Phys 7221 Hwk # 7

Gabriela González

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Derivation 4-4

Show that if $\mathbf{A}$ is a real 3x3 antisymmetric matrix, then the matrices $\mathbf{1} \pm \mathbf{A}$ are non-singular, and the matrix $\mathbf{B} = (\mathbf{1} + \mathbf{A})(\mathbf{1} - \mathbf{A})^{-\mathbf{1}}$ is orthogonal.

If $\mathbf{A}^\dagger = -\mathbf{A}$, then $\mathbf{A}$ has only three independent components:

$$\mathbf{A} = \begin{bmatrix} 0 & a & b \\ -a^* & 0 & c \\ -b^* & -c^* & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are obtained from the equation $\det(\lambda\mathbf{1} - \mathbf{A}) = 0$:

$$\begin{aligned} |\lambda\mathbf{1} - \mathbf{A}| &= \begin{vmatrix} \lambda & -a & -b \\ a^* & \lambda & -c \\ b^* & c^* & \lambda \end{vmatrix} \\ &= \lambda(\lambda^2 + |c|^2) + a(a^*\lambda + cb^*) - b(a^*c^* - \lambda b^*) \\ &= \lambda(\lambda^2 + |a|^2 + |b|^2 + |c|^2) + acb^* - ba^*c^* \end{aligned}$$

If a, b, c are real, then the eigenvalue equation is $\lambda(\lambda^2 + r^2) = 0$; with $r^2 = a^2 + b^2 + c^2$ a positive number. The solutions are $\lambda = 0, \pm ir$. Then, there exists a (complex!) matrix $\mathbf{R}$ that diagonalize the matrix A:

$$\mathbf{A}' = \mathbf{R}^{-\mathbf{1}}\mathbf{A}\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ir & 0 \\ 0 & 0 & -ir \end{bmatrix}$$

The determinant of the matrix $\mathbf{1} \pm \mathbf{A}$ is equal to

$$|\mathbf{1} \pm \mathbf{A}| = |\mathbf{R}^{-\mathbf{1}}(\mathbf{1} \pm \mathbf{A})\mathbf{R}| = |\mathbf{1} \pm \mathbf{A}'| = (1 \pm ir)(1 \mp ir) = 1 + r^2$$

and is positive definite: the matrix $\mathbf{1} \pm \mathbf{A}$ is then non-singular, and invertible.

To prove that $\mathbf{B} = (\mathbf{1}+\mathbf{A})(\mathbf{1}-\mathbf{A})^{-\mathbf{1}}$ is orthogonal, we need to prove that $\mathbf{B}^\dagger = \mathbf{B}^{-\mathbf{1}}$, or that $\mathbf{B}\mathbf{B}^\dagger = \mathbf{1}$. We first start from the deifnition of $\mathbf{B}$ to find an expression for $\mathbf{B}^\dagger$:

$$
\begin{aligned}
\mathbf{B} &= (\mathbf{1}+\mathbf{A})(\mathbf{1}-\mathbf{A})^{-\mathbf{1}} \\
\mathbf{B}(\mathbf{1}-\mathbf{A}) &= \mathbf{1}+\mathbf{A} \\
(\mathbf{1}-\mathbf{A})^\dagger\mathbf{B}^\dagger &= (\mathbf{1}+\mathbf{A})^\dagger \\
(\mathbf{1}-\mathbf{A}^\dagger)\mathbf{B}^\dagger &= \mathbf{1}+\mathbf{A}^\dagger \\
(\mathbf{1}+\mathbf{A})\mathbf{B}^\dagger &= \mathbf{1}-\mathbf{A} \\
\mathbf{B}^\dagger &= (\mathbf{1}+\mathbf{A})^{-\mathbf{1}}(\mathbf{1}-\mathbf{A})
\end{aligned}
$$

