Phys 7221, Fall 2006: Homework # 6

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Problem 3-7

In the laboratory system, the scattering angle of the incident particle is ϑ, and that of the initially stationary target particle, which recoils, is ϕ; see Fig. 3.24, or Fig.1 below: what are the differences in the figures?

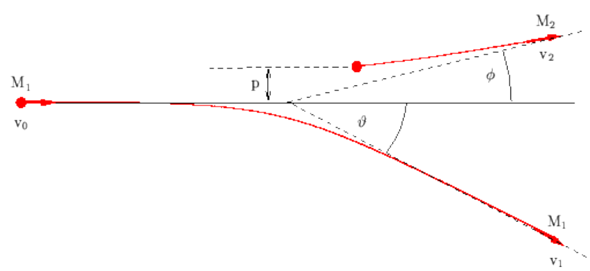


Figure 1: A mass M_1 scattered by an initially stationary target M_2, in the laboratory system, from `www.iue.tuwien.ac.at/phd/hoessinger/node38.html`.

In the center of mass system, the scattering angle of the incident particle is Θ, and that of the target particle (initially stationary) is $\Phi = \pi - \Theta$: see Fig 3.25, or Fig.2.

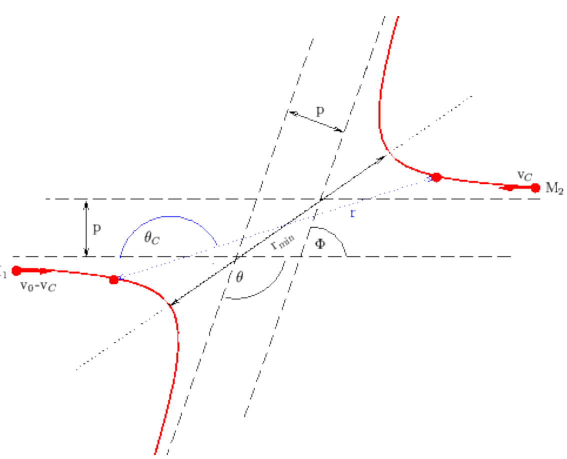


Figure 2: A mass M_1 scattered by an initially stationary target M_2, in the center of mass system, from `www.iue.tuwien.ac.at/phd/hoessinger/node38.html`

If $\mathbf{V}$ is the velocity of the center of mass, $\mathbf{v}_2$ is the velocity of the target particle in the laboratory system, and $\mathbf{v}_2'$ is the velocity of the target particle in the center of mass system, then $\mathbf{v}_2 = \mathbf{V} + \mathbf{v}_2'$, represented in Fig.3.

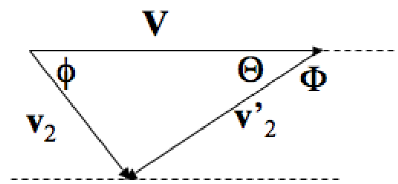


Figure 3: Graphical vector sum $\mathbf{v}_2 = \mathbf{v}_2' + \mathbf{V}$.

The components of the vector sum are equations relating the magnitudes of the vectors and the angles ϕ, Θ, similar to the equaitons (3.106) for m_1:

$$\begin{aligned} v_2 \sin\phi &= v_2' \sin\Theta \\ v_2 \cos\phi &= V - v_2' \cos\Theta \end{aligned}$$

From these equations, we obtain an equation similar to (3.107):

$$\tan\phi = -\frac{\sin\Theta}{\cos\Theta - (V/v_2')} \tag{1}$$

The magnitude of $\mathbf{v}_2'$ is related to the magnitude of the relative velocity v: $v_2' = \mu v/m_2$. The magnitude of the velocity of the center of mass V is related to the speed of m_1 before the collision: $V = \mu v_0/m_2$ (Eq. 3.105); the ratio is then simply $V/v_2' = v_0/v$. For inelastic scattering, this ratio is determined by the Q value of the inelastic collision (see 3.113), but for elastic collisions this ratio is unity.

For elastic collisions, we see that since $V = v_2'$, the triangle in Fig.3 is isosceles, and then $\phi = (\pi - \Theta)/2$. We can also obtain this result from Eq.1, using some trig magic:

$$\tan\phi = -\frac{\sin\Theta}{\cos\Theta - 1} = -\frac{2\sin\Theta/2\ \cos\Theta/2}{2\sin^2\Theta/2} = -\frac{1}{\tan\Theta/2} = \tan(\pi/2 - \Theta/2)$$

Problem 3-30: Rutherford scattering for an attractive force

We consider an attractive gravitational force of the form $\mathbf{F} = -k\mathbf{e}_r/r^2$, and follow the derivation of the Rutherford formula in (3.98)-(3.102).

