Prob 3-11: Collapse of an orbital system

Consider two particles falling into each other due to gravitational forces, starting from rest at a distance $a$. The system has zero angular momentum, with the energy given by

$$E = T + V = \frac{1}{2} m \dot{r}^2 - \frac{k}{r} = -\frac{k}{a}$$

where $m$ is the reduced mass of the system, and $r$ is the distance between the masses. Notice that the value of the energy, $-k/a$, calculated from the initial condition $\dot{r} = 0, r = a$, is not that of a Kepler's orbit, $-k/2a$, because $l = 0$.

We can derive an equation for $r$ as usual:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m} \sqrt{E - V}} = \sqrt{\frac{2}{m} \frac{k}{r} - \frac{k}{a}}$$

$$dt = -\frac{ma}{2k} \frac{\sqrt{r} dr}{\sqrt{a - r}} = \sqrt{\frac{2ma}{k}} \sqrt{a - u^2} du$$

where we used the substitution $u^2 = a - r$, and used the fact that $dr/dt < 0$ to add a negative sign when taking the square root of $\dot{r}^2$. We can integrate the equation from the initial time when $u = 0$, to the collapse time when $u = \sqrt{a}$, obtaining the time of the fall:

$$t_0 = \sqrt{\frac{2ma}{k}} \int_0^{\sqrt{a}} \sqrt{a - u^2} du = \sqrt{\frac{2ma}{k} \pi a \frac{4}{4}} = \pi \sqrt{\frac{ma^3}{8k}}$$

If the masses were in a circular orbit of radius $a$, the period is $\tau = 2\pi \sqrt{ma^3/k}$, so the time of the fall can be expressed as $t_0 = \tau/4\sqrt{2}$.  

1
**Prob 3-21: A modified Kepler’s potential**

Consider a central potential of the form $V(r) = -k/r + h/r^2$. The orbit equation (3.34) for $u(\theta) = 1/r(\theta)$ is

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{l^2} \frac{d}{du}V = \frac{m}{l^2} \frac{d}{du}(-ku + hu^2) = \frac{km}{l^2} - \frac{2mh}{l^2}u$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{2mh}{l^2}\right)u = \frac{km}{l^2}$$

The solution to this equation is of the form

$$u = \frac{km}{l^2} + A \cos(\beta(\theta - \theta_0))$$

with $\beta^2 = 1 + 2mh/l^2$.

This is the equation of a Kepler orbit (parabola, ellipse or hyperbola) in a coordinate system where the angular coordinate is $\theta' = \beta \theta$.

A revolution around the origin sweeps an $\theta$ angle equal to $2\pi$. If $\beta \gg 1$, there are many radial oscillations in one revolution; if $\beta \approx 1$, the orbit shows a slow precession. If the energy is negative and $2mh/l^2 \ll 1$, the orbit is a precessing ellipse. In a cycle of the periodic motion with period $\tau$, the radial coordinate returns to the original value when $\beta(\theta - \theta_0) = 2\pi$, or

$$\theta - \theta_0 = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{1 + 2mh/l^2}} = 2\pi - \dot{\Omega} \tau$$

The precession speed is then

$$\dot{\Omega} = \frac{2\pi}{\tau} \left(1 - \frac{1}{\sqrt{1 + 2mh/l^2}}\right) \approx \frac{2\pi mh}{l^2 \tau}$$

This means orbit precession can be used as a test of Newton’s theory for the gravitational force being derived from a potential $-k/r$. Using $l^2 = mka(1 - e^2)$, we obtain an expression for $\dot{\Omega}$ in terms of the perturbation parameter of Kepler’s potential $\eta = h/ka$, and orbital parameters:

$$\dot{\Omega} \approx \frac{2\pi mh}{l^2 \tau} = \frac{2\pi mh}{mka(1 - e^2)\tau} = \frac{2\pi \eta}{(1 - e^2)\tau}$$

The effect is more pronounced for eccentric and long orbits. The perihelion of Mercury is observed to precess (after correcting for known planetary perturbations) by 43 arc-seconds per century:

$$\dot{\Omega} = \frac{43 \times (2\pi/360) \times (1/3600) \text{ rad}}{100 \text{ yr}} = 2.1 \times 10^{-6} \text{ rad/yr}$$
and thus

\[ \eta \approx \frac{(1 - e^2)\tau \Omega}{2\pi} \approx 7.6 \times 10^{-8} \]

This discrepancy is also (and better) explained by General Relativity.

**Prob 3-23: Mass ratio of Sun and Earth**

The period and the semi-major axis of elliptical orbits in Kepler’s potential are related by \((\tau/2\pi)^2 = a^3/\mu/k\) where \(\mu = m_1 m_2/(m_1 + m_2)\) is the reduced mass of the system, and \(k = Gm_1 m_2\). When \(m_1 \gg m_2\), we have \((\tau/2\pi) \approx a^3/Gm_1\). For the Earth-Sun system,

\[ \left( \frac{\tau_{es}}{2\pi} \right)^2 = \frac{a_{es}^3}{GM_s}. \]

For the Earth-Moon system,

\[ \left( \frac{\tau_{em}}{2\pi} \right)^2 = \frac{a_{em}^3}{GM_e}. \]

Taking ratios, we obtain

\[ \frac{M_s}{M_e} = \left( \frac{a_{es}}{a_{em}} \right)^3 \left( \frac{\tau_{em}}{\tau_{es}} \right)^2 = \left( \frac{1.5 \times 10^8}{3.8 \times 10^5} \right)^3 \left( \frac{27.3}{365} \right)^2 = 3.4 \times 10^5 \]

The measured value, from [http://ssd.jpl.nasa.gov/?constants](http://ssd.jpl.nasa.gov/?constants), is 328900.56±0.02; our estimate is within 3% of this value.

