Phys 7221, Fall 2005
Midterm exam

October 19, 2005

NAME:

TOTAL NUMBER OF PAGES:

- Please write your name on every page.
- You don't need to explain your answers to the Question in page 2 (worth 20 points).
- Write down as much as you can when solving the problems, explaining your steps. Each problem is worth 40 points.
- Problem 2 has an optional "bonus question" in page 5, worth 5 points of extra credit.
- If you use extra pages, please number them, staple all the pages in your exam together, and write int he front page how many total pages you are submitting (if no extra pages, this number is 5).

Question (20 points)

Consider a charged particle moving in the electrostatic field produced by the different source charge configurations described below. What components of the particle's linear momentum vector $\mathbf{P} = m\mathbf{v}$, and of the particle's angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{P}$ will be conserved? State your answers listing P_x, P_y, P_z and/or L_x, L_y, L_z, as appropriate. You may find useful to draw the source charge configuration in each case.

If the potential (and thus the Lagrangian function) has a translational symmetry along a given direction, the component of the linear momentum along that direction is conserved. If the potential has a rottional symmetry about some direction, the component of the angular momentum along the axis of rotation is conserved.

See drawings in next page to relate the symmetries to the conserved quantities in each case.

(a) An infinite homogenous plane of charge, at $x = 0$.

P_y, P_z, L_x

(b) A semi-infinite homogenous plane, at $x = 0$, with $y > 0$.

P_z

(c) An infinite homogenous solid charged cylinder, with its axis along the y-axis.

P_y, L_y

(d) A finite homogenous solid charged cylinder, with its axis along the y-axis, and its center at the origin.

L_y

(e) A homogeneous circular torus, with its axis along the z axis, centered in the $x - y$ plane.

L_z

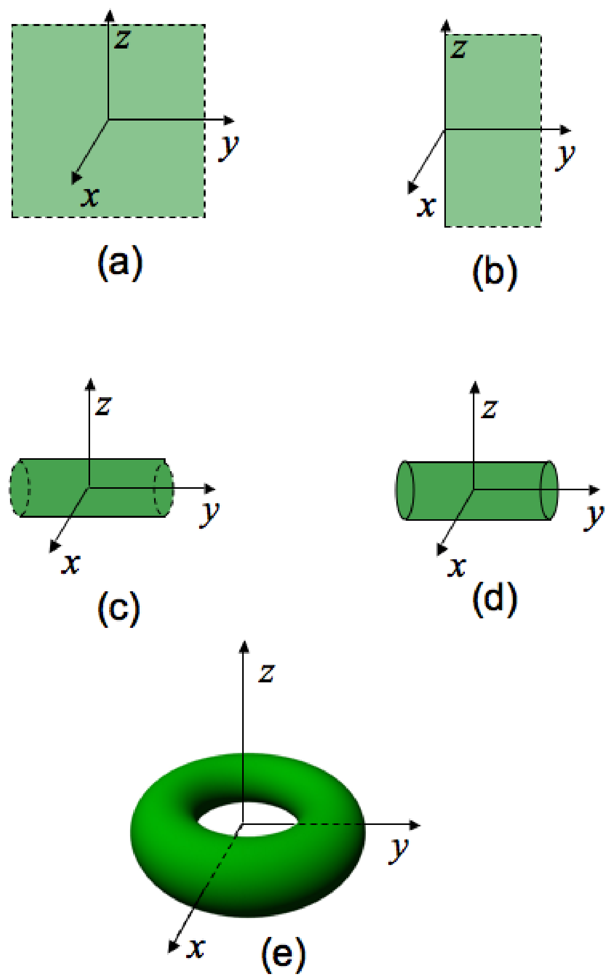


Figure 1: Question

Problem 1

Consider a mass of mass m, suspended from a suspension point, with a string of length l, whose position is described by the vector $\mathbf{r}(t)$. The suspension point is not fixed: its position vector $\mathbf{R}(t)$ has a constant vertical acceleration $\mathbf{a} = a_z\hat{\mathbf{k}}$. (For example, the pendulum is held inside an elevator).