and now we can calculate the product

$$
\begin{aligned}
\mathbf{B}^\dagger\mathbf{B} &= (\mathbf{1}+\mathbf{A})^{-\mathbf{1}}(\mathbf{1}-\mathbf{A})(\mathbf{1}+\mathbf{A})(\mathbf{1}-\mathbf{A})^{-\mathbf{1}} \\
&= (\mathbf{1}+\mathbf{A})^{-\mathbf{1}}(\mathbf{1}-\mathbf{A}^{\mathbf{2}})(\mathbf{1}-\mathbf{A})^{-\mathbf{1}} \\
&= (\mathbf{1}+\mathbf{A})^{-\mathbf{1}}(\mathbf{1}+\mathbf{A})(\mathbf{1}-\mathbf{A})(\mathbf{1}-\mathbf{A})^{-\mathbf{1}} \\
&= \mathbf{1}
\end{aligned}
$$

Derivation 4-15

Calculate the components of the angular velocity vector $\vec{\omega}$ in terms of Euler's angles.

The angular velocity vector $\vec{\omega}$ of a rigid body is defined through the transformation between vectors in an inertial space system and a body system:

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{r}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{r}_{\text{body}}.$$

For infinitesimal rotations, this is

$$d\vec{r} = \vec{r}_{\text{space}} - \vec{r}_{\text{body}} = d\vec{\Omega}\times\vec{r}_{\text{body}}$$

If we perform two such infinitesimal rotations, their infinitesimal rotation vectors will add:

$$
\begin{aligned}
\vec{r}_{\text{space}} &= \vec{r'}_{\text{body}} + d\vec{\Omega}'\times\vec{r'}_{\text{body}} \\
&= \vec{r}_{\text{body}} + d\vec{\Omega}\times\vec{r}_{\text{body}} + d\vec{\Omega}'\times\left(\vec{r}_{\text{body}} + d\vec{\Omega}\times\vec{r}_{\text{body}}\right) \\
&\approx \vec{r}_{\text{body}} + \left(d\vec{\Omega} + d\vec{\Omega}'\right)\times\vec{r}_{\text{body}}
\end{aligned}
$$

Since the angular velocity is defined as $\vec{\omega} = d\vec{\Omega}/dt$, then the angular velocity of consecutive rotations will also add.

Consider a rigid body undergoing the infinitesimal rotations defining Euler's angles: first by $d\phi$ about $\hat{z}$, then by $d\theta$ about $\hat{\xi}$, and finally by $d\psi$ about $\hat{\zeta}' = \hat{z}'$. Each of these rotations will have angular velocities $\vec{\omega}_\phi = \dot{\phi}\hat{z}$, $\vec{\omega}_\theta = \dot{\theta}\hat{\xi}$, and $\vec{\omega}_\psi = \dot{\psi}\hat{z}'$, and the total angular velocity is

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi = \dot{\phi}\hat{z} + \dot{\theta}\hat{\xi} + \dot{\psi}\hat{z}'.$$

If we want the components of the vector $\vec{\omega}$ in the space set of axes, we need to find the components of vectors $\hat{\xi}$ and $\hat{z}'$ in terms of $\hat{x}, \hat{y}, \hat{z}$ (it helps to look at Figure 4.7 in the textbook while doing this exercise).

The vector $\hat{\xi}$ was obtained by a rotation of the vector $\hat{x}$, by an angle ϕ about $\hat{z}$:

$$\hat{\xi} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

We could also obtain this expression using (4.62) in the textbook, with $\vec{r} = \hat{x}$, $\hat{n} = \hat{z}$, $\vec{r}' = \hat{\xi}$, and $\Phi = -\phi$ (the minus sign arises from the fact that we are rotating a vector, not a system of coordinates):

$$\hat{\xi} = \hat{x}\cos\phi + \hat{z}(\hat{z}\cdot\hat{x})(1-\cos\phi) - (\hat{x}\times\hat{z})\sin\phi = \hat{x}\cos\phi + \hat{y}\sin\phi$$

