf{a}_1 \cdot \mathbf{r})}{r^3} - 3r(\mathbf{a}_1 \cdot \mathbf{a}_2) + \frac{(\mathbf{a}_1 \cdot \mathbf{r})(\mathbf{a}_2 \cdot \mathbf{r})}{r}\right]. \end{aligned} \tag{40}$$

The Lagrangian $\mathscr{L}'_{\rm D}$ is the Darwin [14] Lagrangian in standard form. In Appendix A we shall discuss the Darwin Lagrangian in non-standard form. The fourth-order term in the Lagrangian $\mathscr{L}''_{(4)}$ (for the equal mass-equal charge n-body case) has been given by Golubenkov and Smordinskii [8] with, however, some serious sign misprints.

Previously in this section we used the equations of motion to show that the c^{-3} term in Eq. (19) was of order c^{-5}. Since we would have obtained the same result if we had waited to show this in the equations of motion of Eq. (29), what we did was in order.

III. Results from the Lagrangian

A. *Acceleration-Dependent n-Body Lagrangian*

Let us consider an n-body Lagrangian of the form $\mathscr{L} = \mathscr{L}(\mathbf{r}_{ij}, \mathbf{v}_k, \mathbf{a}_l)$, where $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ and $\mathscr{L}$ is a scalar in three dimensions. Let us also define

$$\mathbf{P}_i \equiv \partial\mathscr{L}/\partial\mathbf{v}_i, \tag{41}$$

$$\mathbf{\Phi}_i \equiv \partial\mathscr{L}/\partial\mathbf{a}_i, \tag{42}$$

$$\mathbf{\Pi}_i \equiv \mathbf{P}_i - \dot{\mathbf{\Phi}}_i, \tag{43}$$

$$\boldsymbol{\lambda}_i \equiv \mathbf{r}_i \times \mathbf{\Pi}_i + \mathbf{v}_i \times \mathbf{\Phi}_i. \tag{44}$$

Then, extending the derivations of Landau and Lifshitz [15] to include acceleration terms [16] it is quite easy to verify the following. The equations of motion are

$$\frac{\partial\mathscr{L}}{\partial\mathbf{r}_i} - \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{v}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{a}_i}\right) = 0, \tag{45}$$

which can also be put in the form

$$\dot{\mathbf{\Pi}}_i = \partial\mathscr{L}/\partial\mathbf{r}_i. \tag{46}$$

The total energy

$$\mathscr{E} = \sum_{i=1}^{n} (\mathbf{\Pi}_i \cdot \mathbf{v}_i + \mathbf{\Phi}_i \cdot \mathbf{a}_i) - \mathscr{L} \tag{47}$$

is conserved since $\mathscr{L}$ is not an explicit function of time. The total momentum

$$\mathbf{\Pi} = \sum_{i=1}^{n} \mathbf{\Pi}_i \tag{48}$$

is conserved since $\mathscr{L}$ is a function of the *differences* in coordinates $\mathbf{r}_{ij}$. The total angular momentum

$$\boldsymbol{\lambda} = \sum_{i=1}^{n} \boldsymbol{\lambda}_i \tag{49}$$

is conserved since $\mathscr{L}$ is a scalar in three dimensions.

The two-body Lagrangian $\mathscr{L}''$ of Eq. (38) is clearly a scalar in three dimentions and is not an explicit function of time. Thus, Eqs. (41)–(49) with $n = 2$ apply to $\mathscr{L}''$. Since $\mathbf{P}_i$ and $\mathbf{\Phi}_i$ are explicitly defined there is no problem in finding $\mathscr{E}$, $\mathbf{\Pi}$, or $\boldsymbol{\lambda}$.

B. *Center of Inertia*

We wish to find the center of inertia $\mathbf{r}_{\text{CI}}$ corresponding to the Lagrangian $\mathscr{L}''$ of Eq. (38). The center of inertia must satisfy the equation

$$d(\mathscr{E}\mathbf{r}_{\text{CI}}/c^2)/dt = \mathbf{\Pi}, \tag{50}$$

which implies that $(\mathscr{E}/c^2)\,\mathbf{v}_{\text{CI}} = \mathbf{\Pi}$ and $\mathbf{a}_{\text{CI}} = 0$ since $\mathscr{E}$ and $\mathbf{\Pi}$ are conserved quantities. The momentum $\mathbf{\Pi}$ on the right-hand side of Eq. (50) must be found to order c^{-4} and can be evaluated directly from $\mathscr{L}''$. The quantity $\mathscr{E}\mathbf{r}_{\text{CI}}$ on the left-hand side of Eq. (50) must be found to order c^{-2}.

The energy $\mathscr{E}$ to order c^{-2} can be expressed as

$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2, \tag{51}$$

where

$$\mathscr{E}_1 = m_1c^2 + \frac{1}{2}m_1v_1^2 + \frac{3}{8}m_1v_1^4/c^2 + \frac{e_1e_2}{2r}\left[1 + \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{2c^2} + \frac{(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})}{2c^2r^2}\right], \tag{52}$$

$$\mathscr{E}_2 = m_2c^2 + \frac{1}{2}m_2v_2^2 + \frac{3}{8}m_2v_2^4/c^2 + \frac{e_1e_2}{2r}\left[1 + \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{2c^2} + \frac{(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})}{2c^2r^2}\right], \tag{53}$$

and it should be noted that the interaction energy of $\mathscr{E}$ has been equally split between $\mathscr{E}_1$ and $\mathscr{E}_2$. We call this the $\frac{1}{2}$, $\frac{1}{2}$ split. A guess that $\mathscr{E}\mathbf{r}_{\text{CI}}$ might be equal to $\mathscr{E}_1\mathbf{r}_1 + \mathscr{E}_2\mathbf{r}_2$ turns out to be incorrect. From the results of Sections IV–VII, it turns out that

$$\mathscr{E}\mathbf{r}_{\text{CI}} = \mathscr{E}_1\mathbf{r}_1 + \mathscr{E}_2\mathbf{r}_2 - \mathbf{\Phi}_1c^2 - \mathbf{\Phi}_2c^2 + (e_1e_2/4c^2r)[(\mathbf{v}_2\cdot\mathbf{r})\,\mathbf{v}_1 - (\mathbf{v}_1\cdot\mathbf{r})\,\mathbf{v}_2]. \tag{54}$$

The $\frac{1}{2}$, $\frac{1}{2}$ split part is due to order c^2, c^0 and c^{-2} particle terms, order c^0 electric terms, and order c^{-2} magnetic terms. The $-\mathbf{\Phi}_1c^2 - \mathbf{\Phi}_2c^2$ part is due to c^{-2} electric terms. The remaining part is due to c^{-2} magnetic terms. We have explicitly verified that Eq. (54) satisfies Eq. (50). In order to check this it is necessary to use the equations of motion to eliminate acceleration terms that occur in Eq. (50).

