Phys 7221 Hwk #9: Small Oscillations

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Prob 6-4: Double Pendulum

We follow the conventions for angles in Figure 1.4 (notice that θ_1 is counterclockwise, and θ_2 is clockwise!). We set up a coordinate system with the origin at the top suspension point, the x-axis pointing towards the right and the y-vertical axis pointing down. We will use the angles θ_1, θ_2 as the generalized coordinates for the system. We are then looking for two normal modes, with eigenfrequencies satisfying the equation $|\mathbf{V} - \omega_k^2 \mathbf{T}| = 0$, and eigenvectors $\mathbf{a}_1 = (a_{11}, a_{12})$ and $\mathbf{a}_2 = (a_{21}, a_{22})$, that satisfy the equation $(\mathbf{V} - \omega_k^2 \mathbf{T}) \cdot \mathbf{a}_k = 0$.

We first find expressions for the kinetic and potential energy matrices $\mathbf{T}$ and $\mathbf{V}$.

The position vector of each mass is

$$\begin{aligned}
\mathbf{r}_1 = (x_1, y_1) &= l(\sin\theta_1, \cos\theta_1) \\
\mathbf{r}_2 = (x_2, y_2) &= \mathbf{r}_1 + l(-\sin\theta_2, \cos\theta_2) = l(\sin\theta_1 - \sin\theta_2, \cos\theta_1 + \cos\theta_2)
\end{aligned}$$

The velocity vectors are

$$\begin{aligned}
\mathbf{v}_1 = (\dot{x}_1, \dot{y}_1) &= l\dot{\theta}_1(\cos\theta_1, -\sin\theta_1) \\
\mathbf{v}_2 = (\dot{x}_2, \dot{y}_2) &= l\dot{\theta}_1(\cos\theta_1, -\sin\theta_1) - l\dot{\theta}_2(\cos\theta_2, \sin\theta_2)
\end{aligned}$$

The kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\
&= \frac{1}{2}m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2}l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 - 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2)\right) \\
&= \frac{1}{2}(m_1 + m_2)l^2\dot{\theta}_1^2 + \frac{1}{2}l^2\dot{\theta}_2^2 - l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2)
\end{aligned}$$

For small oscillations, we use $\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2) \approx \dot{\theta}_1\dot{\theta}_2$, and then the (non-diagonal!) kinetic energy matrix is

$$\mathbf{T} = l^2 \begin{pmatrix} m_1 + m_2 & -m_2 \\ -m_2 & m_2 \end{pmatrix}$$

The gravitational potential energy is

$$\begin{aligned}
V &= -m_1 g y_1 - m_2 g y_2 \\
&= -m_1 g l \cos\theta_1 - m_2 g l(\cos\theta_1 + \cos\theta_2) \\
&= -(m_1+m_2) g l \cos\theta_1 - m_2 g l \cos\theta_2 \\
&\approx -(m_1+2m_2) g l + \frac{1}{2}(m_1+m_2) g l \theta_1^2 + \frac{1}{2} m_2 g l \theta_2^2
\end{aligned}$$

and the (diagonal!) potential energy matrix is

$$\mathbf{V} = gl \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

The secular equation for the eigenfrequencies is

$$\begin{aligned}
0 &= \left|\mathbf{V} - \omega^2 \mathbf{T}\right| \\
&= l^2 \begin{vmatrix} (m_1+m_2)(\omega_0^2 - \omega^2) & m_2\omega^2 \\ m_2\omega^2 & m_2(\omega_0^2-\omega^2) \end{vmatrix} \\
&= (m_1+m_2) m_2 (\omega_0^2-\omega^2)^2 - \omega^4 m_2 \\
&= m_2\left(m_1\omega^4 - 2(m_1+m_2)\omega_0^2\omega^2 + \omega_0^4(m_1+m_2)\right)
\end{aligned}$$

where we have defined $\omega_0^2 = g/l$.