The vector $\hat{z}'$ was obtained from a rotation of the vector $\hat{z}$ by an angle θ about the axis $\hat{\xi}$. The rotation matrix can be obtained by a product of rotation matrices, but we can also use (4.62), with $\vec{r} = \hat{z}$, $\vec{r}' = \hat{z}'$, $\hat{n} = \hat{\xi}$ and $\Phi = -\theta$ (since we have an expression for $\hat{\xi}$ ready):

$$\begin{aligned} \hat{z}' &= \hat{z}\cos\theta + \hat{\xi}(\hat{\xi}\cdot\hat{z})(1-\cos\theta) - (\hat{z}\times\hat{\xi})\sin\theta \\ &= \hat{z}\cos\theta - (\hat{z}\times(\hat{x}\cos\phi + \hat{y}\sin\phi))\sin\theta \\ &= \hat{z}\cos\theta - (\hat{y}\cos\phi - \hat{x}\sin\phi)\sin\theta \\ &= \hat{x}\sin\theta\sin\phi - \hat{y}\sin\theta\cos\phi + \hat{z}\cos\theta \end{aligned}$$

Finally, the angular velocity vector is

$$\begin{aligned} \vec{\omega} &= \dot{\phi}\hat{z} + \dot{\theta}\hat{\xi} + \dot{\psi}\hat{z}' \\ &= \hat{x}(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) + \hat{y}(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi) + \hat{z}(\dot{\phi} + \dot{\psi}\cos\theta) \end{aligned}$$

Exercise 21

Calculate the deflection of a particle thrown up to reach a maximum height z_0, and that of a particle dropped from rest from the same height, due to the Coriolis force.

The Coriolis force appears as an "extra" force term in a rotating frame (such as one fixed to Earth), of the form $\vec{F}_C = -2m\vec{\omega} \times \vec{v}$. For a particle moving "up" in the Earth's frame, this force has a horizontal direction. If we choose a coordinate system fixed to the Earth, with the z axis pointing "up", the x axis pointing north, and the y axis pointing west, the Earth's angular velocity will have components $\vec{\omega} = \hat{x}\omega\cos\lambda + \hat{z}\omega\sin\lambda$, where λ is the latitude angle (zero at the Equator, 90^o at the North pole, and 30^o27'N at Baton Rouge).

For a particle moving with vertical velocity $v_z\hat{z}$, the force is

$$\vec{F}_c = -2m\vec{\omega} \times \vec{v} = -2m\omega v_z(\hat{x}\cos\lambda + \hat{z}\sin\lambda) \times \hat{z} = 2m\omega\cos\lambda \; v_z \; \hat{y}$$

so the Coriolis force is only in the east-west direction, is maximum at the Equator, and is in the same direction on both hemispheres.

The motion in the vertical direction is only affected by gravity, so $z(t) = z_0 + v_{0z}t - gt^2/2$, and $v_z = v_{0z} - gt$ as usual. The acceleration in the y direction, however, is

$$m\ddot{y} = 2m\omega\cos\lambda(v_{0z} - gt)$$

and direct integration leads to

$$y(t) = y_0 + v_{0y}t + 2\omega\cos\lambda(v_{0z}t^2/2 - gt^3/6)$$

If the particle is thrown from the ground with a vertical velocity upwards, then it will reach its maximum height at time $t = v_{0z}/g$, and the height will be $z_{\max} = v_{0z}^2/2g$. The particle returns to the ground at time $t = 2v_{0z}/g$, and the total horizontal deflection will be

$$\Delta y = 2\omega\cos\lambda\left(\frac{v_{0z}}{2}\left(\frac{2v_{0z}}{g}\right)^2 - \frac{g}{6}\left(\frac{2v_{0z}}{g}\right)^3\right) = \frac{4}{3}\omega\cos\lambda\frac{v_{0z}^3}{g^2} = \sqrt{2}\frac{8}{3}\cos\lambda \; z_{\max}\sqrt{\frac{\omega^2 z_{\max}}{g}}$$

If the particle is dropped from rest from the same height $z_{\max}$, then $v_{0z} = 0$. The particle will reach the ground at time $t = \sqrt{2z_{\max}/g}$, and the total horizontal deflection will be

$$\Delta y = -\frac{1}{3}\omega\cos\lambda \; gt^3 = -\frac{1}{3}\omega\cos\lambda \; g\left(\frac{2z_{\max}}{g}\right)^{3/2} = -\sqrt{2}\frac{2}{3}\cos\lambda \; z_{\max}\sqrt{\frac{\omega^2 z_{\max}}{g}}$$