The equations of motion (to needed order c^{-2}) can be put in the form

$$m_1\mathbf{a}_1 = e_1e_2\mathbf{r}/r^3 + (e_1e_2/2c^2r^3)\{2(\mathbf{v}_1\cdot\mathbf{r})\,\mathbf{v}_2 - 2(\mathbf{v}_1\cdot\mathbf{r})\,\mathbf{v}_1 + [v_2^2 - v_1^2 - 2\mathbf{v}_1\cdot\mathbf{v}_2 - 3(\mathbf{v}_2\cdot\mathbf{r})^2/r^2]\,\mathbf{r}\} + e_1^2e_2^2\mathbf{r}/m_2c^2r^4, \tag{55}$$

with a similar result for $m_2\mathbf{a}_2$ which can be obtained from Eq. (55) by interchanging indicies 1 and 2 (Note $\mathbf{r} \to -\mathbf{r}$). Let us also put

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_{(0)} + \boldsymbol{\Pi}_{(2)} + \boldsymbol{\Pi}_{(4)}\,, \tag{56}$$

where the subscripts indicate the order of c^{-1}. We shall use the same notation in the expansion of $\mathbf{P}$ and $\boldsymbol{\Phi}$, where $\mathbf{P} \equiv \mathbf{P}_1 + \mathbf{P}_2$ and $\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2$. We then find

$$\boldsymbol{\Pi}_{(0)} = \mathbf{P}_{(0)} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2\,, \tag{57}$$

$$\begin{aligned}\boldsymbol{\Pi}_{(2)} = \mathbf{P}_{(2)} &= \tfrac{1}{2}m_1\mathbf{v}_1 v_1^2/c^2 + \tfrac{1}{2}m_2\mathbf{v}_2 v_2^2/c^2 \\ &+ (e_1e_2/2c^2r)[\mathbf{v}_1 + \mathbf{v}_2 + (\mathbf{v}_1\cdot\mathbf{r})\,\mathbf{r}/r^2 + (\mathbf{v}_2\cdot\mathbf{r})\,\mathbf{r}/r^2],\end{aligned} \tag{58}$$

$$\begin{aligned}\boldsymbol{\Pi}_{(4)} &= \tfrac{3}{8}m_1\mathbf{v}_1 v_1^4/c^4 + \tfrac{3}{8}m_2\mathbf{v}_2 v_2^4/c^4 + (e_1e_2/8r^3c^4)\{r^2[2\mathbf{v}_1\cdot\mathbf{v}_2(\mathbf{v}_1+\mathbf{v}_2) \\ &+ (v_1^2\mathbf{v}_1 + v_2^2\mathbf{v}_2 - v_2^2\mathbf{v}_1 - v_1^2\mathbf{v}_2)] + [(\mathbf{v}_2\cdot\mathbf{r})^2 - (\mathbf{v}_1\cdot\mathbf{r})^2](\mathbf{v}_1 - \mathbf{v}_2) \\ &+ 2(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})(\mathbf{v}_1+\mathbf{v}_2) + [v_1^2(\mathbf{v}_2\cdot\mathbf{r}) + v_2^2(\mathbf{v}_1\cdot\mathbf{r}) + 3v_1^2(\mathbf{v}_1\cdot\mathbf{r}) \\ &+ 3v_2^2(\mathbf{v}_2\cdot\mathbf{r}) - 2(\mathbf{v}_1\cdot\mathbf{v}_2)(\mathbf{v}_1\cdot\mathbf{r} + \mathbf{v}_2\cdot\mathbf{r}) - 3(\mathbf{v}_1\cdot\mathbf{r})(\mathbf{v}_2\cdot\mathbf{r})^2/r^2 \\ &- 3(\mathbf{v}_2\cdot\mathbf{r})(\mathbf{v}_1\cdot\mathbf{r})^2/r^2 - 3(\mathbf{v}_1\cdot\mathbf{r})^3/r^2 - 3(\mathbf{v}_2\cdot\mathbf{r})^3/r^2]\,\mathbf{r}\} \\ &+ (e_1^2e_2^2/4m_2r^4c^4)[2r^2(\mathbf{v}_2 - \mathbf{v}_1) + 5(\mathbf{v}_1\cdot\mathbf{r})\,\mathbf{r} - (\mathbf{v}_2\cdot\mathbf{r})\,\mathbf{r}] \\ &+ (e_1^2e_2^2/4m_1r^4c^4)[2r^2(\mathbf{v}_1 - \mathbf{v}_2) + 5(\mathbf{v}_2\cdot\mathbf{r})\,\mathbf{r} - (\mathbf{v}_1\cdot\mathbf{r})\,\mathbf{r}].\end{aligned} \tag{59}$$

Let us also note that

$$\boldsymbol{\Pi} = \mathbf{P}_1 + \mathbf{P}_2 - \dot{\boldsymbol{\Phi}}_1 - \dot{\boldsymbol{\Phi}}_2\,, \tag{60}$$

where

$$\boldsymbol{\Phi}_1 = (e_1e_2/8c^4r)[2(\mathbf{v}_2\cdot\mathbf{r})\,\mathbf{v}_2 - v_2^2\mathbf{r} + (\mathbf{v}_2\cdot\mathbf{r})^2\,\mathbf{r}/r^2 - 3r^2\mathbf{a}_2 + (\mathbf{a}_2\cdot\mathbf{r})\,\mathbf{r}], \tag{61}$$

and we have a similar result for $\boldsymbol{\Phi}_2$. These values for $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ must be used in Eq. (54) in order to have agreement with the results of Sections IV–VII.