(a) Write the Lagrangian in terms of the generalized variable α, the angle between the string and the vertical axis.

(b) Write Lagrange's equation of motion for the angle α. How does the equation of motion compare with that of a regular pendulum?

(c) If we wanted to find out the tension, we'd need to use Lagrange multipliers. How many, and which coordinates would you use, and how would you write the constraint?

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$$

The potential energy is $V = -m\mathbf{g}\cdot\mathbf{r} = mgz$ where $\mathbf{g} = -g\hat{\mathbf{k}}$ (if we choose the z-axis pointing up), so the Lagrangian is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) - mgz.$$

We have $x = l\sin\alpha$ and $z = Z - l\cos\alpha$, where $Z = Z(t)$ is the vertical coordinate of the suspension point. The kinetic energy is then

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}m\left(l^2\dot{\alpha}^2\cos^2\alpha + \left(\dot{Z} + l\dot{\alpha}\sin\alpha\right)^2\right)$$

$$T = \frac{1}{2}m\left(l^2\dot{\alpha}^2 + \dot{Z}^2\right) + ml\dot{\alpha}\dot{Z}\sin\alpha$$

The Lagrangian is

$$L = \frac{1}{2}m\left(l^2\dot{\alpha}^2 + \dot{Z}^2\right) + ml\dot{\alpha}\dot{Z}\sin\alpha - mg(Z - l\cos\alpha)$$

Lagrange's equations of motion for α is:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} &= 0\\
\frac{d}{dt}\left(ml^2\dot{\alpha} + ml\dot{Z}\sin\alpha\right) - \left(ml\dot{\alpha}\dot{Z}\cos\alpha - mgl\sin\alpha\right) &= 0\\
ml^2\ddot{\alpha} + ml\ddot{Z}\sin\alpha + mgl\sin\alpha &= 0\\
l\ddot{\alpha} = -(g + a_z)\sin\alpha &
\end{aligned}$$

We see that the acceleration of the elevator adds to gravity's acceleration. For small oscillations, the frequency is $\omega^2 = (g + a_z)/l$, and will be larger if the suspension point has an upwards acceleration (opposite of gravity), and will be smaller if the suspension point is accelerating downwards. If the elevator is in free fall, $a_z = -g$, and $\ddot{\alpha} = 0$: the pendulum mass has a constant velocity, and there is no restoring force.

If we wanted to get an expression for the tension, we'd consider the length of the string l as a variable, subject to a constraint $f = l - l_0 = 0$. The Lagrangian then is a function of two variables α, l and their time derivatives. There will be two equations of motion, for three unknown variables α, l and a Lagrange multiplier λ. The third equation is the constraint, so we'll have three equations for three variables.

The mass coordinates are still $x = l\sin\alpha$ and $z = Z - l\cos\alpha$, but now l is not constant, and the velocity has components

$$\dot{x} = \dot{l}\sin\alpha + l\dot{\alpha}\cos\alpha$$

$$\dot{z} = \dot{Z} - \dot{l}\cos\alpha + l\dot{\alpha}\sin\alpha$$

The Lagrangian is then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\alpha}^2 + 2\dot{Z}(l\dot{\alpha}\sin\alpha - \dot{l}\cos\alpha\right) - mg(Z - l\cos\alpha)$$

Lagrange's equation for α is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} &= \lambda\frac{\partial}{\partial\alpha}(l - l_0) \\
\frac{d}{dt}\left(ml^2\dot{\alpha} + m\dot{Z}l\sin\alpha\right) - \left(m\dot{Z}\left(l\dot{\alpha}\cos\alpha + \dot{l}\sin\alpha\right) - mgl\sin\alpha\right) &= 0 \\
ml^2\ddot{\alpha} + m\ddot{Z}l\sin\alpha + m\dot{Z}\dot{l}\sin\alpha + m\dot{Z}l\dot{\alpha}\cos\alpha - m\dot{Z}l\dot{\alpha}\cos\alpha - m\dot{Z}\dot{l}\sin\alpha + mgl\sin\alpha &= 0 \\
l\ddot{\alpha} + (g + a_z)\sin\alpha &= 0
\end{aligned}$$

which is, of course, the same equation we had earlier.

