Phys 7221 Homework # 8

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Derivation 5-6: Torque free symmetric top

In a torque free, symmetric top, with $I_x = I_y = I$, the angular velocity vector ω in body coordinates with axes along the principal axes, is given by

$$\begin{aligned}
\omega_x &= \omega_0 \cos \Omega t \\
\omega_y &= \omega_0 \sin \Omega t \\
\omega_z &= \omega_3
\end{aligned}$$

with $\Omega = \omega_3(I_3?I)/I$. Also in body coordinates, the angular momentum vector $\mathbf{L}$ has components

$$\begin{aligned}
L_x &= I\omega_x = I\omega_0 \cos \Omega t \\
L_y &= I\omega_y = I\omega_0 \sin \Omega t \\
L_z &= I_3\omega_z = I_3\omega_3
\end{aligned}$$

We see that the component of the angular momentum along the top symmetry axis, L_z, is constant, and the the components perpendicular to the symmetry axis rotate about the z axis with angular velocity Ω. Since the top is free of external torques, the angular momentum vector $\mathbf{L}$ is constant in an inertial system. In an inertial system, it is the tops symmetry axis the one that rotates about the $\mathbf{L}$ direction. We can use Euler angles to describe the rotation between body axes and inertial axes. Comparing the expressions for the components of ω in body axes we obtained and the expressions for the same components in terms of Eulers angles in (4.87):

$$\begin{aligned}
\omega_x &= \omega_0 \cos \Omega t = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\
\omega_y &= \omega_0 \sin \Omega t = \dot{\phi} \sin\theta \cos\psi + \dot{\theta} \sin\psi \\
\omega_z &= \omega_3 = \dot{\phi} \cos\theta + \dot{\psi}
\end{aligned}$$

we recognize that the Euler angle θ is constant, the bodys rotation about the symmetry axis is $\psi = \pi/2 - \Omega$, and

$$\begin{aligned}
\dot{\phi} \cos\theta &= \omega_3 - \dot{\psi} = \omega_3 + \Omega = I_3\omega_3/I \\
\dot{\phi} \sin\theta &= \omega_0
\end{aligned}$$

The angle θ is the angle between the symmetry axis and the angular momentum vector, and is determined by initial conditions. The bodys symmetry axis rotates about the angular momentum with constant angular velocity $\dot{\phi} = I_3\omega_3/I\cos\theta$. We now use the Euler angles we obtained to calculate the components of ω in the inertial system, using the expression from Derivation 15 in Chapter 4:

$$\begin{aligned}
\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi &= -\Omega\sin\theta\sin(\dot{\phi}t + \phi_0) \\
\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi &= \Omega sin\theta cos(\dot{\phi}t + \phi_0) \\
\omega_z = \dot{\psi}\cos\theta + \dot{\phi} &= -\Omega\cos\theta + I_3\omega_3/I
\end{aligned}$$

We see that the component of the angular momentum along the z-axis, or the direction of the angular momentum vector **L**, is constant, and the component of the angular velocity perpendicular to **L** is rotating with angular velocity $\dot{\phi}$. The angle between $\vec{\omega}$ and **L** is given by $\sin\theta' = \sqrt{\omega_x^2 + omega_y^2}/\omega = \Omega\sin\theta/\omega$. The angle between ω and the symmetry axis is given by $\sin\theta'' = \sqrt{\omega_{x'}^2 + \omega_{y'}^2}/\omega = \omega_0/\omega$. Then we have

$$\sin\theta' = \Omega\sin\theta/\omega = \Omega\sin\theta\sin\theta''/\omega 0 = \Omega\sin\theta''/\dot{\phi}$$

where we have used $\omega_0 = \dot{\phi}\sin\theta$, which we had obtained when solving for the Euler angles. We can also use moments of inertia for an expression of $\sin\theta'$:

$$\sin\theta' = \Omega\sin\theta''/\dot{\phi} = \Omega I\cos\theta\sin\theta''/I_3\omega_3 = ((I_3 - I)/I_3)\cos\theta\sin\theta''$$

For the Earth considered as a free symmetric top, we have $(I_3 - I)/I \approx 3\times 10^{-3}$, so the angle θ' is very small, independent of values of θ, θ'': the angular velocity vector $\vec{\omega}$ is very close to the angular momentum vector **L**. The measured distance $2R\sin\theta''$ is about 10m, so the distance $R\sin\theta' = (I_3 - I)/IR\sin\theta''\cos\theta \approx 15\text{mm}\cos\theta < 1.5\text{cm}$.