Thus, the particle gets deflected four times more towards the west if it is thrown upwards, than the eastern deflection it experiences if it is dropped from the same height. If a particle were to experience a 1mm deflection when thrown upwards from Baton Rouge, it needs to have enough initial velocity to reach a maximum height equal to

$$z_{\max} = \left(\frac{3\Delta y\sqrt{g}}{8\sqrt{2}\omega\cos\lambda}\right)^{2/3} = 5.6\text{m}$$

or v_{0z}=10.5 m/s = 24 mph.

In reality, the motion of the particle in the Earth rotating coordinates is affected both by the Coriolis force $\vec{F_C} = -2m\vec{\omega} \times \vec{v}$ and by the centrifugal force $\vec{F_C} = -m\vec{\omega} \times \vec{\omega} \times \vec{r}$, which has both horizontal and vertical components. Also, once there is a velocity horizontal velocity in the east-west direction (proportional to ω), the Coriolis force will also have a component in the north-south direction (proportional to ω^2). However, since the Earth's velocity is small compared to the quantities in the system ($\omega R^2 \ll g$), these corrections are even smaller than the the corrections we calculated, proportional to ω.

Exercise 4-23: Foucalt's s pendulum

A straightforward derivation of the equations of motion of a Foucalt pendulum can be found in many textbooks, including Landau and Lifshiftz (Chapter VI, Section 39, Problem 3) or in Marion and Thorne (Example 10.5). If the pendulum has horizontal displacements x, y, we assume small oscillations, with oscillation frequency $\omega_o = \sqrt{g/l}$ much smaller than Earth's rotation angular velocity ω, then the plane of oscillation of the pendulum will rotate in a local coordinate system, with angular frequency $\Omega = \omega \sin\lambda$, where λ is the latitude of the location of the pendulum on Earth.

Here's a derivation of the result. The pendulum mass position is described in a local coordiante system by $\mathbf{r} = (x, y, z)$ (where z is negative in our choice of coordinates axes). In the absence of Earth's rotation, the forces are tension $\mathbf{T} = -T\mathbf{r}$, and gravity $\mathbf{F}_g = -mg\hat{k}$. Newton's equations of motion $\mathbf{F} = m\ddot{\mathbf{r}}$ are three equations, plus the cosntraint equation $x^2 + y^2 + z^2 = l^2$, for the four unknowns x, y, z, T:

$$\begin{aligned} m\ddot{x} &= \mathbf{T}_x = -Tx/l \\ m\ddot{y} &= \mathbf{T}_y = -Ty/l \\ m\ddot{z} &= \mathbf{T}_z + mg = -Tz/l - mg \end{aligned}$$

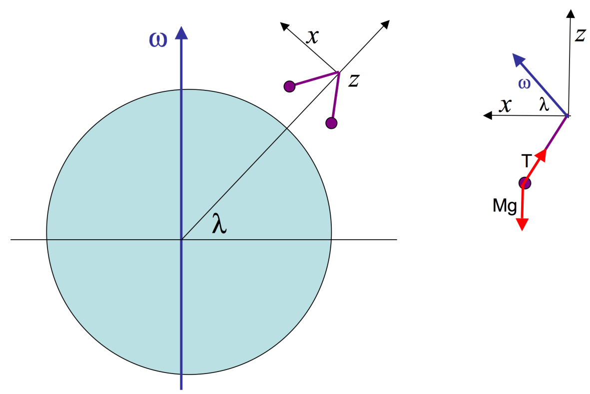
For small oscillations, we have $r^2 = x^2 + y^2 \ll l^2$, and the equations for z, T are solved by $z \approx -l, T \approx mg$ (up to second order in r/l). The equations for the horizontal motion are

$$\ddot{x} + (g/l)x = 0,$$

$$\ddot{y} + (g/l)y = 0.$$

If the pendulum starts with $y_0 = \dot{y}_0 = 0$, then the initial acceleration is also zero, and thus $y = 0$ at all times: the pendulum oscillates in a plane.