Lagrange's equation for l involves the Lagrange multiplier λ:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{l}} - \frac{\partial L}{\partial l} &= \lambda\frac{\partial}{\partial l}(l - l_0) \\
\frac{d}{dt}\left(m\dot{l} - m\dot{Z}\cos\alpha\right) - \left(ml\dot{\alpha}^2 + m\dot{Z}\dot{\alpha}\sin\alpha + mg\cos\alpha\right) &= \lambda \\
m\ddot{l} - m\ddot{Z}\cos\alpha + m\dot{Z}\dot{\alpha}\sin\alpha - ml\dot{\alpha}^2 - m\dot{Z}\dot{\alpha}\sin\alpha - mg\cos\alpha &= \lambda \\
m\ddot{l} - m(a_z + g)\cos\alpha - ml\dot{\alpha}^2 &= \lambda
\end{aligned}$$

We now use the solution to the constraint, $l = l_0$, so $\ddot{l} = 0$ and

$$-\lambda = ml\dot{\alpha}^2 + m(g + a_z)\cos\alpha$$

is the tension in the string. An upwards acceleration of the suspension point will increase the tension in the string, while a downwards acceleration will decrease it.

Problem 2 (40 points)

Consider a particle of mass m moving in a potential of the form

$$V(r) = \frac{\alpha}{r} - \frac{\beta}{r^2}$$

where the constants α, β are positive: the force is repulsive at long distances, and attractive at short distances.

(a) **Sketch the possible curves for effective potential as a function of radial distance.** The potential is a central potential, so the vector $\vec{\mathbf{L}}$ with magnitude l is conserved, and the particle's motion is in a plane perpendicular to $\vec{\mathbf{L}}$. The effective potential for a particle with angular momentum l is

$$V_{\text{eff}} = \frac{l^2}{2mr^2} + V = \frac{l^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2} = \frac{\alpha}{r} - \frac{\beta'}{r^2}$$

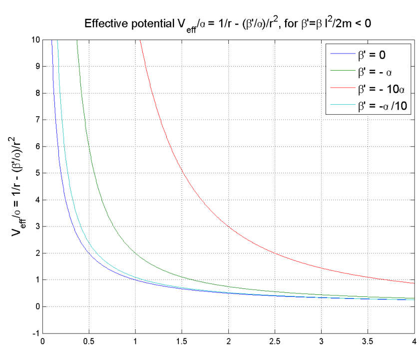
where $\beta' = \beta - l^2/sm$ can be positive, negative or zero depending on the magnitude of the angular momentum. At large distances, $V\text{eff} \approx \alpha/r$: is positive, and has a negative slope. At short distances, $V_{\text{eff}} \approx -\beta'/r^2$. If $\beta' < 0$ ($l^2 > 2m\beta$), V_{eff} at short distances is positive and has a negative slope, just like at long distances. If $\beta' > 0$ ($l^2 < 2m\beta$), V_{eff} at short distances is negative and has a positive slope. In this case, since the slope has a different sign than at long distances, it must be zero at one point at least.

Extrema of the effective potential (local maxima, minima or inflexion points) happen when

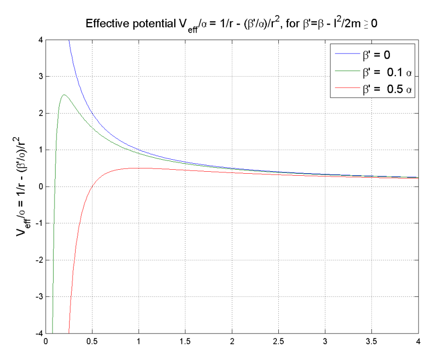
$$\frac{d}{dr}V_{\text{eff}} = 0 \rightarrow -\frac{\alpha}{r^2} + 2\frac{\beta'}{r^3} = 0$$

There is only one solution to this equation, $r_0 = 2\beta'/\alpha$, but it is only positive when $\beta' > 0$. Thus, when $l^2 \geq 2m\beta$, there are no extrema, and when $l^{<}2m\beta$, there is one local maximum: $V_0 = V(r_0) = \alpha^2/4\beta'$.