As seen in the body axes, the angular velocity vector describes a cone of aperture angle θ about the symmetry axis: this is called the body cone. As seen in the inertial frame, the angular velocity vector describes a cone of aperture angle θ , about the angular momentum vector: this is called the space cone. Both cones share the angular momentum vector along their sides at any given instant. The angular velocity vector is the instantaneous axis of rotation, so the cones are rolling without slipping on each other.

A very nice page with animations showing this example, by Prof. Eugene Butikov, can be found in `http://faculty.ifmo.ru/butikov/Applets/Precession.html` , from which Fig.1 is a snapshot.

Exercise 5-15

Consider a flat rigid body in the shape of a right triangle with uniform mass density $\sigma = M/A$, and area $A = a^2/2$, where a is the length of the equal sides of the triangle.

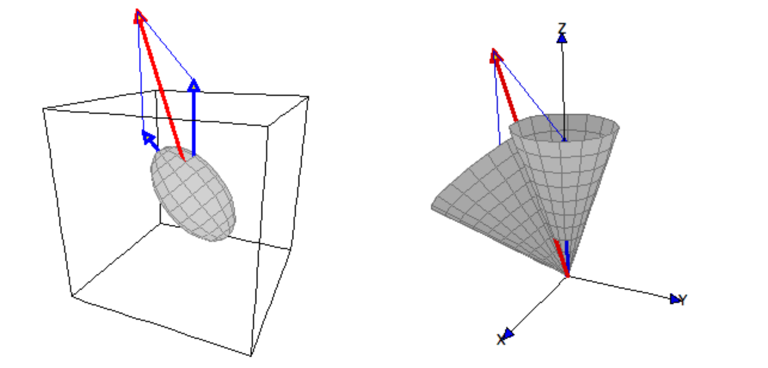

Figure 1: A rotating torque free symmetric top (left), and the associated space and body cones. The conserved angular momentum vector (blue) is along the z axis; the (red) instantaneous angular velocity vector is at the intersection of the cones, precessing about the z axis. From `http://faculty.ifmo.ru/butikov/Applets/Precession.html`

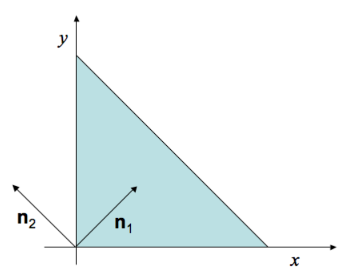

Figure 2: Exercise 5-15

Let us choose the right angle corner of the triangle as the origin of a coordinate system with the x, y axis along the sides of the triangle. The boundary of the triangle is given by $x + y = a$; the mass elements on the surface will have coordinates (x,y,0) with (x,y) within the triangle. The elements of the inertia tensor in such a system are

$$I_{xx} = \int_A \sigma(y^2 + z^2) dA = \sigma \int_0^a dx \int_0^{a-x} y^2 dy = \sigma \int_0^a \frac{(a-x)^3}{3} dx = \sigma \frac{a^4}{12} = \frac{Ma^2}{6}$$

$$I_{yy} = \int_A \sigma(x^2 + z^2) dA = \sigma \int_0^a dy \int_0^{a-y} x^2 dx = \frac{Ma^2}{6}$$

$$I_{xx} = \int_A \sigma(x^2 + y^2) dA = I_{xx} + I_{yy} = \frac{Ma^2}{3}$$

$$I_{zz} = I_{yx} = -\int_A \sigma xy dA = -\sigma \int_0^a x dx \int_0^{a-x} y dy = -\sigma \int_0^a x dx \frac{(a-x)^2}{2} = -\sigma \frac{a^4}{24} = -\frac{Ma^2}{12}$$

$$I_{xz} = -\int_A \sigma xz dA = 0 = I_{yz}$$

$$\mathbf{I} = \frac{Ma^2}{12} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

We look for eigenvalues of the inertia tensor, which will be solutions to the equation $det(\mathbf{I} - I\mathbf{1}) = 0$, or

$$det(\mathbf{I} - I\mathbf{1}) = \begin{vmatrix} 2k - I & -k & 0 \\ -k & 2k - I & 0 \\ 0 & 0 & 4k - I \end{vmatrix} = (4k - I)((2k - I)^2 - k^2)$$

with $k = Ma^2/12$. The three real, positive solutions are the principal moments of inertia:

$$(I_1, I_2, I_3) = (k, 3k, 4k) = (Ma^2/12, Ma^2/4, Ma^2/3).$$

The principal axes are the eigenvectors corresponding to each eigenvalue. If the eigenvectors have components $\vec{n}_i = (n_{ix}, n_{iy}, n_{iz})$, the equations are $\mathbf{I} \cdot \vec{n}_i = \alpha_i k \vec{n}_i$ with $\alpha_i = 1, 3, 4$. The equations for $\mathbf{n}_1$ are:

$$\mathbf{I} \cdot \underline{\mathrm{n}}_1 = k\mathbf{n}_1$$

$$k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} n_{1x} \\ n_{1y} \\ n_{1z} \end{pmatrix} = k \begin{pmatrix} n_{1x} \\ n_{1y} \\ n_{1z} \end{pmatrix}$$

$$\begin{pmatrix} 2n_{1x} - n_{1y} \\ -n_{1x} + 2n_{1y} \\ 4n_{1z} \end{pmatrix} = \begin{pmatrix} n_{1x} \\ n_{1y} \\ n_{1z} \end{pmatrix} \rightarrow n_{1x} = n_{1y},\ n_{1z} = 0.$$

The equations for $\mathbf{n}_2$ are:

$$\mathbf{I} \cdot \underline{\mathrm{n}}_2 = 3k\mathbf{n}_2$$

$$k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} n_{2x} \\ n_{2y} \\ n_{2z} \end{pmatrix} = 3k \begin{pmatrix} n_{2x} \\ n_{2y} \\ n_{2z} \end{pmatrix}$$

.

$$\begin{pmatrix} 2n_{2x} - n_{2y} \\ -n_{2x} + 2n_{2y} \\ 4n_{2z} \end{pmatrix} = 3 \begin{pmatrix} n_{2x} \\ n_{2y} \\ n_{2z} \end{pmatrix} \rightarrow n_{1x} = -n_{1y}, \ n_{1z} = 0$$

The equations for $\mathbf{n}_3$ are:

$$\mathbf{I} \cdot \underline{\mathrm{n}}_3 = 4k\mathbf{n}_3$$

$$k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} n_{3x} \\ n_{3y} \\ n_{3z} \end{pmatrix} = 4k \begin{pmatrix} n_{3x} \\ n_{3y} \\ n_{3z} \end{pmatrix}$$

.

$$\begin{pmatrix} 2n_{2x} - n_{2y} \\ -n_{2x} + 2n_{2y} \\ 4n_{2z} \end{pmatrix} = 4 \begin{pmatrix} n_{3x} \\ n_{3y} \\ n_{3z} \end{pmatrix} \rightarrow n_{3x} = -n_{3y} = 0, \ n_{3z} \neq 0$$

The eigenvectors with unit magnitude are then

$$\mathbf{n}_1 = (1/\sqrt{2})(1, 1, 0)$$
$$\mathbf{n}_2 = (1/\sqrt{2})(1, -1, 0)$$
$$\mathbf{n}_3 = (0, 0, 1)$$

Exercise 5-17: A rolling cone

The instantaneous axis of rotation, and thus the direction of the angular velocity $\vec{\omega}$ is along the line of contact of the cone with the surface. The center of mass is at a distance $3h/4$ along the axis of the cone, which then will be at a vertical height $a = 3h \sin\alpha/4$. The total mass of the cone is $\pi\rho h^3 \tan^2\alpha/3$.

Each cross section of the cone is like a disk perpendicular to the cone's axis (and thus tilted with respect to the vertical), moving in a circle. The velocity of the center of mass has magnitude v, and is related to the angular velocity is $\omega = v/a \sin\alpha$. We also know that, if there is no slipping, the velocity f the center of mass is related to the angular velocity as $v = a\dot{\theta}\cos\alpha$, where θ is the angle along the circle, and $\dot{\theta} = 2\pi/\tau$. Then $\omega = \dot{\theta}\cot\alpha$.

One principal axis of the cone is along its axis, say x_3, with moment of inertia I_3; the other two axes are in the plane perpendicular to its axis, with moments of inertia

$I_1 = I_2 = I$. The angular velocity will have a component $\omega \cos\alpha$ along the cone axis, and a component $\omega \sin\alpha$ on the plane perpendicular to the axis.

The moments of inertia with respect to the cone's axis are $I'_3 = 3MR^2/10$, $I' = 3M(h^2+R^2/4)/5$, where $R = h\tan\alpha$ is the radius of the base. The moments of inertia with respect to the center of mass at a distance $a = 3h/4$ along axis, are $I_3 = I'_3 = 3MR^2/10$, $I = I' - Ma^2 = 3M(R^2 + h^2/4)/20$.

If we choose the center of mass as the origin of the body axes, the kinetic energy is

$$T = (1/2)Mv^2 + (1/2)I_3\omega^2\cos^2\alpha + (1/2)I_1\omega^2\sin^2\alpha = 3Mh^2\dot{\theta}^2(1 + 5\cos^2\alpha)/40.$$

Exercise 5-18

A weightless bar of length l has two masses of mass m at the two ends, and is rotating uniformly about an axis passing through the bar's center, making an angle θ with the bar.