The solutions are

$$w_\pm^2 = \omega_0^2 \frac{m_1+m_2}{m_1}\left(1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right) = \omega_0^2\left(1 + \frac{m_2}{m_1} \pm \sqrt{\frac{(m_1+m_2)m_2}{m_1^2}}\right)$$

The equations for the normal modes are $(\mathbf{V} - \omega_\pm^2 \mathbf{T}) \cdot \mathbf{a}_\pm = 0$, with $\mathbf{a}_\pm = (a_{\pm 1}, a_{\pm 2})$. We can only get a solution for the ratio of the vector components:

$$\begin{aligned}
-m_2\omega_\pm^2 a_{\pm 2} &= (m_1+m_2)(\omega_0^2 - \omega_\pm^2) a_{\pm 1} \\
\omega_\pm^2 \frac{a_{\pm 2}}{a_{\pm 1}} &= \frac{m_1+m_2}{m_2}\omega_0^2\left(\frac{m_2}{m_1} \pm \sqrt{\frac{(m_1+m_2)m_2}{m_1^2}}\right) \\
&= \pm\frac{m_1+m_2}{m_2}\omega_0^2\sqrt{\frac{(m_1+m_2)m_2}{m_1^2}}\left(1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right) \\
\frac{a_{\pm 2}}{a_{\pm 1}} &= \pm\sqrt{\frac{m_1+m_2}{m_2}}
\end{aligned}$$

Thus, up to a normalization constant $a_\pm$, the modes are

$$\mathbf{a}_\pm = a_\pm\left(1, \pm\sqrt{\frac{m_1+m_2}{m_2}}\right)$$

We normalize the normal modes so that $\tilde{\mathbf{a}}_\pm \cdot \mathbf{T} \cdot \mathbf{a}_\pm = 1$, so

$$1 = a_\pm^2 (m_1 + m_2 + m_2 \frac{m_1 + m_2}{m_2}) = 2(m_1 + m_2) a_\pm^2$$

and

$$\mathbf{a}_\pm = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{m_1 + m_2}}, \pm \frac{1}{\sqrt{m_2}} \right)$$

Since the normal modes represent the solutions for the angles θ_1, θ_2 in each case, we see than the lower frequency mode has angles with opposite sign, and according to our convention has then the two masses moving "in phase"; the higher frequency mode has the angles with the same sign, which means the masses are "out of phase".

Beats

The general motion of the system is given by a combination of the normal modes:

$$\begin{aligned}
\theta_1(t) &= \frac{1}{\sqrt{2(m_1 + m_2)}} \left(C_+ \cos(\omega_+ t + \phi_+) + C_- \cos(\omega_- t + \phi_-) \right) \\
\theta_2(t) &= \frac{1}{\sqrt{2 m_2}} \left(C_+ \cos(\omega_+ t + \phi_+) - C_- \cos(\omega_- t + \phi_-) \right)
\end{aligned}$$

where the constants $C_\pm, \phi_\pm$ are chosen according to the initial conditions of the system. If the system is set in motion pulling the top mass slightly away from vertical and releasing it from rest, the initial conditions are $\theta_1(0) = \theta_0, \theta_2(0) = 0, \dot{\theta}_1(0) = 0, \dot{\theta}_2(0) = 0$:

$$\begin{aligned}
\theta_1(0) = \theta_0 &= \frac{C_+ \cos\phi_+ + C_- \cos\phi_-}{\sqrt{2(m_1 + m_2)}} \\
\theta_2(0) = 0 &= \frac{C_+ \cos\phi_+ - C_- \cos\phi_-}{\sqrt{2 m_2}} \\
\dot{\theta}_1(0) = 0 &= -\frac{\omega_+ C_+ \sin\phi_+ + \omega_- C_- \sin\phi_-}{\sqrt{2(m_1 + m_2)}} \\
\dot{\theta}_2(0) = 0 &= -\frac{\omega_+ C_+ \sin\phi_+ - \omega_- C_- \sin\phi_-}{\sqrt{2 m_2}}
\end{aligned}$$

The velocity conditions are solved by $\phi_+ = \phi_- = 0$; then the condition $\theta_2(0) = 0$ tells us that $C_+ = C_- = C$. Finally, the $\theta_1(0) = \theta_0$ condition tells us that $C = \theta_0 \sqrt{m_1 + m_2}/\sqrt{2}$, and the motion is

$$\begin{aligned}
\theta_1(t) &= \frac{1}{\sqrt{2(m_1 + m_2)}} \left(C_+ \cos(\omega_+ t + \phi_+) + C_- \cos(\omega_- t + \phi_-) \right) \\
&= \frac{\theta_0}{2} (\cos(\omega_+ t) + \cos(\omega_- t))
\end{aligned}$$

$$\begin{aligned}\theta_2(t) &= \frac{1}{\sqrt{2m_2}}\left(C_+\cos(\omega_+ t+\phi_+) - C_-\cos(\omega_- t+\phi_-)\right)\\ &= \frac{\theta_0}{2}\sqrt{\frac{m_1+m_2}{m_2}}(\cos(\omega_+ t)-\cos(\omega_- t))\end{aligned}$$