Let us now include the Coriolis force, and see why it makes the plane of a planar pendulum precess. We choose, like in the previous problem and shown in the figure, the x axis pointing North, so Earth's angular velocity in a local coordinate system is $\vec{\omega} = \omega(\cos\lambda, 0, \sin\lambda)$ where λ is the latitude angle ($\lambda = \pi/2$ at the North pole, $\lambda = 0$ at the Equator). Coriolis force is then



$$\begin{aligned}\vec{F}_C &= -2m\vec{\omega}\times\vec{v}\\ &= -2m\omega(\cos\lambda, 0, \sin\lambda)\times(\dot{x},\dot{y},\dot{z})\\ &= -2m\omega(-\dot{y}\sin\lambda, \dot{x}\sin\lambda-\dot{z}\cos\lambda, \dot{y}\cos\lambda)\end{aligned}$$

and the equations of motion are

$$\begin{aligned}m\ddot{x} &= -Tx/l + (2m\omega\sin\lambda)\dot{y}\\ m\ddot{y} &= -Ty/l - (2m\omega\sin\lambda)\dot{x} + (2m\omega\cos\lambda)\dot{z}\\ m\ddot{z} &= -Tz/l - mg - (2m\omega\cos\lambda)\dot{y}\end{aligned}$$

For small oscillations, again the z equation is solved by $z\approx -l$, $T\approx mg$ (but only to first order in r/l). The horizontal equations, however, are now coupled:

$$\begin{aligned}\ddot{x} &\approx -(g/l)x + (2\omega\sin\lambda)\dot{y}\\ \ddot{y} &\approx -(g/l)y - (2\omega\sin\lambda)\dot{x}\end{aligned}$$

If initially $y_0 = \dot{y}_0 = 0$, but $\dot{x}_0 \neq 0$ (the pendulum is let go from some initial angle in the North-South vertical plane), the initial acceleration $\ddot{y}_0 = -(2\omega\sin\lambda)\dot{x}_0 \neq 0$ and the plane of the pendulum is not constant: the mass will deviate into the East-West plane. If we define a complex function $\xi = x + iy$, then the equations can be combined in a complex equation of the form

$$\ddot{\xi} + (g/l)\xi + (2i\omega\sin\lambda)\dot{\xi} = 0$$

which has a solution of the form $\xi = \xi_+ e^{i\Omega_+ t} + \xi_- e^{i\Omega_- t}$, where $\Omega_\pm$ are the solutions to the equation $-\Omega^2 + (g/l) - 2\omega\Omega\sin\lambda = 0$, or

$$\Omega_\pm = -\omega\sin\lambda \pm \omega_0\sqrt{1+\frac{\omega^2\sin^2\lambda}{\omega_0^2}}$$

If $\omega \ll \omega_0 = \sqrt{g/l}$ (that is, the period of the pendulum is much shorter than a day, a very reasonable assumption), then the solutions are $\Omega_\pm \approx \pm\omega_0 - \omega\sin\lambda$, and then

$$\xi = x + iy \approx e^{-i\omega\sin\lambda t}\left(\xi_+ e^{-i\omega_0 t} + \xi_- e^{i\omega_0 t}\right) = e^{-i\omega\sin\lambda t}\left(A\cos(\omega_0 t + \phi_A) + iB\sin(\omega_0 t + \phi_B)\right)$$

Using initial conditions $\mathbf{r}(t=0) = x_0\hat{i}$, we have

$$\xi(t=0) = x_0 = A\cos\phi_A + iB\sin\phi_B.$$

Using the initial condition $\dot{\mathbf{r}}(t=0) = 0$, we have

$$\dot{\xi}(t=0) = 0 = \omega\sin\lambda(A\cos\phi_A + iB\sin\phi_B) + \omega_0(-A\sin\phi_A + iB\cos\phi_B).$$