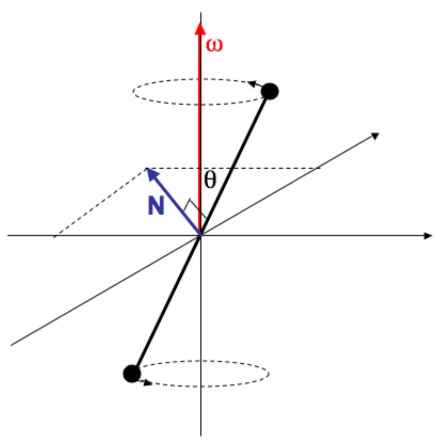

Figure 3: Exercise 5-18

The principal body axes for a bar are one along the bar itself, say the z' axis, and the other two axes, x' and y', in the plane perpendicular to the bar. The principal moments of inertia are then $I_3 = 0$, and $I_1 = I_2 = 2ml^2$. Since the axis of rotation is not perpendicular to the bar, the angular velocity (along the axis of rotation by definition) will have a component along the bar, $\omega_{//} = \omega_3 = \omega\cos\theta$ and a component in the plane perpendicular to the the bar, $\omega_\perp = \omega\sin\theta$. We can choose the x' axis along the component of the angular velocity in the plane perpendicular to the bar, so $\omega_1 = \omega\sin\theta$ and $\omega_2 = 0$. For uniform velocity, Euler equations are then

$$
\begin{aligned}
N_1 &= I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0 \\
N_2 &= I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = 2ml^2\omega^2 \sin\theta\cos\theta \\
N_3 &= I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = 0
\end{aligned}
$$

The torque is then along the y' axis: a direction perpendicular to the bar, and perpendicular to the angular velocity (which is in the $x'z'$ plane).

We can also calculate the torque from the time derivative of the angular momentum vector *in an inertial frame*. If we set up an inertial frame with the origin in the center of the bar, and the z axis along the rotation axis, then the masses have position vectors

$$
\begin{aligned}
\mathbf{r}_1 &= l(\sin\theta\cos(\omega t), \sin\theta\sin(\omega t), \cos\theta) \\
\mathbf{r}_2 &= l(-\sin\theta\cos(\omega t), -\sin\theta\sin(\omega t), -\cos\theta) = -\mathbf{r}_1.
\end{aligned}
$$

The velocities are

$$
\begin{aligned}
\mathbf{v}_1 &= l\omega(-\sin\theta\sin(\omega t), \sin\theta\cos(\omega t), 0) \\
\mathbf{v}_2 &= -\mathbf{v}_1.
\end{aligned}
$$

The angular momentum of each mass is

$$
\begin{aligned}
\mathbf{L}_1 &= \mathbf{r}_1 \times m\mathbf{v}_1 \\
&= ml^2\omega(\sin\theta\cos(\omega t), \sin\theta\sin(\omega t), \cos\theta) \times (-\sin\theta\sin(\omega t), \sin\theta\cos(\omega t), 0) \\
&= ml^2\omega\sin\theta(-\cos\theta\cos(\omega t), -\cos\theta\sin(\omega t), \sin\theta) \\
\mathbf{L}_2 &= \mathbf{r}_2 \times m\mathbf{v}_2 = \mathbf{L}_1
\end{aligned}
$$

The total angular momentum is

$$
\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = -2ml^2\omega\sin\theta(\cos\theta\cos(\omega t), \cos\theta\sin(\omega t), -\sin\theta)
$$

and the total external torque

$$
\mathbf{N} = \frac{d\mathbf{L}}{dt} = 2ml^2\omega^2\sin\theta\cos\theta(\sin(\omega t), -\cos(\omega t), 0)
$$

The torque is a rotating vector, perpendicular to the angular velocity (which is along the z axis), and perpendicular to the bar (which is along $\mathbf{r}_1$), just as we had obtained from Euler's equations.

5-20: A plane physical pendulum

Consider a uniform rod of length l and mass m, suspended in a vertical plane by one end. At the other end, there is a uniform disk of radius a and mass M attached, which can rotate freely in the vertical plane. The systems configuration can be described at any time with two angles: the angle θ the rod makes with the vertical direction, and the angle ϕ a reference direction on the the disk makes with the vertical. If the disk is rigidly attached to the rod, then $\theta = \phi$. Let us take an inertial system with the origin at the suspension point, a horizontal x axis, and the y axis pointing down along the vertical direction. The velocity of a mass element dm a distance s along the rod will be $s\dot{\theta}$. The kinetic energy of the rod is then

$$Trod = (1/2)(m/l)\int_0^l s^2\dot{\theta}^2 ds = (1/2)(m/l)(l^3/3)\dot{\theta}^2 = ml^2\dot{\theta}^2/6.$$