It is convenient to describe the motion in terms of the sum and difference of the eigenfrequencies: we define $\bar{\omega} = (\omega_+ + \omega_-)/2$ and $\Delta\omega = (\omega_+ - \omega_-)/2$. Then, $\omega_\pm = \bar{\omega} \pm \Delta\omega$ and

$$\begin{aligned}\cos(\omega_+ t) \pm \cos(\omega_- t) &= \cos(\bar{\omega}t + \Delta\omega t) \pm \cos(\bar{\omega}t - \Delta\omega t)\\ &= (1\pm 1)\cos(\bar{\omega}t)\cos(\Delta\omega t) - (1\mp 1)\sin(\bar{\omega}t)\sin(\Delta\omega t)\end{aligned}$$

The solutions for θ_1, θ_2 are then

$$\begin{aligned}\theta_1(t) &= \theta_0\cos(\bar{\omega}t)\cos(\Delta\omega t)\\ \theta_2(t) &= -\theta_0\sqrt{\frac{m_1+m_2}{m_2}}\sin(\bar{\omega}t)\sin(\Delta\omega t)\end{aligned}$$

Since $\bar{\omega} > \Delta\omega$, the solutions can be considered oscillations at the higher frequency $\bar{\omega}$, with a periodic amplitude modulation with frequency $\Delta\omega$. The amplitude modulation for θ_1, θ_2 is 90^o out of phase, so when the amplitude of the oscillation is zero for θ_1 (when $t = (2n+1)\pi/2\Delta\omega$), the amplitude modulation for θ_2 will be maximum: this is the phenomenon of "beats". If the eigenfrequencies are very close, the beat frequency is very low; if the eigenfrequencies are very different, the beat frequency is very high. If $m_1 = m_2$, the eigenfrequencies are $\omega_\pm^2 = \omega_0^2(2\pm\sqrt{2})$, $\bar{\omega} = 1.31\omega_0$, and $\Delta\omega = 0.54\omega_0$: beats happen approximately once every two and a half cycles of $\bar{\omega}$.

Limiting cases: $m_1 \gg m_2$ and $m_1 \ll m_2$

If $m_1 \gg m_2$, the eigenfrequencies are

$$w_\pm = \omega_0^2\frac{m_1+m_2}{m_1}\left(1\pm\sqrt{\frac{m_2}{m_1+m_2}}\right) \approx \omega_0^2\left(1\pm\sqrt{\frac{m_2}{m_1}}\right)$$

only slightly different than the pendulum frequency of the top mass alone. The eigenmodes are

$$\mathbf{a}_\pm = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{m_1+m_2}}, \pm\frac{1}{\sqrt{m_2}}\right) \approx \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{m_1}}, \pm\frac{1}{\sqrt{m_2}}\right)$$

showing that $\theta_2 \ll \theta_1$ and the lower string remains approximately vertical.

If $m_1 \ll m_2$, the eigenfrequencies are

$$w_\pm = \omega_0^2\frac{m_1+m_2}{m_1}\left(1\pm\sqrt{\frac{m_2}{m_1+m_2}}\right)$$

$$
\begin{aligned}
&= \omega_0^2 \frac{m_2}{m_1}\left(1+\frac{m_1}{m_2}\right)\left(1 \pm \sqrt{\frac{1}{1+m_1/m_2}}\right) \\
&\approx \omega_0^2 \frac{m_2}{m_1}\left(1+\frac{m_1}{m_2}\right)\left(1 \pm \left(1-\frac{1}{2}\frac{m_1}{m_2}\right)\right) \\
\omega_+^2 &\approx 2\omega_0^2 \frac{m_2}{m_1} \\
\omega_-^2 &\approx \frac{1}{2}\omega_0^2 = \frac{g}{2l}
\end{aligned}
$$

As m_1/m_2 vanishes, the lower frequency is that one of a simple pendulum of length $2l$, and the higher frequency approaches ∞. The normal modes are

$$
\mathbf{a}_\pm = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{m_1+m_2}}, \pm\frac{1}{\sqrt{m_2}}\right) \approx \frac{1}{\sqrt{2m_2}}(1, \pm 1)
$$

showing that the low frequency mode has $\theta_1 \approx -\theta_2$, with the two strings almost aligned (like in a simple pendulum); the high frequency mode has $\theta_1 \approx \theta_2$, which has the bottom mass almost at rest, and the top (lighter) mass oscillating at high frequency.

