Useful formulae related to spherical coordinate systems

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We use r for the radial coordinate ($r > 0$), ψ for the azimuthal angle with the z-axis ($0 \leq \psi \leq \pi$) and θ for the polar angle in the x-y axis ($0 < \theta \leq 2\pi$).

The formulae for cartesian coordinates in terms of spherical coordinates are:

$$\begin{aligned} x &= r \sin\psi \cos\theta \\ y &= r \sin\psi \sin\theta \\ z &= r \cos\psi \end{aligned}$$

The spherical unit vectors in terms of cartesian unit vectors are:

$$\begin{aligned} \hat{\mathbf{e}}_r &= \sin\psi \cos\theta \; \hat{\mathbf{i}} + \sin\psi \sin\theta \; \hat{\mathbf{j}} + \cos\psi \; \hat{\mathbf{k}} \\ \hat{\mathbf{e}}_\psi &= \cos\psi \cos\theta \; \hat{\mathbf{i}} + \cos\psi \sin\theta \; \hat{\mathbf{j}} - \sin\psi \; \hat{\mathbf{k}} \\ \hat{\mathbf{e}}_\theta &= -\sin\theta \; \hat{\mathbf{i}} + \cos\theta \; \hat{\mathbf{j}} \end{aligned}$$

These vectors are orthonormal: $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ and cyclical: $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \epsilon_{ijk}\hat{\mathbf{e}}_k$, but they are functions of coordinates:

$$\begin{array}{lll} \frac{\partial}{\partial r}\hat{\mathbf{e}}_r = 0 & \frac{\partial}{\partial r}\hat{\mathbf{e}}_\psi = 0 & \frac{\partial}{\partial r}\hat{\mathbf{e}}_\theta = 0 \\ \frac{\partial}{\partial \psi}\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\psi & \frac{\partial}{\partial \psi}\hat{\mathbf{e}}_\psi = -\hat{\mathbf{e}}_r & \frac{\partial}{\partial \psi}\hat{\mathbf{e}}_\theta = 0 \\ \frac{\partial}{\partial \theta}\hat{\mathbf{e}}_r = \sin\psi \hat{\mathbf{e}}_\theta & \frac{\partial}{\partial \theta}\hat{\mathbf{e}}_\psi = \cos\psi \hat{\mathbf{e}}_\theta & \frac{\partial}{\partial \theta}\hat{\mathbf{e}}_\theta = -\sin\psi \hat{\mathbf{e}}_r - \cos\psi \hat{\mathbf{e}}_\psi \end{array}$$

and thus, they are also functions of time if the coordinates are functions of time:

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{e}}_r &= \dot{\psi}\hat{\mathbf{e}}_\psi + \dot{\theta}\sin\psi \hat{\mathbf{e}}_\theta \\ \frac{d}{dt}\hat{\mathbf{e}}_\psi &= -\dot{\psi}\hat{\mathbf{e}}_r + \dot{\theta}\cos\psi \hat{\mathbf{e}}_\theta \\ \frac{d}{dt}\hat{\mathbf{e}}_\theta &= -\dot{\theta}(\sin\psi \hat{\mathbf{e}}_r + \cos\psi \hat{\mathbf{e}}_\psi) \end{aligned}$$