The velocity of points in the disk are equal equal to $\dot{\mathbf{r}}_a + \boldsymbol{\omega}_a \times \mathbf{r}'$, where $\mathbf{r}'$ is the position vector of the mass element in the reference frame fixed to the rotating disk, $\mathbf{r}_a$ is the position of the center of mass of of the disk, and the the angular velocity vector $\boldsymbol{\omega}_a$ describes the rotation of the disk in an inertial system. The magnitude of the angular velocity vector $\boldsymbol{\omega}_a$ is $=\dot{\phi}$, and its direction is perpendicular to the motion plane. The squared speed of mass elements will then be $v^2 = v_a^2 + 2\dot{\mathbf{r}}_a \cdot (\boldsymbol{\omega}_a \times \mathbf{r}') + \omega_a^2 r'^2$, and the kinetic energy of the disk will be

$$Tdisk = \frac{1}{2}\int v^2 dm = \frac{1}{2}\int (v_a^2 + 2\dot{\mathbf{r}}_a \cdot (\boldsymbol{\omega}_a \times \mathbf{r}') + \omega_a^2 r'^2) dm.$$

We recognize that the third term will lead to a term $(1/2)I_0\omega_a^2$ in the kinetic energy, with $I_0 = Ma^2/2$ the moment of inertia of the disk with respect to the center of mass.

The integral of the second term will vanish, since $\int \mathbf{r}' dm$ is the position of the center of mass in a coordinate system where the center of mass is at the origin.

The velocity of the center of mass of the disk $\dot{\mathbf{r}}_a$ is equal to $\dot{\mathbf{r}}_0 + \boldsymbol{\omega}_a \times \mathbf{a}$, where $\mathbf{r}_0$ is the position vector of the attachment point, and $\mathbf{a}$ is the position of the center of mass of the disk with respect to the attachment point. Then

$$\begin{aligned} v_a^2 &= (\dot{\mathbf{r}}_0 + \boldsymbol{\omega}_a \times \mathbf{a}) \cdot (\dot{\mathbf{r}}_0 + \boldsymbol{\omega}_a \times \mathbf{a}) \\ &= v_0^2 + 2\mathbf{v}_0 \cdot (\boldsymbol{\omega}_a \times \mathbf{a}) + \omega_a^2 a^2 \\ &= l^2\dot{\theta}^2 + 2\boldsymbol{\omega}_a \cdot (\mathbf{a} \times \mathbf{v}_0) + a^2\dot{\phi}^2 \end{aligned}$$

The velocity of the attachment point $\mathbf{v}_0$ is tangent to the disk, so the direction of the cross product $\mathbf{a} \times \mathbf{v}_0$ is perpendicular to the plane of motion, and $\boldsymbol{\omega}_a \cdot (\mathbf{a} \times \mathbf{v}_0) = \omega_a|\mathbf{a} \times \mathbf{v}_0|$. The position vector of the attachment point is $\mathbf{r}_0 = l(\cos\theta, \sin\theta, 0)$, and its velocity vector is $\mathbf{v}_0 = l\dot{\theta}(-\sin\theta, \cos\theta, 0)$. The position of the center of mass of the disk with respect to the attachment point is $\mathbf{a} = a(\cos\phi, \sin\phi, 0)$. Thus,

$$\mathbf{a} \times \mathbf{v}_0 = a(\cos\phi, \sin\phi, 0) \times l\dot{\theta}(-\sin\theta, \cos\theta, 0)$$

$$\begin{aligned} &= al\dot{\theta}(0,0,\cos\phi\cos\theta+\sin\phi\sin\theta) \\ &= al\dot{\theta}\cos(\phi-\theta)\hat{\mathbf{k}} \end{aligned}$$

The kinetic energy of the disk is then

$$\begin{aligned} T_{disk} &= \frac{1}{2}Mv_a^2 + I_0\omega_a^2 \\ &= \frac{1}{2}M(l^2\dot{\theta}^2 + 2al\dot{\theta}\dot{\phi}\cos(\phi-\theta) + a^2\dot{\phi}^2) + \frac{1}{2}\frac{Ma^2}{2}\dot{\phi}^2 \\ &= \frac{1}{2}Ml^2\dot{\theta}^2 + Mal\dot{\theta}\dot{\phi}\cos(\phi-\theta) + \frac{3}{4}Ma^2\dot{\phi}^2 \end{aligned}$$

and the total kinetic energy is

$$\begin{aligned} T &= T_{rod} + T_{disk} \\ &= \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}Ml^2\dot{\theta}^2 + Mal\dot{\theta}dot\phi cos(\phi-\theta) + \frac{3}{4}Ma^2\dot{\phi}^2 \\ &= \frac{1}{6}(3M+m)l^2\dot{\theta}^2 + Mal\dot{\theta}\dot{\phi}cos(\phi-\theta) + \frac{3}{4}Ma^2\dot{\phi}^2 \end{aligned}$$