C. *Incorrect Lagrangian*

Using the lowest-order equations of motion in the Lagrangian $\mathscr{L}''$ of Eq. (38) to eliminate the higher-order acceleration terms as was done by Golubenkov and Smorodinskii [8] and Landau and Lifshitz [9] gives us the Lagrangian $\mathscr{L}''_{\mathrm{GS}}$. Since the lowest-order equations of motion do not contain $\mathbf{v}_1$ or $\mathbf{v}_2$ it is clear that $\mathbf{P}$ is the same for both $\mathscr{L}''_{\mathrm{GS}}$ and $\mathscr{L}''$. However, the conserved momentum $\mathbf{P}$ corresponding to $\mathscr{L}''_{\mathrm{GS}}$ is not the same as the conserved momentum $\boldsymbol{\Pi}$ [see Eq. (60)] corresponding to $\mathscr{L}''$. We conclude that the equations of motion corresponding to $\mathscr{L}''_{\mathrm{GS}}$ and $\mathscr{L}''$ must also be different and that, therefore, the Lagrangian $\mathscr{L}''_{\mathrm{GS}}$ must be incorrect.

IV. Electromagnetic Field Energy

The total energy density of the electromagnetic field is given by [10]

$$W_T = W_{12} + W_S , \tag{62}$$

where W_{12} is the energy density due to the retarded fields of both particles 1 and 2, that is

$$W_{12} = W_{1,2\text{ret}} , \tag{63}$$

and

$$W_S = -\tfrac{1}{2}(W_{1\text{ret}} + W_{1\text{adv}}) - \tfrac{1}{2}(W_{2\text{ret}} + W_{2\text{adv}}), \tag{64}$$

where $W_{1\text{ret}}$, $W_{1\text{adv}}$ and $W_{2\text{ret}}$, $W_{2\text{adv}}$ are due to the fields of only particle 1 and only particle 2, respectively. Similar results hold for the other components of the electromagnetic energy–momentum tensor.

Since

$$W_{12} = (E^2 + H^2)/8\pi, \tag{65}$$

where

$$\mathbf{E} = \mathbf{E}_{1\text{ret}} + \mathbf{E}_{2\text{ret}} , \qquad \mathbf{H} = \mathbf{H}_{1\text{ret}} + \mathbf{H}_{2\text{ret}} , \tag{66}$$

we find that

$$\begin{aligned} W_T = [&2\mathbf{E}_{1\text{ret}} \cdot \mathbf{E}_{2\text{ret}} + \tfrac{1}{2}(E_{1\text{ret}}^2 - E_{1\text{adv}}^2) + \tfrac{1}{2}(E_{2\text{ret}}^2 - E_{2\text{adv}}^2) \\ &+ 2\mathbf{H}_{1\text{ret}} \cdot \mathbf{H}_{2\text{ret}} + \tfrac{1}{2}(H_{1\text{ret}}^2 - H_{1\text{adv}}^2) + \tfrac{1}{2}(H_{2\text{ret}}^2 - H_{2\text{adv}}^2)]/8\pi. \end{aligned} \tag{67}$$

In this section we shall be interested in finding W_T to order c^{-2}. For ϕ' and $\mathbf{A}'/c$ to order c^{-2} the retarded potentials equal the advanced potentials so that Eq. (67) now becomes

$$W_T = [\mathbf{E}_1 \cdot \mathbf{E}_2 + \mathbf{H}_1 \cdot \mathbf{H}_2]/4\pi, \tag{68}$$

where we have dropped the subscripts ret and adv.

From the results of Section II it is easy to see that the potentials and fields due to particle 1 are

$$\phi_1'(\mathbf{x}, t) = e_1/| \mathbf{x} - \mathbf{r}_1 | , \tag{69}$$

$$\mathbf{A}_1'(\mathbf{x}, t) = \frac{e_1\mathbf{v}_1}{2c \mid \mathbf{x} - \mathbf{r}_1 \mid} + \frac{e_1[\mathbf{v}_1 \cdot (\mathbf{x} - \mathbf{r}_1)](\mathbf{x} - \mathbf{r}_1)}{2c \mid \mathbf{x} - \mathbf{r}_1 \mid^3} , \tag{70}$$

$$\mathbf{E}_1(\mathbf{x}, t) = -\text{grad}(e_1/| \mathbf{x} - \mathbf{r}_1 |) - \frac{e_1}{2c^2} \frac{\partial}{\partial t} \left[\frac{\mathbf{v}_1}{| \mathbf{x} - \mathbf{r}_1 |} + \frac{[\mathbf{v}_1 \cdot (\mathbf{x} - \mathbf{r}_1)](\mathbf{x} - \mathbf{r}_1)}{| \mathbf{x} - \mathbf{r}_1 |^3}\right] , \tag{71}$$

$$\mathbf{H}_1(\mathbf{x}, t) = \text{curl}(e_1\mathbf{v}_1/c \mid \mathbf{x} - \mathbf{r}_1 |), \tag{72}$$

with similar results due to particle 2. We then have

$$\int W_{\rm T}\, dV = \frac{1}{4\pi}\int \mathbf{E}_1 \cdot \mathbf{E}_2\, dV + \frac{1}{4\pi}\int \mathbf{H}_1 \cdot \mathbf{H}_2\, dV, \tag{73}$$

with

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2\, dV = I_0 + I_1 + I_2\,, \tag{74}$$

$$\int \mathbf{H}_1 \cdot \mathbf{H}_2\, dV = I_{3A} + I_{3B}\,, \tag{75}$$

and

$$I_0 = e_1 e_2 \int \frac{(\mathbf{x} - \mathbf{r}_1)\cdot(\mathbf{x} - \mathbf{r}_2)}{|\,\mathbf{x} - \mathbf{r}_1\,|^3\,|\,\mathbf{x} - \mathbf{r}_2\,|^3}\, dV, \tag{76}$$

$$I_1 = \left[\frac{\partial I^{(1)}}{\partial t}\right]_{\mathbf{r}_1}, \qquad I_2 = \left[\frac{\partial I^{(2)}}{\partial t}\right]_{\mathbf{r}_2}, \tag{77}$$

$$I^{(1)} = -\frac{e_1 e_2}{2c^2}\int \left(\frac{\mathbf{x} - \mathbf{r}_1}{|\,\mathbf{x} - \mathbf{r}_1\,|^3}\right)\cdot\left(\frac{\mathbf{v}_2}{|\,\mathbf{x} - \mathbf{r}_2\,|} + \frac{[\mathbf{v}_2 \cdot (\mathbf{x} - \mathbf{r}_2)](\mathbf{x} - \mathbf{r}_2)}{|\,\mathbf{x} - \mathbf{r}_2\,|^3}\right) dV, \tag{78}$$