Problem 6-8,9: Right triangle molecule

We will use generalized coordinates measuring the deviation from equilibrium of each mass:

$$
\begin{aligned}
\mathbf{r}_1 &= (x_1, y_1) \\
\mathbf{r}_2 &= (l + x_2, y_2) \\
\mathbf{r}_3 &= (x_3, l + y_3)
\end{aligned}
$$

The kinetic energy will be

$$
T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2)
$$

so the kinetic energy matrix is the mass times the identity matrix: $\mathbf{T} = m\mathbf{I}$.

The potential energy has three terms, one from each spring. For small oscillations, we wil assume $x_i, y_i \ll l$.

The potential energy of the spring along the x-axis is

$$
\begin{aligned}
V_{12} &= \frac{1}{2}k(|\mathbf{r}_2 - \mathbf{r}_1| - l)^2 \\
|\mathbf{r}_2 - \mathbf{r}_1| &= \sqrt{(l + x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= l\sqrt{1 + 2\frac{x_2 - x_1}{l} + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{l^2}}
\end{aligned}
$$

$$
\begin{aligned}
&\approx l\left(1+\frac{x_2-x_1}{l}\right)\\
&\approx l+x_2-x_1\\
V_{12} &\approx \frac{1}{2}k(x_2-x_1)^2
\end{aligned}
$$

Similarly,

$$V_{13}=\frac{1}{2}k(y_3-y_1)^2$$

For the spring in the x-y plane, the potential energy is

$$
\begin{aligned}
V_{23} &= \frac{1}{2}k(|\mathbf{r}_3-\mathbf{r}_2|-\sqrt{2}l)^2\\
|\mathbf{r}_3-\mathbf{r}_2| &= \sqrt{(x_3-x_2-l)^2+(l+y_3-y_3)^2}\\
&\approx \sqrt{2}l\sqrt{1-\frac{x_3-x_2}{l}+\frac{y_3-y_2}{l}}\\
&\approx \sqrt{2}l\left(1-\frac{1}{2}\frac{x_3-x_2}{l}+\frac{1}{2}\frac{y_3-y_2}{l}\right)\\
&\approx \sqrt{2}l-\frac{x_3-x_2}{\sqrt{2}}+\frac{y_3-y_2}{\sqrt{2}}\\
V_{13} &\approx \frac{1}{4}k(x_3-x_2)^2+\frac{1}{4}k(y_3-y_2)^2-\frac{1}{2}k(x_3-x_2)(y_3-y_2)
\end{aligned}
$$

The total potential energy is then

$$V\approx\frac{1}{2}k(x_2-x_1)^2+\frac{1}{2}k(y_3-y_1)^2+\frac{1}{4}k(x_3-x_2)^2+\frac{1}{4}k(y_3-y_2)^2-\frac{1}{2}k(x_3-x_2)(y_3-y_2)$$

and the potential energy matrix, a 6x6 matrix for $x_1, y_1, x_2, y_2, x_3, y_3$ is

$$
\mathbf{V}=k\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1\\
-1 & 0 & 3/2 & -1/2 & -1/2 & 1/2\\
0 & 0 & -1/2 & 1/2 & 1/2 & -1/2\\
0 & 0 & -1/2 & 1/2 & 1/2 & -1/2\\
0 & -1 & 1/2 & -1/2 & -1/2 & 3/2
\end{pmatrix}
$$

The secular equation is

$$
|\mathbf{V}-\omega^2\mathbf{T}|=k\begin{vmatrix}
1-m\omega^2/k & 0 & -1 & 0 & 0 & 0\\
0 & 1-m\omega^2/k & 0 & 0 & 0 & -1\\
-1 & 0 & 3/2-m\omega^2/k & -1/2 & -1/2 & 1/2\\
0 & 0 & -1/2 & 1/2-m\omega^2/k & 1/2 & -1/2\\
0 & 0 & -1/2 & 1/2 & 1/2-m\omega^2/k & -1/2\\
0 & -1 & 1/2 & -1/2 & -1/2 & 3/2-m\omega^2/k
\end{vmatrix}
$$