The possible curves for V_{eff} are shown below:

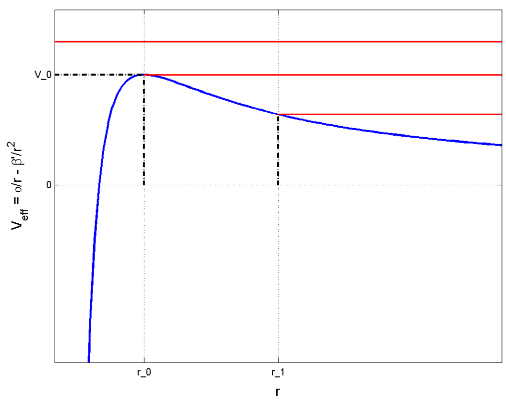


Effective potential for $\beta \leq l^2/2m$.



Effective potential for $\beta \geq l^2/2m$.

(b) **Assume the angular momentum satisfies $l^2 < 2m\beta$. Classify and sketch the possible orbits (bound, unbound, with or without turning points) for a particle approaching the force center from infinity, with impact parameter s.** If the particle is approaching from infinity, its orbit is unbound. Whether it has turning points or not depends on the particle's energy, which is equal to the kinetic energy of the particle at $r = \infty$, and thus is positive. There are three qualitative different cases for the energy values: $E > V_0$, $E = V_0$, and $E < V_0$, as shown in the figure below.



If $E > V_0$, there are no turning points, and the particle falls into the center. The particle does not go through the center force because the force is infinite at that point, and thus the potential cannot be a good description of the actual physical force. In general, we would consider the potential a good descriptor for $r > r_{min}$ (such as the radius of a planet for gravitational forces), and assume that other forces act for smaller distances (like contact forces with the surface of the planet).

If $E = V_0$, the radial velocity will be zero at $r = r_0 = 2\beta'/\alpha$, and the particle will remain in an unstable circular orbit, with constant angular velocity $\dot{\theta} = l/mr_0^2$.

If $E < V_0$, there will be a turning point at $r = r_1 > r_0$ when $E = V_eff(r_1)$. The radial distance r_1 indicates a minimum distance from the force center. The particle will move away from the center with positive radial velocity after the point of closest approach.

There are two interesting cases, when $E = V_0(1+\epsilon)$, If $\epsilon > 0$, the energy is sightly larger than V_0. The particle will slow down at r_0, and circle around the center with slowly decreasing radius, until the radial velocity increases and the particle moves on towards its fall into the center. If $\epsilon < 0$, the energy is slightly smaller than V_0. The particle will have a small radial velocity before and after the point of closest approach: the particle will circle around the center many times with decreasing radius, it will reach the minimum radius, then circle again many times with increasing radius, then moving away with increasing radial velocity and decreasing angular velocity, approaching a linear trajectory with a scattering angle.

(c) **Extra credit (5 points): What are the conditions on energy, angular momentum and initial postion for a particle to have a bound orbit? Sketch such an orbit, assuming it starts with $\dot{r} > 0$.**

There are no bound orbits if $l^2 \geq 2m\beta$, so the condition on the angular momentum is $l^2, 2m\beta$.

Looking at the effective potential from part (b), we see that there are bound orbits if the energy is smaller than the maximum of the effective potential: $E < V_0 = \alpha^2/4(\beta - l^2/2m)$.

For these energies, there are two kinds of orbits allowed with two different ranges of radial distances; the bound orbits are for particles with an initial position $r < r_1$, where $V_{\text{eff}}(r_1) = E$ and $r_1 < 2(\beta - l^2/2m)/\alpha$.

The orbits will have a turning point if they start with a positive radial velocity, but will eventually turn around and fall into the center. As the energy gets closer to V_0, the particle may go around the center many times before and after the turning point.