$$I^{(2)} = -\frac{e_1 e_2}{2c^2}\int \left(\frac{\mathbf{x} - \mathbf{r}_2}{|\,\mathbf{x} - \mathbf{r}_2\,|^3}\right)\cdot\left(\frac{\mathbf{v}_1}{|\,\mathbf{x} - \mathbf{r}_1\,|} + \frac{[\mathbf{v}_1 \cdot (\mathbf{x} - \mathbf{r}_1)](\mathbf{x} - \mathbf{r}_1)}{|\,\mathbf{x} - \mathbf{r}_1\,|^3}\right) dV, \tag{79}$$

$$I_{3A} = (\mathbf{v}_1 \cdot \mathbf{v}_2/c^2)\, I_0\,, \tag{80}$$

$$I_{3B} = -\frac{e_1 e_2}{c^2}\int \frac{[\mathbf{v}_1 \cdot (\mathbf{x} - \mathbf{r}_2)][\mathbf{v}_2 \cdot (\mathbf{x} - \mathbf{r}_1)]}{|\,\mathbf{x} - \mathbf{r}_2\,|^3\,|\,\mathbf{x} - \mathbf{r}_1\,|^3}\, dV. \tag{81}$$

The above integrals have been evaluated (with the use of tables) by integrating over an infinite sphere centered at the position of particle 2. We find that

$$I_0 = 4\pi e_1 e_2/r, \tag{82}$$

$$I^{(1)} = I^{(2)} = 0, \qquad I_1 = I_2 = 0, \tag{83}$$

$$I_{3A} = 4\pi\,\frac{e_1 e_2}{r}\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right), \tag{84}$$

$$I_{3B} = 4\pi\,\frac{e_1 e_2}{r}\left[-\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{2c^2} + \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{2c^2 r^2}\right]. \tag{85}$$

The total energy (to order c^{-2}) is thus

$$\mathscr{E} = m_1 c^2 + \tfrac{1}{2} m_1 v_1^2 + \tfrac{3}{8} m_1 v_1^4/c^2 + m_2 c^2 + \tfrac{1}{2} m_2 v_2^2 + \tfrac{3}{8} m_2 v_2^4/c^2 + \int W_{\rm T}\, dV, \tag{86}$$

where

$$\int W_{\rm T}\, dV = \frac{e_1 e_2}{r}\left[1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{2c^2} + \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{2c^2 r^2}\right], \tag{87}$$

which is in agreement with Eq. (51). It should be noted that the static part of $\int W_T dV$ comes from order c^0 electric terms while the velocity-dependent part comes from order c^{-2} magnetic terms; there is no contribution from order c^{-2} electric terms.

V. Moment of Electromagnetic Field Energy

In this section we wish to find (to order c^{-2})

$$\int W_T \mathbf{x}\, dV = \frac{1}{4\pi} \int \mathbf{E}_1 \cdot \mathbf{E}_2 \mathbf{x}\, dV + \frac{1}{4\pi} \int \mathbf{H}_1 \cdot \mathbf{H}_2 \mathbf{x}\, dV, \tag{88}$$

where

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2 \mathbf{x}\, dV = \mathbf{J}_0 + \mathbf{J}_1 + \mathbf{J}_2\,, \tag{89}$$

$$\int \mathbf{H}_1 \cdot \mathbf{H}_2 \mathbf{x}\, dV = \mathbf{J}_{3A} + \mathbf{J}_{3B}\,, \tag{90}$$

and

$$\mathbf{J}_0 = \mathbf{J}_0^* + I_0 \mathbf{r}_2\,, \tag{91}$$

$$\mathbf{J}^{(1)} = \mathbf{J}^{(1)*} + I^{(1)} \mathbf{r}_2 = \mathbf{J}^{(1)*}, \tag{92}$$

$$\mathbf{J}^{(2)} = \mathbf{J}^{(2)*} + I^{(2)} \mathbf{r}_2 = \mathbf{J}^{(2)*}, \tag{93}$$

$$\mathbf{J}_{3A} = \mathbf{J}_{3A}^* + I_{3A} \mathbf{r}_2\,, \tag{94}$$

$$\mathbf{J}_{3B} = \mathbf{J}_{3B}^* + I_{3B} \mathbf{r}_2\,, \tag{95}$$

with

$$\mathbf{J}_0^* = e_1 e_2 \int \left[\frac{(\mathbf{x} - \mathbf{r}_1) \cdot (\mathbf{x} - \mathbf{r}_2)}{|\mathbf{x} - \mathbf{r}_1|^3 \, |\mathbf{x} - \mathbf{r}_2|^3} \right] (\mathbf{x} - \mathbf{r}_2)\, dV, \tag{96}$$

$$\mathbf{J}_1 = \left[\frac{\partial \mathbf{J}^{(1)}}{\partial t} \right]_{\mathbf{r}_1} = \left[\frac{\partial \mathbf{J}^{(1)*}}{\partial t} \right]_{\mathbf{r}_1}, \tag{97}$$

$$\mathbf{J}_2 = \left[\frac{\partial \mathbf{J}^{(2)}}{\partial t} \right]_{\mathbf{r}_2} = \left[\frac{\partial \mathbf{J}^{(2)*}}{\partial t} \right]_{\mathbf{r}_2}, \tag{98}$$

$$\mathbf{J}^{(1)*} = -\frac{e_1 e_2}{2c^2} \int \left[\left(\frac{\mathbf{x} - \mathbf{r}_1}{|\mathbf{x} - \mathbf{r}_1|^3} \right) \cdot \left(\frac{\mathbf{v}_2}{|\mathbf{x} - \mathbf{r}_2|} + \frac{[\mathbf{v}_2 \cdot (\mathbf{x} - \mathbf{r}_2)](\mathbf{x} - \mathbf{r}_2)}{|\mathbf{x} - \mathbf{r}_2|^3} \right) \right] (\mathbf{x} - \mathbf{r}_2)\, dV, \tag{99}$$