The potential energy of the rod is $V_{rod} = -mg(l/2)\cos\theta$, and the potential energy of the disk is $V_{disk} = -Mg(l\cos\theta + a\cos\phi)$, so the total potential energy is

$$V = -\frac{1}{2}mgl\cos\theta - Mg(l\cos\theta + a\cos\phi) = -\frac{1}{2}(m+2M)gl\cos\theta - Mga\cos\phi$$

The Lagrangian is

$$L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 + Mal\dot{\theta}\dot{\phi}\cos(\phi-\theta) + \frac{3}{4}Ma^2\dot{\phi}^2 + \frac{1}{2}(m+2M)gl\cos\theta + Mga\cos\phi$$

Lagrange's equation for θ is:

$$\frac{d}{dt}\left(Ml^2\dot{\theta} + Mal\dot{\phi}\cos(\phi-\theta)\right) - \left(Mal\dot{\theta}\dot{\phi}\sin(\phi-\theta) - \frac{1}{2}(m+2M)glsin\theta\right) = 0$$

$$Ml^2\ddot{\theta} + Mal\ddot{\phi}\cos(\phi-\theta) - Mal\dot{\phi}^2\sin(\phi-\theta) + \frac{1}{2}(m+2M)gl\sin\theta = 0$$

Lagrange's equation for ϕ is

$$\frac{d}{dt}\left(\frac{3}{2}Ma^2\dot{\phi} + Mal\dot{\theta}\cos(\phi-\theta)\right) - \left(-Mal\dot{\theta}\dot{\phi}\sin(\phi-\theta) - Mgasin\phi\right) = 0$$

$$\frac{3}{2}Ma^2\ddot{\phi} + Mal\ddot{\theta}\cos(\phi-\theta) + Mal\dot{\theta}^2\cos(\phi-\theta) + mga\sin\phi = 0$$

Exercise 25: A rolling sphere

The sphere is a rigid body described by the position of its center of mass $\mathbf{r} = (x, y, z)$, and its orientation, defined by Euler angles θ, ϕ, ψ. The constraint of the sphere remaining on the surface (the effect of the normal force responding to gravity) is

$$z = R$$

(if the origin of the coordinate system is on the surface). At any instant, the sphere is rotating about an axis on its point of contact with the surface, which is the instantaneous axis of rotation and the direction of the angular velocity vector $\boldsymbol{\omega}$, and thus $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$. If the sphere is not slipping, the angular velocity vector is perpendicular to the velocity of the center of mass, $\boldsymbol{\omega} \cdot \mathbf{v} = 0$, which we can express as a condition for an angle Θ to exist (defining the direction of the horizontal component of the angular velocity) such that $\boldsymbol{\omega} = (\Omega \cos\Theta, \Omega \sin\Theta, \omega_z)$ and $\mathbf{v} = v(\sin\Theta, -\cos\Theta)$. Moreover, not slipping also means that the instantaneous velocity of the contact point is zero:

$$\begin{aligned}
0 = \mathbf{v}_c &= \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}' \\
&= v(\sin\Theta, -\cos\Theta, 0) + (\Omega \cos\Theta, \Omega \sin\Theta, \omega_z) \times (0, 0, -R) \\
&= (v - R\Omega)(\sin\Theta, -\cos\Theta, 0)
\end{aligned}$$

which then means

$$v = R\Omega.$$

The angular velocity vector is related to Euler angles, and thus our constraints are

$$\begin{aligned}
R\omega_x = v\cos\Theta = -\dot{y} &= R(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) \\
R\omega_y = v\sin\Theta = \dot{x} &= R(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)
\end{aligned}$$

It may be useful to compare these conditions with the problem of a rolling disk considered in Chapter 1: our equations are the same as (1.39), since the disk would have $\dot{\psi} = 0$.

We also know that $\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$, but that is not a constraint, of course. The angular velocity ω_z is the instantaneous "spinning" velocity of the sphere about a vertical axis.

