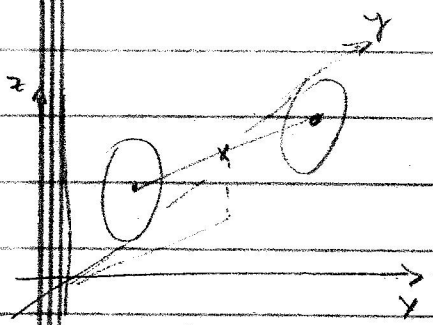
Two wheels + axle



COM: $\vec{R} = (x, y, a)$

Rotation of each wheel about axle: ϕ_1, ϕ_2

Rotation of axle about $\hat{k}$: θ

Constraints:

$$\dot{x}\cos\theta + \dot{y}\sin\theta = 0$$

$$\dot{x}\sin\theta - \dot{y}\cos\theta - \frac{1}{2}a(\dot{\phi}_1 + \dot{\phi}_2) = 0$$

} semi holonomic

$$b\dot{\theta} + a(\dot{\phi}_2 - \dot{\phi}_1) = 0 \quad : \text{holonomic}$$

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_z\dot{\theta}^2 + \frac{1}{2}I_1\dot{\phi}_1^2 + \frac{1}{2}I_2\dot{\phi}_2^2$$

(no V on the plane!)

Use $\dot{\theta} = -\frac{a}{b}(\dot{\phi}_2 - \dot{\phi}_1)$

$$\Rightarrow \quad L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_z\frac{a^2}{b^2}(\dot{\phi}_2 - \dot{\phi}_1)^2 + \frac{1}{2}I_1\dot{\phi}_1^2 + \frac{1}{2}I_2\dot{\phi}_2^2$$

Coordinates: $\{x, y, \phi_1, \phi_2\}$ + Constraints:

$$\mu_1 \longleftrightarrow \dot{x}\cos\theta + \dot{y}\sin\theta = 0$$

$$\mu_2 \longleftrightarrow \dot{x}\sin\theta - \dot{y}\cos\theta - \frac{1}{2}a(\dot{\phi}_1 + \dot{\phi}_2) = 0$$

(where $\theta = C - \frac{a}{b}(\phi_2 - \phi_1)$)

ϕ_1, ϕ_2 eqns:

$$-I_z\frac{a^2}{b^2}(\ddot{\phi}_2 - \ddot{\phi}_1) + I_1\ddot{\phi}_1 = -\frac{a}{2}\mu_2$$

$$+I_z\frac{a^2}{b^2}(\ddot{\phi}_2 - \ddot{\phi}_1) + I_2\ddot{\phi}_2 = -\frac{a}{2}\mu_2$$

$(I_1 = I_2 = I) + \left(\begin{matrix}\text{subtract}\\ \text{eqns}\end{matrix}\right) \Rightarrow -2I_z\frac{a^2}{b^2}(\ddot{\phi}_2 - \ddot{\phi}_1) + I(\ddot{\phi}_2 - \ddot{\phi}_1) = 0$

$$\Rightarrow \quad \ddot{\phi}_1 = \ddot{\phi}_2$$

$$\theta = C - \frac{a}{b}(\phi_2 - \phi_1)$$

$$\dot{\phi}_1 = \dot{\phi}_2 \Rightarrow \ddot{\theta} = 0 \Rightarrow \theta = \omega t + \theta_0$$

Also $\ddot{\phi}_1 = \ddot{\phi}_2 \Rightarrow \mu_2 = -\frac{2I}{a}\ddot{\phi}$

and $\dot{\phi}_1 = \dot{\phi}_2 + \Delta\phi_0$

x, y eqns :

$$M\ddot{x} = \mu_1 \cos\omega t + \mu_2 \sin\omega t$$
$$M\ddot{y} = \mu_1 \sin\omega t - \mu_2 \cos\omega t$$

$+ \quad \dot{x}\cos\omega t + \dot{y}\sin\omega t = 0$

$$\dot{x}\sin\omega t - \dot{y}\cos\omega t = \frac{a}{2}(\dot{\phi}_1 + \dot{\phi}_2) = a\dot{\phi}_2 + a\Delta\phi_0$$

$+ \quad \mu_2 = -\frac{2I}{a}\ddot{\phi}_2$

5 eqns for $\{x, y, \mu_1, \mu_2, \phi_2\}$

Take derivative of constraints:

$$\ddot{x}\cos\omega t + \ddot{y}\sin\omega t - \dot{x}\omega\sin\omega t + \dot{y}\omega\cos\omega t = 0$$
$$\ddot{x}\sin\omega t - \ddot{y}\cos\omega t + \dot{x}\omega\cos\omega t + \dot{y}\omega\sin\omega t = a\ddot{\phi}$$

Use x,y eom : $M(\ddot{x}\cos\omega t + \ddot{y}\sin\omega t) = \mu_1$

$$M(\ddot{x}\sin\omega t - \ddot{y}\cos\omega t) = \mu_2$$

$$\Rightarrow \quad \mu_1/M - \omega(\dot{x}\sin\omega t - \dot{y}\cos\omega t) = 0$$

$$\underbrace{\mu_2/M}_{-\frac{2I}{Ma}\ddot{\phi}} + \omega\underbrace{(\dot{x}\cos\omega t + \dot{y}\sin\omega t)}_{=0} = a\ddot{\phi}$$

$$\Rightarrow \ddot{\phi} = 0 \qquad (\mu_2 = 0)$$

$$\mu_1/M = \omega(\dot{x}\sin\omega t - \dot{y}\cos\omega t)$$

$$= \omega\frac{a}{2}(\dot{\phi}_1 + \dot{\phi}_2)$$

$$= \frac{a}{2}\omega(\Omega_1+\Omega_2) \quad \Rightarrow \quad \mu_1 = \frac{1}{2}aM\omega(\Omega_1+\Omega_2)$$

Finally, x, y :

$$M\ddot{x} = \mu_1\cos\omega t + \mu_2\sin\omega t$$

$$= \frac{1}{2}aM\omega(\Omega_1+\Omega_2)\cos\omega t$$

$$\Rightarrow \quad x = -\frac{a}{2}\left(\frac{\Omega_1+\Omega_2}{\omega}\right)\cos\omega t + V_{x_0}t + x_0$$

$$M\ddot{y} = \mu_1\sin\omega t - \mu_2\cos\omega t$$

$$= \frac{1}{2}aM\omega(\Omega_1+\Omega_2)\sin\omega t$$

$$\Rightarrow \quad y = -\frac{a}{2}\left(\frac{\Omega_1+\Omega_2}{\omega}\right)\sin\omega t + V_{y_0}t + y_0$$

$\Rightarrow$ uniform motion + circular motion with

radius $\quad r = \left|\frac{a}{2}\left(\frac{\Omega_1+\Omega_2}{\omega}\right)\right|$

$$\omega = -\frac{a}{b}(\Omega_2-\Omega_1)$$

$$\Rightarrow \quad r = \frac{a}{2}\frac{\Omega_1+\Omega_2}{+\frac{a}{b}(\Omega_2-\Omega_1)} = \frac{b}{2}\frac{\Omega_1+\Omega_2}{|\Omega_2-\Omega_1|}$$

If $\Omega_1 = \Omega_2$, $\omega = 0$ and $r = \infty$: straight motion

Depending on which wheel turns *faster*, rotation is clockwise or counter-clockwise. ($\omega > 0$ or $\omega < 0$)