$$\mathbf{J}^{(2)*} = -\frac{e_1 e_2}{2c^2} \int \left[\left(\frac{\mathbf{x} - \mathbf{r}_2}{|\mathbf{x} - \mathbf{r}_2|^3} \right) \cdot \left(\frac{\mathbf{v}_1}{|\mathbf{x} - \mathbf{r}_1|} + \frac{[\mathbf{v}_1 \cdot (\mathbf{x} - \mathbf{r}_1)](\mathbf{x} - \mathbf{r}_1)}{|\mathbf{x} - \mathbf{r}_1|^3} \right) \right] (\mathbf{x} - \mathbf{r}_2)\, dV, \tag{100}$$

$$\mathbf{J}_{3A}^* = (\mathbf{v}_1 \cdot \mathbf{v}_2 / c^2)\, \mathbf{J}_0^*, \tag{101}$$

$$\mathbf{J}_{3B}^* = -\frac{e_1 e_2}{c^2} \int \left[\frac{[\mathbf{v}_1 \cdot (\mathbf{x} - \mathbf{r}_2)][\mathbf{v}_2 \cdot (\mathbf{x} - \mathbf{r}_1)]}{|\mathbf{x} - \mathbf{r}_2|^3 \, |\mathbf{x} - \mathbf{r}_1|^3} \right] (\mathbf{x} - \mathbf{r}_2)\, dV. \tag{102}$$

The above integrals have been evaluated (again with the use of tables) by integrating over a sphere of radius R, with $R \to \infty$, centered at the position of particle 2. We find that

$$\mathbf{J}_0 = 4\pi(e_1e_2/2r)(\mathbf{r}_1 + \mathbf{r}_2), \tag{103}$$

$$\mathbf{J}^{(1)*} = -\pi(e_1e_2/2c^2)\left[\mathbf{v}_2\left(\frac{8}{3}R - 3r\right) + (\mathbf{v}_2 \cdot \mathbf{r})\,\mathbf{r}/r\right], \tag{104}$$

$$\mathbf{J}^{(2)*} = -\pi(e_1e_2/2c^2)\left[\mathbf{v}_1\left(\frac{8}{3}R - 3r\right) + (\mathbf{v}_1 \cdot \mathbf{r})\,\mathbf{r}/r\right], \tag{105}$$

$$\begin{aligned}\mathbf{J}_1 &= -4\pi(e_1e_2/8c^2r)\Big[\frac{8}{3}Rr\mathbf{a}_2 - 3r^2\mathbf{a}_2 + (\mathbf{a}_2 \cdot \mathbf{r})\,\mathbf{r} - v_2{}^2\mathbf{r} + 2(\mathbf{v}_2 \cdot \mathbf{r})\,\mathbf{v}_2 \\ &\quad + (\mathbf{v}_2 \cdot \mathbf{r})^2\,\mathbf{r}/r^2\Big] = -4\pi[(e_1e_2/3c^2)\,R\mathbf{a}_2 + \boldsymbol{\Phi}_1c^2],\end{aligned} \tag{106}$$

$$\begin{aligned}\mathbf{J}_2 &= -4\pi(e_1e_2/8c^2r)\Big[\frac{8}{3}Rr\mathbf{a}_1 - 3r^2\mathbf{a}_1 + (\mathbf{a}_1 \cdot \mathbf{r})\,\mathbf{r} + v_1{}^2\mathbf{r} - 2(\mathbf{v}_1 \cdot \mathbf{r})\,\mathbf{v}_1 \\ &\quad - (\mathbf{v}_1 \cdot \mathbf{r})^2\,\mathbf{r}/r^2\Big] = -4\pi[(e_1e_2/3c^2)\,R\mathbf{a}_1 + \boldsymbol{\Phi}_2c^2],\end{aligned} \tag{107}$$

$$\mathbf{J}_{3A} = 4\pi\left[\frac{e_1e_2}{2r}\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)(\mathbf{r}_1 + \mathbf{r}_2)\right], \tag{108}$$

$$\begin{aligned}\mathbf{J}_{3B} &= 4\pi\,\frac{e_1e_2}{2r}\left[\left(-\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{2c^2} + \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{2c^2r^2}\right)(\mathbf{r}_1 + \mathbf{r}_2)\right. \\ &\quad \left. + \frac{(\mathbf{v}_2 \cdot \mathbf{r})\,\mathbf{v}_1}{2c^2} - \frac{(\mathbf{v}_1 \cdot \mathbf{r})\,\mathbf{v}_2}{2c^2}\right].\end{aligned} \tag{109}$$

We thus have

$$\frac{1}{4\pi}\int \mathbf{E}_1 \cdot \mathbf{E}_2\mathbf{x}\,dV = \frac{e_1e_2}{2r}(\mathbf{r}_1 + \mathbf{r}_2) - \boldsymbol{\Phi}_1c^2 - \boldsymbol{\Phi}_2c^2 - \frac{e_1e_2}{3c^2}R(\mathbf{a}_1 + \mathbf{a}_2), \tag{110}$$

$$\begin{aligned}\frac{1}{4\pi}\int \mathbf{H}_1 \cdot \mathbf{H}_2\mathbf{x}\,dV &= \frac{e_1e_2}{2r}\left[\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{2c^2} + \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{2c^2r^2}\right)(\mathbf{r}_1 + \mathbf{r}_2)\right. \\ &\quad \left. + \frac{(\mathbf{v}_2 \cdot \mathbf{r})\,\mathbf{v}_1}{2c^2} - \frac{(\mathbf{v}_1 \cdot \mathbf{r})\,\mathbf{v}_2}{2c^2}\right].\end{aligned} \tag{111}$$

The term $-(e_1e_2/3c^2)\,R(\mathbf{a}_1 + \mathbf{a}_2)$ in Eq. (110) is a new type of divergent term. When its removal is justified in Section VII, Eqs. (110) and (111) lead to the result of Eq. (54).

VI. Conservations Laws

Let us start with the energy–momentum tensor $T_{\mu\nu}$ for both *particles* and *fields*. We have

$$T_{\mu\nu} = T_{\nu\mu}\,, \qquad \partial T_{\mu\nu}/\partial x_\nu = 0. \tag{112}$$

Let us next define the four-dimensional angular momentum tensor as

$$M_{\rho,\mu\nu} \equiv x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho}\,. \tag{113}$$

It then follows from Eq. (112) that

$$M_{\rho,\mu\nu} = -M_{\rho,\nu\mu}\,, \qquad \partial M_{\rho,\mu\nu}/\partial x_\rho = 0. \tag{114}$$

In the above we are using the notation that $x_\mu = (\mathbf{x}, ict)$, and $x_4 = ix_0$.