This information is enough to sketch the orbits for a particle approaching from infinity with impact parameter s, but we can do better and solve the orbit equation (this was, of course, not needed in the exam). If we use $u = 1/r$, the potential is $V(u) = \alpha u - \beta u^2$, and the orbit equation is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dV}{du} = -\frac{m}{l^2}(\alpha - 2\beta u)$$

$$\frac{d^2u}{d\theta^2} + \omega^2 u = -\frac{m\alpha}{l^2}$$

where $\omega^2 = 1 - 2m\beta/l^2$. Notice that $\omega^2 > 0$ only when $l^2 > 2m\beta$ (and we know that orbits are unbound, with a single turning point). We also have $w^2 < 1$. In this case, the general solution is

$$u(\theta) = \frac{1}{r(\theta)} = -\frac{m\alpha}{l^2\omega^2} + u_0\cos(\omega\theta - \phi_0) = \frac{\epsilon\cos(\omega\theta - \phi_0) - 1}{A}$$

where $A = l^2\omega^2/m\alpha$ and ϵ, ϕ_0 (or u_0, ϕ_0) are constants of integration. We can relate the constant ϵ to the energy:

$$\begin{aligned}
E &= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \\
&= \frac{1}{2}\frac{m}{u^4}\dot{u}^2 + \frac{l^2u^2}{2m} + V(u) \\
&= \frac{1}{2}\frac{m}{u^4}\dot{\theta}^2\left(\frac{du}{d\theta}\right)^2 + \frac{l^2u^2}{2m} + \alpha u - \beta u^2 \\
&= \frac{1}{2}\frac{m}{u^4}\left(\frac{lu^2}{m}\right)^2\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}\omega^2u^2 + \alpha u \\
&= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}\omega^2u^2 + \alpha u \\
&= \frac{l^2}{2m}\left(\frac{-\epsilon\omega\sin\Theta}{A}\right)^2 + \frac{l^2\omega^2}{2m}\left(\frac{\epsilon\cos\Theta - 1}{A}\right)^2 + \alpha\frac{\epsilon\cos\Theta - 1}{A} \\
&= \frac{\alpha}{2A}\epsilon^2 - \frac{\alpha}{2A} \\
\epsilon^2 &= 1 + \frac{2EA}{\alpha} = 1 + \frac{2El^2\omega^2}{m\alpha^2}
\end{aligned}$$

(We have used a shorthand $\Theta = \omega\theta - \phi_0$.) The distance approaches infinity when $\cos(\omega\theta - \phi_0) \to 1/\epsilon$. The closest approach (max u) is when $\omega\theta - \phi_0 = 0$.

If $l^2 < 2m\beta$, the orbit equation can be written as

$$\frac{d^2u}{d\theta^2} - \omega^2 u = -\frac{m\alpha}{l^2}$$

where $\omega^2 = 2m\beta/l^2 - 1$. Now $\omega^2 > 0$ only when $l^2 < 2m\beta$. The range of w is not limited, as in the previous case, it can be close to zero or infinity. In this case, the general solution is

$$u(\theta) = u_0\cosh(\omega\theta - \phi_0) + \frac{m\alpha}{l^2\omega^2} = \frac{1 + \epsilon\cosh(\omega\theta - \phi_0)}{A}$$

where $A = l^2\omega^2/m\alpha$. We can relate the constant of integration ϵ to the energy:

$$\begin{aligned}
E &= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2u^2}{2m} + \alpha u - \beta u^2 \\
&= \frac{l^2}{2m}\left(\frac{\epsilon\omega\sinh\Theta}{A}\right)^2 - \frac{l^2\omega^2}{2m}\left(\frac{1+\epsilon\cosh(\omega\theta-\phi_0)}{A}\right)^2 + \alpha\frac{1+\epsilon\cosh(\omega\theta-\phi_0)}{A} \\
&= -\frac{l^2\omega^2\epsilon^2}{2mA^2} - \frac{l^2\epsilon^2}{2mA^2} + \frac{\alpha}{A} \\
&= \frac{\alpha}{2A}\left(\epsilon^2 - 1\right) \\
\epsilon^2 &= 1 + \frac{2EA}{\alpha} = 1 + \frac{2El^2\omega^2}{m\alpha^2}
\end{aligned}$$

This is the same expression as before, except that now E can be negative, so ϵ might be smaller than unity.