For a particle approaching an initially stationary target with impact parameter s, with velocity v_0 at infinity, the angular momentum is $l = mv_0 s$. The initial energy, if the target is stationary, is $E = mv_0^2/2$, so $l = s\sqrt{2mE}$.

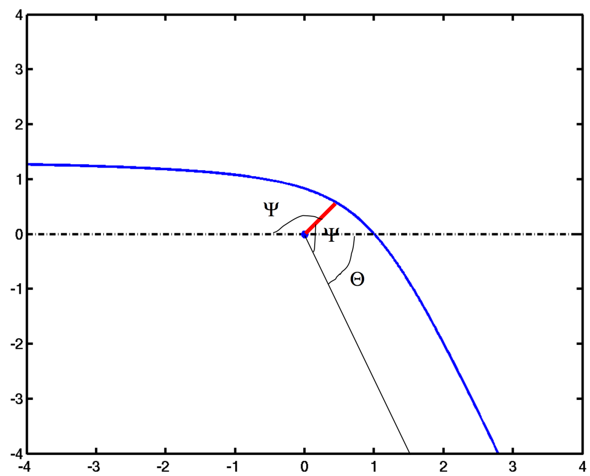
When the energy E is greater than zero, the orbit is a hyperbola given by

$$1/r = (\mu k/l^2)(1 - e\cos\theta),$$

with

$$e^2 = 1 + 2El^2/\mu k^2 = 1 + (m/\mu)(2Es/k)^2 > 1.$$

We have chosen $\theta' = \pi$, like in the textbook, so $\theta = 0$ corresponds to periapsis. Notice that the eccentricity is the same as in repulsive scattering, since it depends on k^2, but the orbit equation is different, since it is proportional to k.



Since $e > 1$, the angular values are restricted so that $\cos\theta < 1/e$ and the radial coordinate is positive. The minimum radial distance will be when $\theta = 0$, and $r = 1/(1+e)$; the asymptotes with $r = \infty$ are at $\cos\Psi = -1/e$ (since $\Psi > \pi/2$), as shown in Fig.. For large energies and thus large eccentricities, the asymptotes are close to $\pi/2$ (the orbit is almost straight); for small energies, and eccentricities close to unity, the asymptotes are close to π (the orbit has almost a 90deg angle).

The scattering angle Θ is related to Ψ as $\Theta = 2\Psi - \pi$ (similar, but different than 3.94), and then $1/e = -\cos\Psi = -\cos((\Theta+\pi)/2) = \sin(\Theta/2)$, and $\cot^2(\Theta/2) = e^2 - 1 = (m/\mu)(2Es/k)^2$, which results in

$$s = \frac{k}{2E}\sqrt{\frac{\mu}{m}}\cot\frac{\Theta}{2},$$

of similar form than 3.101, and therefore the scattering cross section is

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{k}{2E}\right)^2 \frac{\mu}{m}\csc^4\frac{\Theta}{2}$$

Problem 3-32

We consider a potential equal to zero for $r > a$, and equal to a negative constant $V = -V_0$ for $r \geq a$.

If the potential energy V is a constant, the force $\mathbf{F} = \nabla V$ is zero, and the motion is a stright line with constant velocity. If the potential has different constant values in different regions, the particle is "refracted" across the boundary (a sphere in this case), traveling in a straight line in all regions, but changing slope (velocity direction) and speed as the particle goes in and out of the sphere. Since $\Delta V/\Delta r > 0$ across the sphere, the change in potential represents a an attractive central force, acting only across the sphere boundary.

If the particle approaches the sphere with horizontal velocity of magnitude v_0, the "incident" angle (angle between velocity and the normal to the sphere) is α, such that $\sin\alpha = s/a$ (see figure). As it gets refracted, the particle is in the sphere with velocity v_1, with the angle between the velocity and the normal to the sphere β.

The angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = l\,\hat{\mathbf{k}}$ is conserved. The magnitude of the angular momentum before the particle enters the sphere is $l = mav_0 \sin\alpha$, and the angular momentum just after it enters the sphere is $l = mav_1 \sin\beta$. Since the angular momentum is conserved, we obtain "Snell's law":

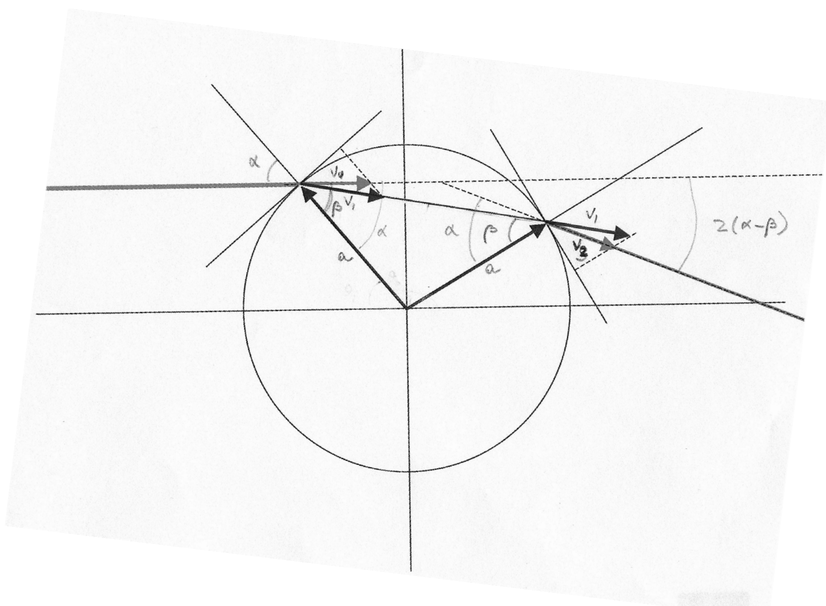
$$\sin\alpha = \frac{v_1}{v_0}\sin\beta = n\sin\beta,$$

with an "index of refraction" $n = v_1/v_0$.

Energy $E = T + V$ is conserved too. The energy before the particle enters the sphere is $E = mv_0^2/2$, and after it enters the sphere, is $E = mv_1^2/2 - V_0$, so

$$n = \frac{v_1}{v_0} = \sqrt{\frac{E + V_0}{E}}.$$

Since $n > 1$, $\beta < \alpha$: the particle gets refracted "down", like expected from an attractive force.



When the particle goes out of the sphere, it gets refracted again, exiting the sphere with velocity $v_2 = v_0$, at a direction with angle α with respect to the normal to the sphere, at the exiting point. The scattering angle is then

$$\Theta = 2(\alpha - \beta)$$

In order to get the scattering cross section, we need the function $s(\Theta)$, where s is the impact parameter. We use $\sin\alpha = s/a$ and $\sin\beta = \sin\alpha/n = s/an$ to get what we need:

$$\frac{s}{a} = \sin\alpha = \sin(\Theta/2 + \beta)$$

$$
\begin{aligned}
&= \sin\Theta/2\cos\beta + \cos\Theta/2\sin\beta \\
&= \sin\Theta/2\cos\beta + \frac{s}{an}\cos\Theta/2 \\
\frac{s}{a}(n-\cos\Theta/2) &= n\sin\Theta/2\cos\beta \\
\left(\frac{s}{a}\right)^2(n-\cos\Theta/2)^2 &= n^2\sin^2\Theta/2\cos^2\beta \\
\left(\frac{s}{a}\right)^2(n-\cos\Theta/2)^2 &= n^2\sin^2\Theta/2(1-\frac{s^2}{n^2a^2}) \\
\left(\frac{s}{a}\right)^2\left((n-\cos\Theta/2)^2+\sin^2\Theta/2\right) &= n^2\sin^2\Theta/2 \\
s^2 &= \frac{a^2n^2\sin^2\Theta/2}{1+n^2-2n\cos\Theta/2} \\
s &= \frac{an\sin\Theta/2}{\sqrt{1+n^2-2n\cos\Theta/2}}
\end{aligned}
$$

This formula gives the impact parameter as a function of the scattering angle Θ and the index n, itself a function of the energy E and the potential parameter V_0. The function $s(\Theta)$ is antisymmetric on Θ: a particle with negative impact parameter $-s$ will scatter downwards with a negative angle $-\Theta$, equal to minus the upwards scatter angle for a particle approaching with a positive impact parameter s and the same energy.