Integrating $\partial T_{\mu\nu}/\partial x_\nu$ over a volume and converting the 3 divergence into a surface integral we obtain

$$\int_S N_l\, dS_l = -\frac{d}{dt}\int_V W\, dV \qquad \text{ENERGY CONSERVATION,} \tag{115}$$

$$\int_S T_{il}\, dS_l = -\frac{d}{dt}\int_V G_i\, dV \qquad \text{MOMENTUM CONSERVATION,} \tag{116}$$

where $W \equiv T_{00}$ is the energy density, $N_l \equiv cT_{0l}$ is the rate of flow of energy along the x_l axis per unit area, $G_i \equiv T_{i0}/c$ is the momentum density, and T_{il} is the rate of flow of the ith component of momentum along the x_l axis per unit area.

We now define

$$M_i \equiv \epsilon_{ijk} x_j G_k = \epsilon_{ijk} M_{0,jk}/2c, \tag{117}$$

$$M_{l,i} \equiv \epsilon_{ijk} x_j T_{kl}/c = \epsilon_{ijk} M_{l,jk}/2c. \tag{118}$$

Integrating $\partial M_{\rho,\mu\nu}/\partial x_\rho$ Over a volume and converting the 3 divergence into a surface integral we obtain

$$\int_S cM_{l,i}\, dS_l = -\frac{d}{dt}\int_V M_i\, dV \qquad \text{ANGULAR MOMENTUM CONSERVATION,} \tag{119}$$

$$\int_S M_{l,0i}\, dS_l = -\frac{d}{dt}\int_V \frac{1}{c} M_{0,0i}\, dV, \tag{120}$$

where M_i is the angular momentum density, and $cM_{l,i}$ is the rate of flow of the ith component of angular momentum along the x_l axis per unit area. Next, using

$$M_{l,0i} = ctT_{il} - x_i N_l/c, \tag{121}$$

$$M_{0,0i} = c^2 t G_i - x_i W, \tag{122}$$

together with Eq. (116) in Eq. (120) we obtain

$$\frac{d}{dt}\int_V x_i W\, dV + \int_S x_i N_l\, dS_l = c^2 \int_V G_i\, dV, \tag{123}$$

which is the conservation law for the center of inertia. It should be noted that the total number of components of the scalar equation (115) and the vector equations (116), (119), (123) add up to 10, which correspond to the 10 arbitrary parameters in the inhomogeneous Lorentz transformation.

For the electromagnetic field we have in addition to Eq. (65)

$$\mathbf{N}_{12} = c\mathbf{E} \times \mathbf{H}/4\pi, \tag{124}$$

$$\mathbf{G}_{12} = \mathbf{E} \times \mathbf{H}/4\pi c, \tag{125}$$

$$T_{il12} = [-E_i E_l - H_i H_l + \tfrac{1}{2}\delta_{il}(E^2 + H^2)]/4\pi, \tag{126}$$

and results for $\mathbf{N}_{\mathrm{T}}$, $\mathbf{G}_{\mathrm{T}}$ and $T_{il\mathrm{T}}$ can be obtained in the same manner that W_{T} was obtained from W_{12}. We shall also put

$$T_{\mu\nu} = T_{\mu\nu\mathrm{T}} + T_{\mu\nu\mathrm{p}}, \tag{127}$$

where $T_{\mu\nu\mathrm{T}}$ is the total energy–momentum tensor for the electromagnetic field and $T_{\mu\nu\mathrm{p}}$ is the particle energy–momentum tensor.

VII. Removal of Divergent Terms

The moment of particle energy $\mathcal{M}_{\mathrm{p}}$, and the particle momentum $\mathbf{P}_{\mathrm{p}}$ are both finite and are given by

$$\mathcal{M}_{\mathrm{p}} \equiv \int_V W_{\mathrm{p}}\mathbf{x}\, dV = \frac{m_1 c^2 \mathbf{r}_1}{(1 - v_1^2/c^2)^{1/2}} + \frac{m_2 c^2 \mathbf{r}_2}{(1 - v_2^2/c^2)^{1/2}}, \tag{128}$$

$$\mathbf{P}_{\mathrm{p}} \equiv \int_V \mathbf{G}_{\mathrm{p}}\, dV = \frac{m_1 \mathbf{v}_1}{(1 - v_1^2/c^2)^{1/2}} + \frac{m_2 \mathbf{v}_2}{(1 - v_2^2/c^2)^{1/2}}, \tag{129}$$

which can be expanded to the appropriate order in c. From our results of Eq. (110) and (111) we can define the finite moment of electromagnetic energy $\mathcal{M}_{f\mathcal{E}\mathcal{M}}$ to order c^{-2} as

$$\int_V W_{\mathrm{T}}\mathbf{x}\, dV = \mathcal{M}_{f\mathcal{E}\mathcal{M}} - (e_1 e_2/3c^2)\, R(\mathbf{a}_1 + \mathbf{a}_2). \tag{130}$$

We have calculated, to order c^{-2}, that

$$\int_S x_i N_{l\mathrm{T}}\, dS_l = 0, \tag{131}$$

$$c^2 \int_V \mathbf{G}_\mathrm{T}\, dV = c^2 \mathbf{P}_{f\mathscr{E}\mathscr{M}} - (e_1 e_2/3c^2)\, R(\dot{\mathbf{a}}_1 + \dot{\mathbf{a}}_2), \tag{132}$$

where $\mathbf{P}_{f\mathscr{E}\mathscr{M}}$ is the finite electromagnetic momentum which was not explicitly found. To the order we calculated Eqs. (131) and (132) the ret fields $\doteq$ the adv fields.

We have also calculated, to order c^{-4}, that

$$\int_S T_{il\mathrm{T}}\, dS_l = (e_1 e_2/3c^4)\, R[(\ddot{\mathbf{a}}_1)_i + (\ddot{\mathbf{a}}_2)_i]. \tag{133}$$

In this case the ret fields $\neq$ adv fields, and we used $e_1/m_1 = e_2/m_2$ to eliminate a finite c^{-3} term. For Eqs. (130)–(133) we had a sphere of radius R, with $R \to \infty$, centered at the position of particle 2. However, there would be no change in the results if the center were at the position of particle 1.

