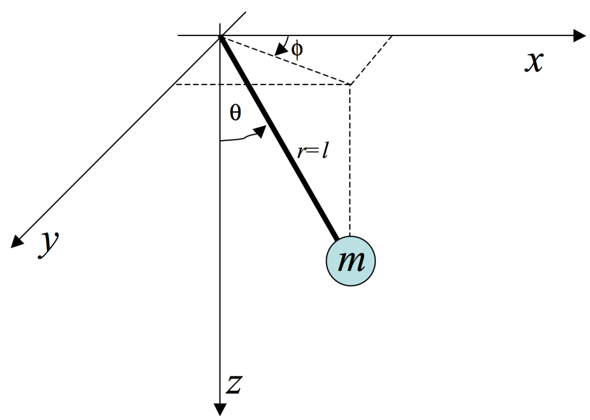
Phys 7221, Fall 2006: Midterm exam

October 20, 2006

Problem 1 (40 pts)

Consider a spherical pendulum, a mass m attached to a rod of length l, as a constrained system with $|\vec{\mathbf{r}}| = l$, as shown in the figure.



a) (15 pts) **What are the conserved quantities of the system, and symmetries associated with each?**

Since there are no dissipative forces, and the gravitational potential does not depend on velocities, the energy will be conserved. The conservation of energy is associated with the Lagrangian not being explicitly dependent on time.

There is a rotational symmetry about a vertical axis through the suspension point, so the z component of the angular momentum is conserved. This is because the tension does not produce a torque, and the gravitational torque only has x, y components (because the gravitational force is vertical). Since is no torque in the z direction, $\mathbf{N} = \dot{\mathbf{L}}$ means that $\dot{L}_z = N_z = 0$.

The gravitational force breaks the symmetry in z, and the suspension point breaks the symmetry in all three directions, so no components of the linear momentum are conserved. The gravitational force has a z component, and the tension will have in general all three components (unless it is a planar pendulum), so $\mathbf{F} = \dot{\mathbf{P}}$ means no component of $\mathbf{P}$ is constant.

b) (30 pts) **Find out an expression for the tension in the rod using the method of Lagrange multipliers.**

We write the Lagrangian in terms of all three spherical coordinates r, ϕ, θ:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) + mgr\cos\theta$$

The constraint, associated with a multiplier λ, is

$$f(r) = r - l = 0$$

There are three Lagrange's equations, plus the constraint, making a set of four equations for the four unknowns, $q_i = \{r, \theta, \phi\}$, and λ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda \frac{\partial f}{\partial q_i}$$

The equation for r is the one involving λ. We use the solution to the constraint, $r = l, \dot{r} = 0$ in the equation:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} &= \lambda \frac{\partial f}{\partial r} \\ -ml(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mg\cos\theta &= \lambda \end{aligned}$$

The multiplier λ is the force of constraint in the direction $\mathbf{e}_r$, or minus the tension:

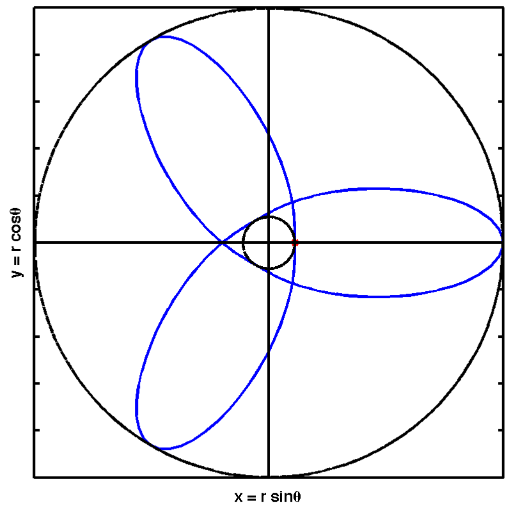
$$T = -\lambda = ml\dot{\theta}^2 + ml\dot{\phi}^2\sin^2\theta + mg\cos\theta$$

For a planar pendulum, $\dot{\phi} = 0$ and we recover the usual expression, $T = ml\dot{\theta}^2 + mg\cos\theta$. For small angular displacements, $\theta \ll 1$, and $T \approx mg$.

Problem 2 (40 pts)

Consider an orbiting system with an orbit of the form $r(\theta) = 1/(u_0 + u_1 \cos(3\theta/2))$, with $0 < u_1 < u_0$.

a) (10 pts) Draw the orbit in the physical $x = r\cos\theta, y = r\sin\theta$ axes, for $0 < \theta < 4\pi$. Is the orbit bound? Is it closed?



The figure shows of the orbit $r(\theta) = 1/(u_0 + u_1 \cos(3\theta/2))$, in blue. The black circles have radii $1/(u_0 - u_1), 1/u_0, 1/(u_0 + u_1)$. The red point indicates the point with $\theta = 0, r = 1/(u_0 + u_1)$, and also $\theta = 4\pi, r = 1/(u_0 + u_1)$.