At $s = 0$, $\Theta = 0$ and the particle is not refracted at all, since it enters the sphere in a normal direction. The function $s(\Theta)$ has a maximum $s = a$ at $\cos\Theta_{\max}/2 = 1/n$, $n\sin\Theta_{\max}/2 = \sqrt{n^2-1}$. So, there is a maximum angle $\Theta_{\max} = 2\cos^{-1}(1/n)$, which is the scatter angle of particles grazing the spherical region (particles wills scatter down from the top of the sphere, and up from the bottom of the sphere).

As the energy of the approaching particle increases, the index of refraction $n = \sqrt{V_0/E+1} \approx 1$, and $\Theta_{\max} = 2\cos^{-1}(1/n) \approx 0$: the particles get only slightly deflected.

For small incident energies, $n = \sqrt{V_0/E+1}$ is large, and $\Theta_{\max} = 2\cos^{-1}(1/n) \to \pi$: the particle is refracted by $\beta = \pi/2$ across the first boundary, and then goes back in the direction it approached.

From the function $s(\Theta)$, we obtain the scattering cross section (not a pretty formula!):

$$
\begin{aligned}
\frac{ds}{d\Theta} &= \frac{1}{2}an\frac{(\cos\Theta/2-n)(1-n\cos\Theta/2)}{(1+n^2-2n\cos\Theta/2)^{3/2}} \\
\sigma(\Theta) = \frac{s}{\sin\Theta}\frac{ds}{d\Theta} &= \frac{a^2n^2}{4\cos\Theta/2}\frac{(\cos\Theta/2-n)(1-n\cos\Theta/2)}{(1+n^2-2n\cos\Theta/2)^2}
\end{aligned}
$$