Using Eqs. (132) and (133) in Eq. (116) we obtain (correct to order c^{-4})

$$\frac{d}{dt}\,[\mathbf{P}_\mathrm{p} + \mathbf{P}_{f\mathscr{E}\mathscr{M}}] = 0, \tag{134}$$

and thus $\mathbf{P}_\mathrm{p} + \mathbf{P}_{f\mathscr{E}\mathscr{M}}$ is the conserved momentum. It should be noted that the particle energy–momentum tensor cannot contribute to infinite surface integrals. Using Eqs. (130)–(132) in Eq. (123) we obtain (correct to order c^{-2})

$$\frac{d}{dt}\,[\mathscr{M}_\mathrm{p} + \mathscr{M}_{f\mathscr{E}\mathscr{M}}] = c^2(\mathbf{P}_\mathrm{p} + \mathbf{P}_{f\mathscr{E}\mathscr{M}}). \tag{135}$$

Since we have used

$$\mathscr{E}\mathbf{r}_\mathrm{CI} \equiv \mathscr{M}_\mathrm{p} + \mathscr{M}_{f\mathscr{E}\mathscr{M}}, \tag{136}$$

we must have (correct to order c^{-4})

$$\mathbf{\Pi} = \mathbf{P}_\mathrm{p} + \mathbf{P}_{f\mathscr{E}\mathscr{M}}. \tag{137}$$

A. *Conjectures*

We conjecture, that to order c^{-4}, we will have

$$\int_V W_\mathrm{T}\, dV = \mathscr{E}_{f\mathscr{E}\mathscr{M}} + \textit{divergent part}, \tag{138}$$

$$\int_V \mathbf{M}_\mathrm{T}\, dV = \mathbf{L}_{f\mathscr{E}\mathscr{M}} + \textit{divergent part}, \tag{139}$$

as well as

$$\mathscr{E} = \mathscr{E}_{\mathrm{p}} + \mathscr{E}_{f\mathscr{E}\mathscr{M}}, \tag{140}$$

$$\boldsymbol{\lambda} = \mathbf{L}_{\mathrm{p}} + \mathbf{L}_{f\mathscr{E}\mathscr{M}}. \tag{141}$$

VIII. Conclusions

We have found that the Lagrangian to order c^{-4} in electromagnetic theory is acceleration dependent and have shown why elimination of the acceleration terms by using the equations of motion in the Lagrangian is an incorrect procedure. However, there is nothing wrong with using the equations of motion in themselves to eliminate higher-order acceleration terms [as was done in Eq. (55) and could also have been done if Eq. (55) were given to order c^{-4}]. We have also given a detailed treatment of the center of inertia result and have shown how some new type of divergent terms cancel out in the conservation laws. *In a future publication* [17], *we will present a new method which enables us to replace the higher-order acceleration terms in the Lagrangian with velocity- and coordinate-dependent terms in such a manner that the equations of motion (and also the conserved energy, momentum, and angular momentum, and center of inertia result) are unaltered to order* c^{-4}.

There are two other two-body Lagrangians to order c^{-4} (and their corresponding center of inertia results, etc.) that would be very interesting to investigate. They are:

(a) the post-post-Newtonian uncharged two-body Lagrangian in general relativity (an extension of the post-Newtonian Einstein–Infeld–Hoffmann Lagrangian) and

(b) the post-post-Newtonian charged two-body Lagrangian in general relativity (an extention of the post-Newtonian Bażański Lagrangian), where the condition $e_1/m_1 = e_2/m_2$ must be imposed.

We do not believe that a correct Lagrangian of either (a) or (b) exists in the literature. A proper investigation of (a) or (b) would have to look into the problem of self-force in gravitation and since gravitation is non-linear this problem may be much more difficult than in the corresponding problem in electromagnetism [10, 11]. It also seems likely that if a two-body problem can be solved the corresponding n-body problem can also be solved.

Finally, let us note that we cannot have two-body (or n-body) Lagrangians of higher order than c^{-4} since radiation effects occur at order c^{-5}.

Appendix A

The Darwin Lagrangian, Eq. (39), in the Coulomb gauge (standard form) is

$$\mathscr{L}_{\mathrm{D}}' = \mathscr{L}_0 - \frac{e_1 e_2}{r}\left[1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{2c^2} - \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{2c^2 r^2}\right], \tag{A1}$$

where $\mathscr{L}_0$ is the free particle term. If we go back to the original Lorentz gauge we have

$$\mathscr{L}_{\mathrm{D}} = \mathscr{L}_0 - \frac{e_1 e_2}{r}\left[1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} + \frac{v_2^2}{2c^2} - \frac{(\mathbf{v}_2 \cdot \mathbf{r})^2}{2c^2 r^2} - \frac{\mathbf{a}_2 \cdot \mathbf{r}}{2c^2}\right]. \tag{A2}$$

Since $\mathscr{L}_{\mathrm{D}}$ and $\mathscr{L}'_{\mathrm{D}}$ differ by a total time derivative they have the same equations of motion. The Lagrangian $\mathscr{L}_{\mathrm{D}}$ does not contain $\mathbf{a}_1$ since we started with $\mathscr{L}_1$, the Lagrangian of particle 1 in an electromagnetic field. While Eq. (A2) is not symmetrical under interchange of indicies 1 and 2 the resulting equations of motion are. Interchanging the indices 1 and 2 in Eq. (A2) will give us another valid Lagrangian which could have been obtained by starting with $\mathscr{L}_2$, the Lagrangian of particle 2 in an electromagnetic field. We thus have

$$\mathscr{L}_{\mathrm{D}}{}^{\mathrm{I}} = \mathscr{L}_0 - \frac{e_1 e_2}{r}\left[1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} + \frac{v_1^2}{2c^2} - \frac{(\mathbf{v}_1 \cdot \mathbf{r})^2}{2c^2 r^2} + \frac{\mathbf{a}_1 \cdot \mathbf{r}}{2c^2}\right]. \tag{A3}$$

Finally, the symmetrical Lagrangian $\mathscr{L}_{\mathrm{D}}{}^{\mathrm{S}} = \frac{1}{2}\mathscr{L}_{\mathrm{D}} + \frac{1}{2}\mathscr{L}_{\mathrm{D}}{}^{\mathrm{I}}$ is also valid. We have

$$\mathscr{L}_{\mathrm{D}}{}^{\mathrm{S}} = \mathscr{L}_0 - \frac{e_1 e_2}{r}\left[1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} + \frac{(v_1^2 + v_2^2)}{4c^2} - \frac{[(\mathbf{v}_1 \cdot \mathbf{r})^2 + (\mathbf{v}_2 \cdot \mathbf{r})^2]}{4c^2 r^2} + \frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{r}}{4c^2}\right]. \tag{A4}$$

In the same manner as we have done for the Darwin Lagrangian, the Lagrangian $\mathscr{L}''$ of Eq. (38) can be put in different forms containing time derivatives of acceleration.