The sphere then has 6 coordinates, and three constraints, so the system only has three degrees of freedom. We'd like to choose these as the three Euler angles, for example, using the two costraints to solve for the x, y. However, the constraints are non-holonomic and do not allow us to integrate them for x, y. To prove the constraints are non-holonomic, we consider the differentials

$$df_x(x, \theta, \psi, \phi) = R(d\theta \sin\phi - d\psi \sin\theta\cos\phi) - dx$$

$$df_y(y, \theta, \psi, \phi) = R(d\theta \cos\phi + d\psi \sin\theta\sin\phi) + dy$$

and we prove they are not exact differentials (they are not derivatives of a function):

$$\frac{d}{d\phi}\frac{df_x}{d\theta} = \frac{d}{d\phi} R\sin\phi = R\cos\phi \neq \frac{d}{d\theta}\frac{df_x}{d\phi} = 0$$

$$\frac{d}{d\phi}\frac{df_y}{d\theta} = \frac{d}{d\phi} R\cos\phi = -R\sin\phi \neq \frac{d}{d\theta}\frac{df_y}{d\phi} = 0$$

We now want to write Lagrange's equations of motion, using Lagrange multipliers. The potential energy is constant, so we only have kinetic energy. The kinetic energy has a translational part, and a rotational part. The rotational energy is especially simple since the sphere has identical moments of inertia about the principal axes:

$$\begin{aligned}
L = T &= \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}I\boldsymbol{\omega}\cdot\boldsymbol{\omega} \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I((\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi)^2 + (\dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi)^2 + (\dot\psi\cos\theta + \dot\phi)^2) \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I(\dot\theta^2 + \dot\psi^2 + \dot\phi^2 + 2\dot\phi\dot\psi\cos\theta)
\end{aligned}$$

The non-holonomic constraints are

$$\begin{aligned}
F_x &= R(\dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi) - \dot{x} \\
F_y &= R(\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi) + \dot{y}
\end{aligned}$$

Lagrange's equations are

$$\begin{aligned}
m\ddot{x} &= \mu_x\frac{\partial F_x}{\partial\dot{x}} = -\mu_x \\
m\ddot{y} &= \mu_y\frac{\partial F_y}{\partial\dot{y}} = \mu_y \\
I\ddot\theta + I\dot\phi\dot\psi\sin\theta &= \mu_x\frac{\partial F_x}{\partial\dot\theta} + \mu_y\frac{\partial F_y}{\partial\dot\theta} \\
&= R(\mu_x\sin\phi + \mu_y\cos\phi) \\
I\ddot\psi + I(\ddot\phi\cos\theta - \dot\phi\dot\theta\sin\theta) &= \mu_x\frac{\partial F_x}{\partial\dot\psi} + \mu_y\frac{\partial F_y}{\partial\dot\psi} \\
&= R\sin\theta(-\mu_x\cos\phi + \mu_y\sin\phi) \\
I\ddot\phi + I(\ddot\psi\cos\theta - \dot\psi\dot\theta\sin\theta) &= 0 = I\frac{d}{dt}(\dot\phi + \dot\psi\cos\theta) = I\frac{d\omega_z}{dt}
\end{aligned}$$

We have then 7(!) differential equations (five Lagrange equations and two constraints) for the eight unknowns $x, y, \phi, \theta, \psi, \mu_x, \mu_y$.

The last equation (Lagrange's equation for ψ) says that the angular velocity ω_z is constant. This is because ϕ is a cyclical variable in the Lagrangian *and* because the constraints do not depend on $\dot{\phi}$. Notice that ψ is also a cyclical variable in the Lagrangian, but it is not associated with a conserved quantity because the constraints depend on $\dot{\psi}$.

Since there is no dissipation or forces doing any work, the energy (equal to the kinetic energy) is conserved:

$$E = T + V = T_{tr} + T_{rot} + 0 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}I\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \frac{1}{2}mv^2 + \frac{1}{2}I(\Omega^2 + \omega_z^2)$$

Due to the constraint $v = R\Omega$, the energy is equal to $E = \frac{1}{2}(mR^2+I)\Omega^2+\frac{1}{2}I\omega_z^2$. Since ω_z is constant, then we know that Ω (and thus also v) are constant. Therefore, the translational kinetic energy $\frac{1}{2}mv^2$ and the rotational energy $\frac{1}{2}I(\Omega^2 + \omega_z^2)$ are separately conserved.

Exercise 30: Closing tilted door

The door's axis of rotation is along the hinges, which make an angle θ with the vertical (normally, on firm ground and for a well aligned door, of course, $\theta = 0$). The position of the door as a rigid body is determined by the angle about its axis of rotation. The principal axes of the door are an axis perpendicular to the door, and two axes in the door plane, parallel to the door's sides (assumed to be straight) .

We choose a body coordinate system with an origin in the bottom door's corner along the hinged side. We choose an inertial coordinate system with the same origin, a vertical axis z pointing up, and two perpendicular axis x, y in the horizontal plane. We choose body axes x' along the short side of the door, z' along its hinged side (the axis of rotation), and y' perpendicular to the door.