The orbit is bound, enclosed within a circle of radius $r_{max} = 1/(u_0 - u_1)$. The orbit is also closed: when $\theta = 4\pi$ (after two turns about the origin), $r(4\pi) = r(0) = r_{min} = 1/(u_0 + u_1)$.

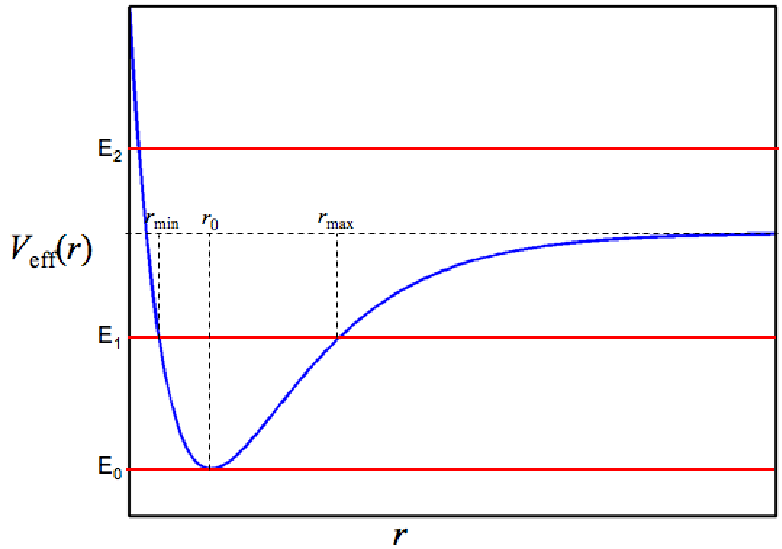
b) (15 pts) Find out the potential $V(r)$ that produces this orbit.

We use the orbit equation in terms of $u(\theta) = 1/r(\theta) = u_0 + u_1 \cos(3\theta/2)$:

$$\begin{aligned}
-\frac{m}{l^2}\frac{dV}{du} &= \frac{d^2u}{d\theta^2} + u = -u_1\left(\frac{3}{2}\right)^2 \cos(3\theta/2) + u_0 + u_1\cos(3\theta/2) \\
&= u_0 - \frac{5}{4}u_1\cos(3\theta/2) = -\frac{5}{4}(u_0 + u_1\cos(3\theta/2)) + \frac{9}{4}u_0 \\
&= -\frac{5}{4}u + \frac{9}{4}u_0 \\
\frac{dV}{du} &= \frac{5l^2}{4m}u - \frac{9l^2u_0}{4m} \\
V(u) &= \frac{5l^2}{8m}u^2 - \frac{9l^2u_0}{4m}u \\
V(r) &= \frac{k_1}{r^2} - \frac{k_0}{r}
\end{aligned}$$

The potential has an attractive "Kepler term" $-k_0/r$, and a repulsive term k_1/r^2. This particular orbit has initial conditions given by $u_0 l^2 = 4k_0 m/9, l^2 = 8k_1 m/5$.

c) (15 pts) Draw a diagram of the effective potential $V_{\text{eff}}(r)$. Describe qualitatively the possible orbits, depending on values of the angular momentum and the energy of the system. Identify the orbit described in part (a).



The effective potential is $V_{\text{eff}}(r) = V(r) + l^2/2mr^2 = -k_0/r + (k_1 + l^2/2m)/r^2$, qualitatively similar to the effective potential of a Kepler potential, but with the minimum potential at a different radial coordinate. Systems will have a minimum energy E_0; systems with exactly that energy will have circular orbits with $r = r_0 = 2(k_1 + l^2/2m)/k_0$.

Like Kepler's orbits, orbits of this potential will be unbound, with a turning point, for positive energy (like E_1 in the figure); and bound with two turning points for negative energy (like E_2 in the figure). Unlike Kepler's orbits, the orbits are not ellipses and hyperbolas, since the orbit equation is different. The orbit in part (a) has negative energy like E_2, with a minimum and a maximum radial coordinate.

Unlike Kepler's potential, the effective potential is qualitative the same even for zero angualr momentum (colinear motion). The orbits in this potential for head-on approach of two masses are still bound even if they have negative energy: the repulsive term k_1/r^2 overcomes the attractive term $-k_0/r$, and avoids a collision, but the attractive force is strong enough to keep the masses bound (in Kepler's potential, the masses collide with each other when $r = 0$). Orbits with zero angular momentum and positive energy are unbound, with a turning point: if the masses are initially moving towards each other, they will reach a minimum distance from each other, and then move away; in Kepler' potential, systems with positive energy will have the masses either colliding, or moving away from each other for ever, depending on the initial conditions.