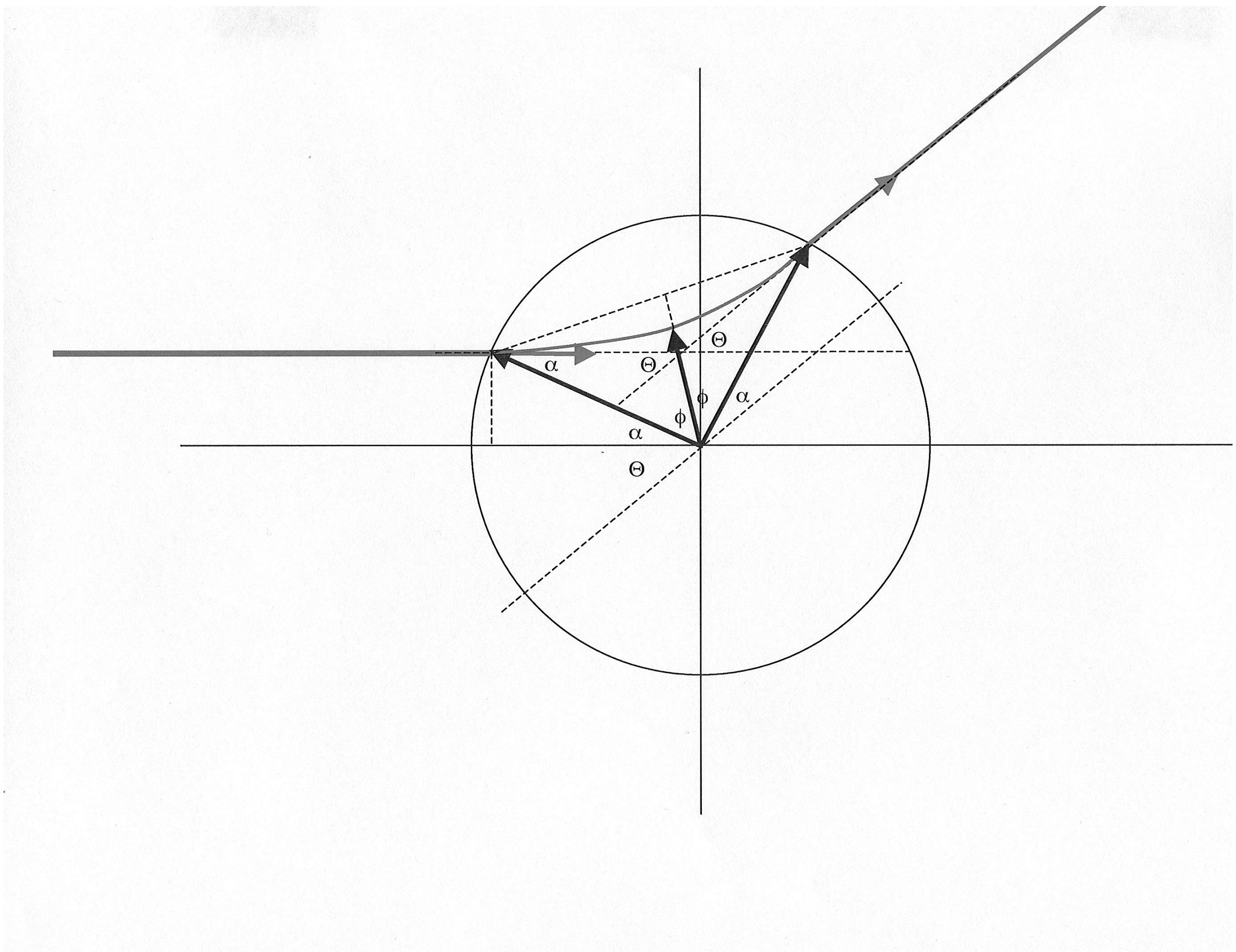
The total cross section is

$$
\begin{aligned}
\sigma_T &= \int_{4\pi}\sigma(\mathbf{\Omega})d\mathbf{\Omega} = 2\pi\int_0^{\Theta_{\max}}\sigma(\Theta)\sin\Theta d\Theta \\
&= 2\pi\int s\frac{ds}{d\Theta}d\Theta = \pi(s^2(\Theta_{\max})-s^2(0)) \\
&= \pi\frac{a^2n^2\sin^2\Theta_{\max}/2}{1+n^2-2n\cos\Theta_{\max}/2} = \pi a^2
\end{aligned}
$$

which is of course the cross section area of the incoming beam of particles incident on the sphere, the only ones that are scattered.

Problem 3-35

Consider a potential of the form $V(r) = k/r - k/a$ if $r \leq a$, and $V(r) = 0$ if $r \geq a$, a "truncated" Coulomb potential. This is a central potential so energy and momentum are conserved. Since the potential is continuous across the spherical boundary, there is no refraction as there was in Problem 3.32.



Outside the sphere, the potential is zero, so the particle moves in a straight line. Inside the sphere, the orbit is an hyperbola , as shown in the figure:

$$r(\theta) = \frac{l^2}{mk} \frac{1}{e \cos\theta - 1} = \frac{1}{u(\theta)} \tag{2}$$

where we have chosen the origin for angle *theta* at the closest point to the center, where $r = r_{min} = (l^2/mk)1/(e-1)$.

When the particle enters the sphere, $r = a$, $\theta = -\phi$, with the angle ϕ shown in the figure. When the particle exits the sphere, $r = a$ and $\theta = \phi$. The scattering angle is $\Theta = \pi - 2(\alpha + \phi)$, or

$$\sin \Theta/2 = \cos(\alpha + \phi).$$

We need then to find expressions for ϕ and α relating those angles to the scattering parameter s and the energy E. From the drawing, we see that $\sin\alpha = s/a$.

The angular momentum is $l^2 = m^2 v_0^2 s^2 = 2mEs^2$. From the orbit equation, evlauted at $\theta = \pm\phi$ where $r = a$, we have

$$\frac{1}{a} = \frac{mk}{l^2}(e\cos\phi - 1) = \frac{k}{2Es^2}(e\cos\phi - 1)$$

The x and y coordinates of the trajectory inside the sphere are given by $x = r\cos(-\theta + \phi + \alpha + \Theta) = r\sin(\theta - \Theta/2)$, $y = r\sin(-\theta + \phi + \alpha + \Theta) = r\cos(\theta - \Theta/2)$. When the particle enters the sphere, $r = a, \theta = -\phi$, and the velocity is horizontal, so $\dot{y} = (dy/d\theta)\dot{\theta} = 0$, providinga nother relationship between ϕ and α:

$$\begin{aligned}
\frac{dy}{d\theta} &= \frac{dr}{d\theta}\cos(\theta - \Theta/2) - r\sin(\theta - \Theta/2) \\
&= r\left(\frac{e\sin\theta}{e\cos\theta - 1}\cos(\theta - \Theta/2) - \sin(\theta - \Theta/2)\right) \\
\left.\frac{dy}{d\theta}\right|_{\theta=-\phi} = 0 &= \frac{-e\sin\phi}{e\cos\phi - 1}\cos(-\phi - \Theta/2) - \sin(-\phi - \Theta/2) \\
\tan(\phi + \Theta/2) &= \frac{e\sin\phi}{e\cos\phi - 1} \\
\tan(\pi/2 + \alpha) &= \frac{e\sin\phi}{e\cos\phi - 1} \\
\tan\alpha &= \frac{e\cos\phi - 1}{e\sin\phi} \\
e\sin\phi\sin\alpha &= e\cos\phi\cos\alpha - \cos\alpha \\
\cos\alpha &= e\cos(\phi + \alpha) = e\cos(\pi/2 - \Theta/2) \\
\cos\alpha &= e\sin\Theta/2
\end{aligned}$$

This provides the desired relationship between s and Θ, since $\cos\alpha = \sqrt{1 - (s/a)^2}$, and the eccentricity e is related to the energy $E = T + V$ and angular momentum $l = mr^2\dot{\theta}$:

$$\begin{aligned}
E &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \\
&= \frac{1}{2}m\left(\frac{dr}{d\theta}\right)^2\dot{\theta}^2 + \frac{1}{2}\frac{l^2}{mr^2} + \frac{k}{r} - \frac{k}{a} \\
&= \frac{1}{2}\frac{l^2}{mr^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{2}\frac{l^2}{mr^2} + \frac{k}{r} - \frac{k}{a} \\
&= \frac{1}{2}\frac{l^2}{m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}u^2 + ku - \frac{k}{a}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{l^2}{m}\left(\frac{mk}{l^2}\right)^2 e^2\sin^2\theta + \frac{l^2}{2m}\left(\frac{mk}{l^2}\right)^2 (e\cos\theta - 1)^2 + \frac{mk^2}{l^2}(e\cos\theta - 1) - \frac{k}{a} \\
&= \frac{1}{2}\frac{mk^2}{l^2}(e^2 - 1) - \frac{k}{a} \\
e^2 &= 1 + \frac{2(E + k/a)l^2}{mk^2} = 1 + \lambda(s/a)^2
\end{aligned}
$$

with $\lambda = 4Ea/k(1 + Ea/k)$.

We finally have then a formula for $s(\Theta)$:

$$
\begin{aligned}
\cos^2\alpha &= e^2\sin^2\Theta/2 \\
1 - \frac{s^2}{a^2} &= \left(1 + \lambda\frac{s^2}{a^2}\right)\sin^2\Theta/2 \\
s^2 &= a^2\frac{1 - \sin^2\Theta/2}{1 + \lambda\sin^2\Theta/2} = a^2\frac{\cos^2\Theta/2}{1 + \lambda\sin^2\Theta/2}
\end{aligned}
$$

The function $s(\Theta)$ is monotonically decreasing, from $s = a$ at $\Theta = 0$, to $s = 0$ at $\Theta = \pi$.

A particle approaching with impact parameter $s = a(1 - \epsilon)$, grazing the sphere, will be only slightly scattered up, with scattering angle $\sin\Theta/2 \approx \epsilon/\lambda$.

A particle approaching with no angular momentum ($s = 0$), will have a scattering angle $\Theta = \pi$: the particle is bounced back ("back scattered") from the sphere. Particles approaching with very small angular momentum will also be backscattered. The minimum impact parameter for which there will be forward scattering is given by

$$
s_0 = s(\Theta = \pi/2) = a\frac{\cos\pi/4}{\sqrt{1 + \lambda\sin^2\pi/4}} = \frac{a/\sqrt{2}}{\sqrt{1 + \lambda/2}}
$$

If the particle's initial kinetic energy E is very large compared with the maximum potential energy k/a: $E \gg k/a$, then $\lambda \gg 1$ and most particles will have forward scattering. If the energy is very low, $\lambda \ll 1$, and the particles with an impact parameter $s \geq s_0 \approx a/\sqrt{2}$ will be forwards scattered: half the cross section area of the incident beam will be back scattered, and half will be forward scattered.

The scattering cross section is

$$
\begin{aligned}
\sigma(\Theta) &= \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \\
&= \frac{a^2}{4}\frac{1 + \lambda}{(1 + \lambda\sin^2\Theta/2)^2}
\end{aligned}
$$