Thus, $B = \phi_B = 0$, $A\cos\phi_A = x_0$, $A\sin\phi_A = x_0\omega\sin\lambda/\omega_0$, and

$$\begin{aligned}
\xi &= e^{-i\omega\sin\lambda t}A\cos(\omega_0 t + \phi_A) \\
&= e^{-i\omega\sin\lambda t}x_0(\cos\omega_0 t - (\omega\sin\lambda/\omega_0)\sin\omega_0 t) \\
&\approx e^{-i\omega\sin\lambda t}x_0\cos\omega_0 t \\
x(t) = \Re\xi &= x_0\cos(\omega\sin\lambda t)\cos\omega_0 t \\
y(t) = \Im\xi &= x_0\sin(\omega\sin\lambda t)\cos\omega_0 t
\end{aligned}$$

If we use polar coordinates r, ϕ so that $x = r\cos\phi, y = r\sin\phi$, then the precession angle of the pendulum plane is $\phi(t) = \tan^{-1}(y/x) = (\omega\sin\lambda)t$. At the North pole, the plane "rotates" once a day with respect to Earth's coordinate system, which is rotating once a day itself: the plane of the pendulum is constant in an inertial frame, as seen in the figure below. At the Equator, the plane of the pendulum does not rotate in the local coordinate system: the pendulum co-rotates with the Earth. At Baton Rouge, coordinates are 30.43N -91.15W, so $\sin\lambda = \sin(30.43°) = 0.51$, so the plane of the pendulum takes two days to rotate once, as seen in coordinates tied to Nicholson building.

Problem 4-24

A wagon wheel with spokes is mounted on a vertical axis so it is free to rotate in the horizontal plane. It is rotating with $\omega = 3.0$rad/s. A bug crawls out on one of the spokes with a a velocity of 0.5 cm/s, holding on to the spoke with coefficient of friction $\mu = 0.30$. How far can the bug crawl along the poke before it starts to slip?

In the bug's rotating coordinate system, there is a centripetal force $-m\omega \times \omega \times \mathbf{r}$ in the radial direction (away from the center), which gets larger as the bug crawls out. Since the bug is moving along the spoke with a radial velocity $\mathbf{v} = v\hat{\mathbf{e}}_r$, there is also a Coriolis force

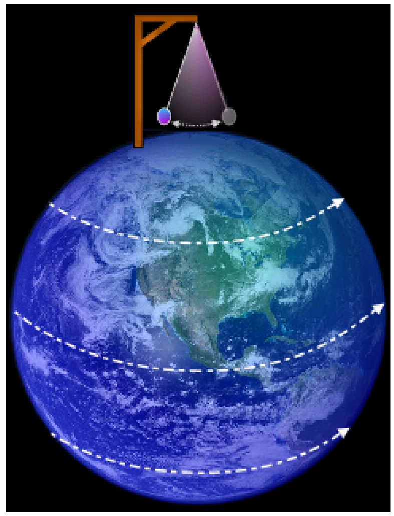


Figure 1: A Foucault pendulum at the north pole. The pendulum swings in the same plane as the Earth rotates beneath it. From `http://en.wikipedia.org/wiki/Foucault_pendulum`

$-2(mv/r)\omega \times \mathbf{r}$, tangent to the circle, and of magnitude independent of the radial distance. The magnitude of the total force is $F = m\omega\sqrt{\omega^2 r^2 + 4v^2}$. The static (!) friction force has magnitude $f = \mu mg$. If $f \geq F$, then $\omega^2 r^2 \leq (\mu g/\omega)^2 - 4v^2 = (0.98\text{m/s})^2 - (0.5\text{cm/s})^2 \approx (0.98\text{m/s})^2$. We see that the Coriolis force is very small compared to the maximum friction force, so it can be neglected. Neglecting the Coriolis force, the condition for not splipping is then $r \leq \mu g/\omega^2 = 0.3\text{m}$. The distance at which the centripetal force is larger than the Coriolis force is $r_0 = 2v/\omega = 3.3\text{mm}$.