After some algebra, or using a computer program (admittedly what I did...), the secular equation is

$$|\mathbf{V} - \omega^2 \mathbf{T}| = -k(m\omega^2/k)^3(m\omega^2/k - 1)(m\omega^2 - 2)(m\omega^2 - 3)$$

which leads to the set of six eigenfrequencies:

$$\omega_k^2 = \{0, 0, 0, k/m, 2k/m, 3k/m\}$$

The null eigenfrequencies correspond to the rigid body motions of the molecule: translation in the x-y plane (2 modes), and rotation about the z-axis (1 mode). We can see that because the translation vectors vectors

$$\mathbf{a}_x = x(1, 0, 1, 0, 1, 0,)$$

$$\mathbf{a}_y = y(0, 1, 0, 1, 0, 1),$$

and the rotation vector about the particle at the origin

$$\mathbf{a}_\theta = l(0, 0, 1 - \cos\theta, \sin\theta, -\sin\theta, 1 - \cos\theta) \approx l\theta(0, 0, 0, 1, -1, 0)$$

all satisfy $(\mathbf{V} - \omega_k^2 \mathbf{T}) \cdot \mathbf{a}_k = \mathbf{V} \cdot \mathbf{a}_k = 0$.

The other three eigenmodes are:

$$\mathbf{a}_1 = (-1, -1, 0, 1, 1, 0)$$

for $\omega_1^2 = k/m$, with springs along the x, y axes stretching and compressing in phase with each other but out of phase with the diagonal spring. The normal mode satisfies satisfying $(\mathbf{V} - \omega_1^2 \mathbf{T}) \cdot \mathbf{a}_1 = (\mathbf{V} - k\mathbf{T}) \cdot \mathbf{a}_1 = 0$.

$$\mathbf{a}_2 = (1, -1, -1, 0, 0, 1)$$

with $\omega_2^2 = 2k/m$, with side springs stretching and compressing out of phase phase with each other.

$$\mathbf{a}_3 = (1, 1, -2, 1, 1, -2)$$

for $\omega^2 = 3k/m$, with all springs stretching and compressing in phase.

Problem 6-11: Rod hanging on springs

We will use as generalized coordinates the position of the center of mass of the rod (two coordinates), and the rotation angle of the rod with respect to the horizontal position. We set up a coordinate system with the origin in the suspension plane, halfway between the suspension points, with the x-axis pointing towards the right and the y-axis pointing down. We express the generalized coordinates in terms of deviation from the equilibrium position.

The equilibrium position will be $x_0 = 0, y_0 = y_0, \theta_0 = 0$, and we then use coordinates x, y, θ for deviations from equilibrium.

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$$

where $I = ml^2/12$ is the moment of inertia about an axis perpendicular to the plane of motion, with respect to the center of mass. In order to work with coordinates with the same units, we choose to work with $x, y, l\theta/2$, and the kinetic energy is then

$$T = \frac{1}{2}\left(m\dot{x}^2 + m\dot{y}^2 + m(l\dot{\theta}/2)^2/3\right)$$

and the kinetic energy matrix is $\mathbf{T} = \mathrm{diag}(m, m, m/3)$.

The potential energy is the sum of gravitational potential energy $V_g = -mg(y + y_0)$, and the potential energy in the springs. The length of the springs are equal to the distance between the suspension points $\mathbf{R}_\pm = (\pm L, 0)$ and the ends of the bar $\mathbf{r}_\pm = (x \pm l\cos\theta/2, y + y_0 \pm l\sin\theta/2)$:

$$\begin{aligned}
l_\pm &= |\mathbf{r}_\pm - \mathbf{R}_\pm| \\
&= \left((x \pm l\cos\theta/2 \mp L)^2 + (y + y_0 \pm l\sin\theta/2)^2\right)^{1/2} \\
&= \left(L^2 + (l/2)^2 + y_0^2 + x^2 + y^2 - lL\cos\theta \pm xl\cos\theta \mp 2xL \pm y_0 l\sin\theta + 2y_0 y \pm yl\sin\theta\right)^{1/2} \\
&\approx \left(L^2 + (l/2)^2 - lL + y_0^2 \pm xl \mp 2xL \pm y_0 l\theta + 2y_0 y\right)^{1/2} \\
&\approx \left((L - l/2)^2 + y_0^2 \mp 2(L - l/2)x + 2y_0 y \pm y_0 l\theta\right)^{1/2} \\
&\approx l_0\sqrt{1 \mp 2\frac{(L - l/2)x}{l_0^2} + 2\frac{y_0 y}{l^2} \pm \frac{y_0 l\theta}{l_0^2}} \\
&\approx l_0\left(1 \mp \frac{(L - l/2)x}{l_0^2} + \frac{y_0 y}{l^2} \pm \frac{y_0 l\theta}{2l_0^2}\right)
\end{aligned}$$