Appendix B

We wish to show in this appendix that using the equations of motion in the Lagrangian to eliminate the higher-order acceleration terms is an incorrect procedure as it leads to different equations of motion. To illustrate our point, let us choose the following very simple one-body Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\mu v^2 - e_1 e_2/r + (e_1 e_2/c^2)\, \mathbf{a} \cdot \mathbf{n}_0, \tag{B1}$$

where $\mathbf{n}_0$ is an arbitrary constant unit vector, and μ is the reduced mass. The last term in Eq. (B1), which is the higher-order term, was chosen purely for mathematical simplicity, rather than physical reality. Using Eq. (B1) in the equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \mathbf{v}}\right) = \frac{\partial \mathscr{L}}{\partial \mathbf{r}} + \frac{d^2}{dt^2}\left(\frac{\partial \mathscr{L}}{\partial \mathbf{a}}\right) \tag{B2}$$

gives us

$$\mu \mathbf{a} = e_1 e_2 \mathbf{r}/r^3. \tag{B3}$$

It should be noted that the higher-order term in Eq. (B1) did not contribute to Eq. (B3). The last term in $\mathscr{L}$ can be written as $d[(e_1e_2/c^2)\,\mathbf{v}\cdot\mathbf{n}_0]/dt$, a total time derivative, and, thus, can be dropped to give us the equivalent Lagrangian

$$\mathscr{L}' = \tfrac{1}{2}\mu v^2 - e_1e_2/r, \tag{B4}$$

which gives us the same equations of motion, Eq. (B3).

We shall now use the equations of motion, Eq. (B3), in the higher-order term of Eq. (B1) to eliminate the acceleration term. The result is

$$\mathscr{L}^* = \tfrac{1}{2}\mu v^2 - \frac{e_1e_2}{r} + \frac{e_1{}^2e_2{}^2}{\mu c^2}\left(\frac{\mathbf{r}\cdot\mathbf{n}_0}{r^3}\right), \tag{B5}$$

which gives us the equations of motion

$$\mu\mathbf{a} = \frac{e_1e_2\mathbf{r}}{r^3} + \frac{e_1{}^2e_2{}^2}{\mu c^2}\left[\frac{\mathbf{n}_0}{r^3} - \frac{3(\mathbf{r}\cdot\mathbf{n}_0)\,\mathbf{r}}{r^5}\right], \tag{B6}$$

which is clearly *not* in agreement with Eq. (B3). Thus we see that the Lagrangian $\mathscr{L}^*$ is *not* equivalent to the Lagrangian $\mathscr{L}$.

It is the *functional form* of $\mathscr{L}$ which is crucial in leading to the correct equations of motion. Substitution into $\mathscr{L}$ changes its functional form, and thus, upon variation, changes the equations of motion. *We conclude that it is not correct to use the equations of motion in the Lagrangian.*

References

1. B. M. Barker and R. F. O'Connell, *J. Math. Phys.* **18** (1977), 1818; **19** (1978), 1231.
2. L. D. Landau and E. M. Lifschitz, "The Classical Theory of Fields," 4th revised English ed., Sect. 14, pp. 168–169, 342, Pergamon, New York, 1975.
3. S. C. Coleman and J. H. Van Vleck, *Phys. Rev.* **171** (1968), 1370.
4. O. Costa de Beauregard, *Phys. Lett. A* **28** (1968), 365.
5. P. Havas and J. Stachel, *Phys. Rev.* **185** (1969), 1637.
6. B. M. Barker and R. F. O'Connell, *J. Math. Phys.* **20** (1979), 1427.
7. See Sect. 67 of Ref. [2].
8. V. N. Golubenkov and Ia. A. Smorodinskii, *Sov. Phys. JETP* **4** (1957), 55.
9. See p. 210 of Ref. [2].
10. S. N. Gupta, *Proc. Phys. Soc. London Ser. A* **64** (1951), 50. Gupta uses the self potentials to give a revised (divergence-free) Lorentz force equation that agrees with the result of Dirac (Ref. [11]). Our present paper presents the Lagrangian formalism with self potentials that leads to this same result [see our Eq. (29)]. Gupta also gives the revised (divrrgence-free) Maxwell's equations and electromagnetic energy–momentum tensor.
11. P. A. M. Dirac, *Proc. Roy. Soc. Sect. A* **167** (1938), 148. See also F. Rohrlich, "Classical Charged Particles," Addison–Wesley, Reading, Mass., 1965.
12. This becomes somewhat clearer if one expands the potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, then carry out the time differentiation, and then set $\mathbf{x}$ equal to $\mathbf{r}_1$.

13. Note the difference in approach that Landau and Lifschitz use (see [2, p. 209–210]). They symmetrize an already symmetrical quantity. Note that the **R** of Landau and Lifschitz is equal to minus our **r**.
14. C. G. DARWIN, *Phil. Mag.* **39** (1920), 537.
15. L. D. LANDAU AND E. M. LIFSCHITZ, "Mechanics," 2nd English ed., Chapt. II, Pergamon, New York, 1969.
16. Lagrangian formalism with acceleration-dependent terms is discussed in: M. OSTROGRADSKY, *Mem. Acad. St.-Pet.* VI (1850), 385; E. T. Whittaker, "Analytical Dynamics of Particles and Rigid Bodies," 4th ed., p. 256, Cambridge Univ. Press, Cambridge, 1964; L. Koenigsberger, "Die Prinzipien der Mechanik," Leipzig, 1901.
17. B. M. BARKER AND R. F. O'CONNELL, *Canad. J. Phys.*, in press.