Problem 3 (20 pts)

Consider a particle of mass m and velocity v_0, approaching from infinity a particle of equal mass m, with impact parameter s. The only forces between the masses are gravitational forces.

a) (6 pts) **What's the energy and angular momentum of the system, before and after the collision?**

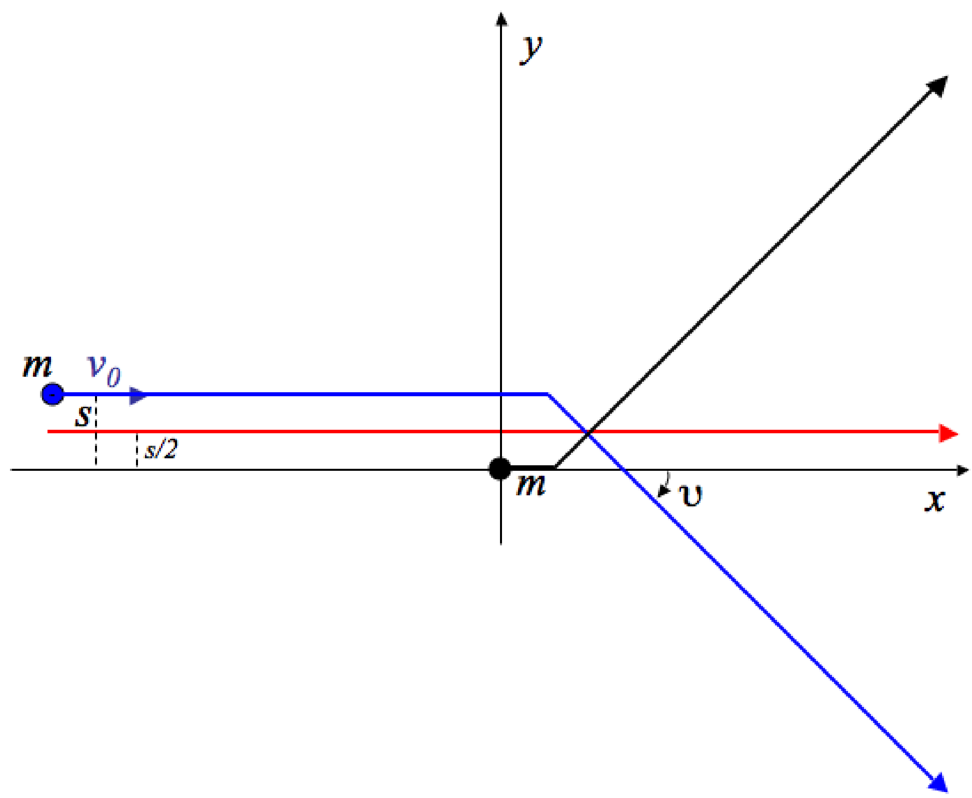
since the system has angular momentum, there will not be a collision: the masses will reach a minimum distance, and will then move away from each other.

At infinite distance, the potential is zero, and the total energy is the kinetic energy of the moving particle, $E = (1/2)mv_0^2$.

Taking the origin of an inertial system at the position of the target particle initially at rest, and a coordiante system as in the figure, the angular momentum is $\mathbf{L} = \mathbf{r}_0 \times \mathbf{p}_0 = (-x, s, 0) \times m(v_0, 0, 0) = (0, 0, mv_0 s)$.

The energy and the angular momentum are conserved, so they will have the same values at all times.

b) (14 pts) **Sketch the trajectory of the two particles in the laboratory system shown in the figure, and the trajectory of the center of mass. Indicate in the figure the scattering angle ϑ.**



The relative distance between the particles will be a hyperbola about the initial position of the target particle. If the target particle has a much larger mass than the scattered particle, that will also be the approximate trajectory of the approaching particle. In this case, since the masses are equal, the target particle will also move significantly.

The center of mass of the system moves with constant velocity $\mathbf{V} = v_0/2$, with a constant y-coordinate $s/2$, and has the red trajectory shown in the figure. The vector $\mathbf{r}$, the relative distance $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, has a hyperbolic trajectory about the origin, with initial position $(s, -\infty)$ and initial velocity $(v_0, 0)$. The approaching particle will have a position vector $\mathbf{r}_1 = \mathbf{R} + \mathbf{r}/2$, which is similar to a hyperbola: this is the blue trajecotry in the figure. Since the total momentum is constant, has horizontal velocity, and the scateered particle is moving down and to the right, we know that the target particle will move up, so that $\mathbf{v}_1 + \mathbf{v}_2$ is horizontal: this is the black trajectory in the figure.

If the initial velocity is small, the particles may whirl around a few turns before moving away, in about the same directions as shown here.