We have defined $l_0^2 = (L - l/2)^2 + y_0^2$, which is the length of the springs in equilibrium. We can use $\sin\theta_0 = (L - l/2)/l_0$ and $\cos\theta_0 = y_0/l_0$, so the spring lengths are

$$l_\pm = l_0 \mp x\sin\theta_0 + y\cos\theta_0 \pm (l\theta/2)\cos\theta_0$$

Notice that the angle θ_0 does not have anything to do with the generalized coordinate θ, and is *not* the equilibrium value of the angle θ ($\theta = 0$ in equilibrium, since the rod is horizontal in equilibrium).

The potential energy is

$$\begin{aligned}
V &= -mg(y + y_0) + \frac{1}{2}k(l_+ - b)^2 + \frac{1}{2}k(l_- - b)^2 \\
&\approx \left[k(l_0 - b)^2 - mgy_0\right] + \left[2k(l_0 - b)\cos\theta_0 - mg\right]y \\
&\quad + \left[k\sin^2\theta_0 x^2 + k\cos^2\theta_0 y^2 + k\cos^2\theta_0(l\theta_0/2)^2 - 2k\sin\theta_0\cos\theta_0 x(l\theta/2)\right]
\end{aligned}$$

The constant term can be ignored; the linear term tells us about the equilibrium length of the springs, balancing gravity and springs. The quadratic terms are the ones determining the equations of motion, with the potential matrix being (for the variables $x, y, l\theta/2$):

$$\mathbf{V} = 2k \begin{pmatrix} \sin^2\theta_0 & 0 & -\sin\theta_0\cos\theta_0 \\ 0 & \cos^2\theta_0 & 0 \\ -\sin\theta_0\cos\theta_0 & 0 & \cos^2\theta_0 \end{pmatrix}$$

The secular equation is is then

$$\begin{aligned} |\mathbf{V} - \omega^2\mathbf{T}| &= 2k \begin{vmatrix} \sin^2\theta_0 - m\omega^2/2k & 0 & -\sin\theta_0\cos\theta_0 \\ 0 & \cos^2\theta_0 - m\omega^2/2k & 0 \\ -\sin\theta_0\cos\theta_0 & 0 & \cos^2\theta_0 - m\omega^2/6k \end{vmatrix} \\ &= 2k(\cos^2\theta_0 - m\omega^2/2k)\left((\sin^2\theta_0 - m\omega^2/2k)(\cos^2\theta_0 - m\omega^2/6k) - \sin^2\theta_0\cos^2\theta_0\right) \\ &= m\omega^2(\cos^2\theta_0 - m\omega^2/2k)(m\omega^2(\cos^2\theta_0 - \sin^2\theta_0/3)/2k - 1) \end{aligned}$$

The eigenfrequencies are

$$\omega_i^2 = \{0, 2k\cos^2\theta_0/m, 2k(\cos^2\theta_0 + \sin^2\theta_0/3)/m\}.$$

The equations for the coefficients of the normal modes are

$$(2k\sin^2\theta_0 - m\omega_i^2)a_{i1} = 2k\sin\theta_0\cos\theta_0 a_{i3}$$

$$(2k\cos^2\theta_0 - m\omega_i^2)a_{i2} = 0$$

The null eigenfrequency mode ("DC mode") has the bar rotating so that the vertical position of the center of mass does not move, and the end points describe circles that keep the spring lengths calibrations: the potential energy remains constant. For the null eigenfrequency $\omega_1^2 = 0$, the eigenvector is

$$\mathbf{a}_1 = (\cos\theta_0, 0, \sin\theta_0)/\sqrt{m(\cos^2\theta_0 + \sin^2\theta_0/3)}$$

The eigenmode corresponding to $\omega_2^2 = 2k\cos^2\theta_0/m$ has only vertical motion of the bar, without rotation:

$$\mathbf{a}_2 = (0, 1, 0)/\sqrt{m}$$

The highest frequency eigenmode has no vertical motion of the bar's center of mass, like the DC mode, but the bar is rotating so that one spring is stretched and the other compressed:

$$\mathbf{a}_3 = A(-\sin\theta_0\cos\theta_0, 0, 1 - 5\sin^2\theta_0/3)$$

Problem 6-12: Two masses and three springs

There are two masses, whose position vectors are (using an origin on the left wall)