Following the convention for Euler angles, and placing the door in the $x'z'$ plane in Figure 4.7, we see that the angle θ is the angle between the door and the vertical; the angle $\phi = 0$, and the angle ψ is the one that defines the position (angle) of the door.

The initial position of the door ("lifted" by 90 degrees) has an angle $\psi = 0$. The final equilibrium position of the door (when the door is "shut") has an angle $\psi = -\pi/2$.

The gravitational potential energys is $V = -m\mathbf{g}\cdot\mathbf{r}_0 = mgz$, where $\mathbf{r}_0$ is the position of the center of mass in the inertial system, and z its vertical component.

The position of the center of mass in the body axes system is $\mathbf{r}'_0 = (x', y', z') = (w, 0, h)/2$, where w is the door's width and h is the doors height. In the inertial system, the position of the center of mass is given by the transformation (4.47), $\mathbf{r}_0 = \mathbf{A}^{-1}\mathbf{r}'_0$.

The vertical coordinate will be

$$z = x'\sin\theta\sin\psi + y'\sin\theta\cos\psi + z'\cos\theta = (w/2)\sin\theta\sin\psi + (h/2)\cos\theta$$

The gravitational potential energy is then

$$V = mgz = mg(w\sin\theta sin\psi + h\cos\theta)/2$$

Since the motion is a pure rotation, the kientic energy has the form $T = (1/2)I_0\omega^2$, where I_0 is the moment of inertia about the axis of rotation (the door's hinged side, the z' axis), and $\omega = \dot{\psi}$ is the angular velocity: $T_{rot} = \frac{1}{2}I_{z'}\dot{\psi}^2$. The moment of inertia of the door with respect to the z' axis, is calcualted integrating over the points on the door (all with $y' = 0$ coordinates:

$$I_{z'} = \int (x'^2 + y'^2)\sigma dx'dz' = \frac{m}{wh}\int_0^w x'^2 dx' \int_0^h dz' = \frac{m}{wh}\frac{w^3}{3}h = \frac{1}{3}mw^2.$$

The kinetic energy is then

$$T = \frac{1}{2}I_{z'}\dot{\psi}^2 = \frac{1}{6}mw^2\dot{\psi}^2.$$

The total energy is

$$E = T + V = \frac{1}{6}mw^2\dot{\psi}^2 + \frac{mg}{2}(w\sin\theta sin\psi + h\cos\theta).$$

In the initial position, $\psi = 0$ and $\dot{\psi} = 0$. Since the energy is conserved, we obtain

$$E = \frac{1}{6}mw^2\dot{\psi}^2 + \frac{mg}{2}(w\sin\theta sin\psi + h\cos\theta) = \frac{mgh}{2}\cos\theta,$$

an expression we can use as a differential equation for ψ:

$$\dot{\psi}^2 = -\frac{3g\sin\theta}{w}\sin\psi$$

Notice that since $-\pi/2 < \psi < 0$, the expression on the left hand side is a positive expression. The angular velocity when the door reaches the equilibrium position at $\psi = -\pi/2$ (where it will not stop, but oscillate about, if it can go through the shut position) is then

$$\dot{\psi}_f = \sqrt{\frac{3g\sin\theta}{w}}.$$

The time it will take to reach that position can be obtained the equation for ψ:

$$\begin{aligned}
\frac{d\psi}{dt} &= \sqrt{-\frac{3g\sin\theta}{w}\sin\psi} \\
dt &= \sqrt{\frac{w}{3g\sin\theta}}\frac{d\psi}{\sqrt{-\sin\psi}} \\
\Delta t &= \sqrt{\frac{w}{3g}}\int_0^{-\pi/2}\frac{d\psi'}{\sqrt{-sin\psi}} \\
&= \sqrt{\frac{w}{3g\sin\theta}}\int_{\pi/2}^0\frac{d\psi'}{\sqrt{\sin\psi'}}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{w}{3g\sin\theta}}2F(\pi/4, 2) \\
&= 1.51\sqrt{\frac{w}{g\sin\theta}} \\
\sin\theta &= \frac{w}{g}\left(\frac{1.51}{\Delta t}\right)^2 = \frac{0.9\text{m}}{9.8\text{m/s}^2}\left(\frac{1.51}{3\text{s}}\right)^2 = 1.3^\circ
\end{aligned}
$$

Three seconds is a looong time for a door to close, so we obtain a small hinge angle. The smallest time for the door to close is when the hinges are horizontal, $\theta = \pi/2$ and $\Delta t = 1.51\sqrt{w/g} = 0.45$sec. The wider the door, the longer it takes to close.