$$\mathbf{r}_1 = (a + x_1)\hat{\mathbf{i}},$$

$$\mathbf{r}_2 = (2a + x_2)\hat{\mathbf{i}}.$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

so the kinetic energy matrix is just a 2x2 identity matrix times the mass m:

$$\mathbf{T} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The potential energy is

$$\begin{aligned} V &= \frac{1}{2}k(r_1 - a)^2 + \frac{1}{2}3k(r_2 - r_1 - a)^2 + \frac{1}{2}k(2a - r_2)^2 \\ &= \frac{1}{2}kx_1^2 + \frac{3}{2}(x_1 - x_2)^2 + \frac{1}{2}kx_2^2 \\ &= \frac{1}{2}k(4x_1^2 - 6x_1x_2 + 4x_2^2) \end{aligned}$$

so the potential energy matrix is

$$\mathbf{V} = k\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$$

The eigenvalue equation is

$$|\mathbf{V} - \omega^2\mathbf{T}| = \begin{vmatrix} 4k - m\omega^2 & -3k \\ -3k & 4k - m\omega^2 \end{vmatrix} = (4k - m\omega^2)^2 - 9k^2 = (k - m\omega^2)(7k - m\omega^2)$$

so the eigenvalues are $\omega_\mp^2 = \{k/m, 7k/m\}$. The normal mode with frequency $\omega_-^2 = k/m$ has an eigenvector equation

$$0 = (\mathbf{V} - (k/m)\mathbf{T}) \cdot \mathbf{a}_- = \begin{pmatrix} 3k & -3k \\ -3k & 3k \end{pmatrix}\begin{pmatrix} a_{-1} \\ a_{-2} \end{pmatrix} = 3k\begin{pmatrix} a_{-1} - a_{-2} \\ -a_{-1} + a_{-2} \end{pmatrix}$$

which tells us that $\mathbf{a}_- = a_-(1,1)$ (when : the two masses move in phase, leaving the middle spring unstretched, and compressing the first spring at the same time the last spring is stretched.

The normal mode with frequency $\omega_-^2 = 7k/m$ has an eigenvector equation

$$0 = (\mathbf{V} - (k/m)\mathbf{T}) \cdot \mathbf{a}_+ = \begin{pmatrix} -3k & -3k \\ -3k & -3k \end{pmatrix} \begin{pmatrix} a_{+1} \\ a_{+2} \end{pmatrix} = 3k \begin{pmatrix} -a_{+1} - a_{+2} \\ -a_{+1} - a_{+2} \end{pmatrix}$$

which tells us that $\mathbf{a}_+ = a_+(1, -1)$: the two masses move out of phase, compressing the middle spring, while the first spring and the last spring are stretched.

Normalizing the normal modes with the kinetic energy matrix, we obtain

$$\begin{aligned} \mathbf{a}_- &= \frac{1}{\sqrt{2m}}(1, 1) \\ \mathbf{a}_+ &= \frac{1}{\sqrt{2m}}(1, -1) \end{aligned}$$

Problem 6-18: Spring and electromagnetic forces

The potential energy is

$$V = \frac{1}{2}kr^2 + q\phi - q\mathbf{A} \cdot \mathbf{v}$$

If the electric field is $\mathbf{E} = E\hat{i}$ and the magnetic field is $\mathbf{B} = b\hat{j}$, the electric potential is $\phi = -Ex$ and the vector potential is $\mathbf{A} = -Bx\hat{k}$, so that $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The choice for $\mathbf{A}$ is not unique, but the results are independent of the choice made.

The Lagrangian is then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2) + qEx - QBx\dot{z}$$

The equations of motion are

$$\begin{aligned} m\ddot{x} + kx - qE + qB\dot{z} &= 0 \\ m\ddot{z} - qB\dot{x} + kz &= 0 \\ m\ddot{y} + ky &= 0 \end{aligned}$$

The equations for x, z are coupled, while the equation for y is independent of x, z and has the well known simple harmonic oscillator solution wit frequency $\omega^2 = k/m$.

In equilibrium, the particle is displaced from the origin due to the electric force, balanced by the spring: $\mathbf{r}_0 = (qE/k, 0, 0)$. If we consider small deviatons from equilibrium, with the position vector $\mathbf{r} = (qE/k + x, y, z)$, then the equations for x, z are

$$\begin{aligned} m\ddot{x} + kx + qB\dot{z} &= 0 \\ m\ddot{z} + kz - qB\dot{x} &= 0 \end{aligned}$$

Following the treatment for problems with small oscillations, we propose solutions of the form $x = \Re(Xe^{-i\omega t})$, $z = \Re(Ze^{-i\omega t})$ and look for solutions to the complex constants X, Z:

$$\begin{aligned}(k - m\omega^2)X - iqB\omega Z &= 0\\ iqB\omega X + (k - m\omega^2)Z &= 0\end{aligned}$$

The equations will have non-zero solutions only if the frequency satisfy the secular equation, equal to the vanishing of the determinant of the matrix form of the equations fro X, Z:

$$(k - m\omega^2)^2 - (qB)^2\omega^2 = 0$$

If we define the frequencies $\omega_k^2 = k/m$ (frequency of the mass-spring system n the absence of EM fields), and $\omega_B = qB/m$ (cyclotron frequency for a charge in a magnetic field), then the solutions to the secular equation are

$$\omega_\pm^2 = \omega_k^2 + \frac{1}{2}\omega_B^2\left(1 \pm \sqrt{1 + 4\frac{\omega_k^2}{\omega_B^2}}\right)$$

The equation for the X, Z in each mode is

$$Z_\pm = \frac{1}{2}iX_\pm\frac{w_B}{\omega_\pm}\left(1 \pm \sqrt{1 + 4\frac{\omega_k^2}{\omega_B^2}}\right)$$

The third normal mode (since there must be three modes) is the oscillation along the y-axis with frequency $\sqrt{k/m}$

In the strong field limit, $\omega_B^2 \gg \omega_k^2$ $(B \gg \sqrt{mk}/q)$,

$$\begin{aligned}\omega_\pm^2 &\approx \omega_k^2 + \frac{1}{2}\omega_B^2\left(1 \pm \left(1 + 2\frac{\omega_k^2}{\omega_B^2} - 2\left(\frac{\omega_k^2}{\omega_B^2}\right)^2\right)\right)\\ \omega_- &\approx \frac{w_k^2}{\omega_B} = \frac{k}{qB}\\ \omega_+ &\approx \omega_B = \frac{qB}{m}\end{aligned}$$

For the lower eigenfrequency , the normal mode solution will have $Z \approx (-iqB\omega_-/k)X \approx -iX$, so the solutiosn for x, z are

$$x_-(t) = \Re\left(Xe^{-i\omega_- t}\right) = x_0\cos(kt/qB + \phi_0)$$

$$z_-(t) = \Re\left(-iXe^{-i\omega_- t}\right) = -x_0\sin(kt/qB + \phi_0)$$

The trajectory is a circle in the $x - z$ plane, with the angular velocity vector with magnitude ω_-, along the y-axis. The origin of the circle is displaced from the origin along the x-axis.

For the higher eigenfrequency, the normal mode solution has $Z \approx (iqB/m\omega_+)X \approx iX$ and the solutions for x, z are

$$x_+(t) = \Re\left(Xe^{-i\omega_+ t}\right) = x_0 \cos(qBt/m + \phi_0)$$

$$z_+(t) = \Re\left(iXe^{-i\omega_+ t}\right) = x_0 \sin(qBt/m + \phi_0)$$

The trajectory is again a circle in the x-z plane, with the center displaced from the origin, and with the angular velocity vector with magnitude ω_B, along the -y axis (the particle travels the circle in the opposite direction than in the lower eigenmode).

The third normal mode (since there must be three modes) is the oscillation along the y-axis with frequency $\sqrt{k/m}$, significantly larger than the ω_- and significantly smaller than ω_B.

In the low field limit, $\omega_B \ll \omega_k$, or $B \ll \sqrt{mk}/q$, the eigenfrequencies are

$$\begin{aligned}
\omega_\pm^2 &= \omega_k^2 + \frac{1}{2}\omega_B^2 \left(1 \pm \sqrt{1 + 4\frac{\omega_k^2}{\omega_B^2}}\right) \\
&= \omega_k^2 + \frac{1}{2}\omega_B^2 \left(1 \pm 2\frac{\omega_k}{\omega_B}\sqrt{1 + 4\frac{\omega_B^2}{\omega_k^2}}\right) \\
&\approx \omega_k^2 \pm \omega_k \omega_B
\end{aligned}$$

The magnetic field is splitting the degenerate x, z spring modes, making one frequency slightly higher and the other slightly lower than $\sqrt{k/m}$. The eigenmodes have ceofficients $Z_\pm \approx \pm iX_\pm$, so we see again that they travel in circles, in opposite directions for each mode.