**Prob 3-24: Kepler’s equation**

The energy in Kepler’s motion is

\[ E = \frac{1}{2} m r^2 + \frac{l^2}{2mr^2} - \frac{k}{r} \]

For negative energy and elliptical orbits, the energy is \(E = -\frac{k}{2a}\), and the angular momentum is \(l^2 = kma(1 - e^2)\), thus

\[
\dot{r}^2 = \frac{2}{m} \left( E - \frac{l^2}{2mr^2} + \frac{k}{r} \right) \\
= \frac{2}{m} \left( -\frac{k}{2a} - \frac{ka(1 - e^2)}{2r^2} + \frac{k}{r} \right) \\
= \frac{k}{mar^2} \left( -r^2 - a^2(1 - e^2) + 2ar \right) \\
= \frac{k}{mar^2} \left( a^2 e^2 - (r - a)^2 \right)
\]
Since the period of the motion is \( \tau = 2\pi/\omega \), with \( \omega = \sqrt{k/ma^3} \), we obtain

\[
\dot{r} = \frac{k}{ma^2} (a^2 e^2 - (r - a)^2) \\
= \omega^2 \frac{a^2}{r^2} (a^2 e^2 - (r - a)^2) \\
\frac{dr}{dt} = \omega \frac{a}{r} \sqrt{a^2 e^2 - (r - a)^2} \\
dt = \frac{1}{\omega a} \frac{r dr}{\sqrt{a^2 e^2 - (r - a)^2}}
\]

which we can use to integrate \( t(r) \). Using the orbit equation \( r = a(1 - e \cos \psi) \), and \( dr = ea \sin \psi d\psi \) we obtain

\[
\omega dt = \frac{1}{a} \frac{r dr}{\sqrt{a^2 e^2 - (r - a)^2}} \\
= \frac{1}{a} \frac{a(1 - e \cos \psi)ea \sin \psi d\psi}{\sqrt{a^2 e^2 - (a e \cos \psi)^2}} \\
= (1 - e \cos \psi) d\psi
\]

which can be now trivially integrated into Kepler’s equation:

\[
\omega t = \psi - e \sin \psi
\]

Prob 3-33: A particle in a paraboloid of revolution

A particle with coordinates \( \mathbf{r} = (x, y, z) \) is constrained to move in a paraboloid of revolution, i.e., \( z = r^2/a = (x^2 + y^2)^2/a \). We will use generalized polar coordinates \( r, \theta \) to describe the motion of the particle, with \( z = r^2/a \). The kinetic energy is

\[
T = \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2 + \dot{z}^2) = \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2 + 4 \frac{r^2}{a^2} r^2)
\]

The potential energy is

\[
V = -mg \cdot \mathbf{r} = +mgz = \frac{mg}{a} r^2
\]

The Lagrangian is

\[
L = T - V = \frac{1}{2} mr^2 + \frac{1}{2} mr^2 \dot{\theta}^2 + 2m \frac{r^2}{a^2} \dot{r}^2 - \frac{mg}{a} r^2
\]

We see that the coordinate \( \theta \) is cyclic, so the \( z \)-component of the angular momentum is conserved (associated with the symmetry of rotation about the \( z \)-axis):

\[
mr^2 \ddot{\theta} = l = \text{constant}
\]
Lagrange’s equation for the coordinate $r$ is
\[
\frac{d}{dt} \left( m\dot{r} + 2m\frac{r^2}{a^2}\dot{r} \right) - mr\dot{\theta}^2 - 4m\frac{r\dot{r}^2}{a^2} + \frac{2mg}{a}r \quad = \quad 0
\]
\[
\left( 1 + 2\frac{r^2}{a^2} \right) \ddot{r} - \frac{l^2}{m^2r^3} + \frac{2g}{a}r \quad = \quad 0
\]

There are solutions for circular orbits, with $r^4 = r_0^4 = \frac{l^2a}{2gm^2}$. If the orbit is circular with radius $r_0$, the angular momentum is related to the radius and the speed, $l = mr_0v$. We can then find the condition between the speed and the radius for circular orbits: $v^2 = 2gr_0^2/a$. If the orbit is only approximately circular, we find an approximate equation for the perturbation $\delta r = r - r_0$:
\[
\left( 1 + 2\frac{(r_0 + \delta r)^2}{a^2} \right) \ddot{\delta r} = \frac{l^2}{m^2r_0^3} \frac{1}{1 + (\delta r/r_0)^3} - \frac{2g}{a} (r_0 + \delta r)
\]
\[
\left( 1 + 2\frac{r_0^2}{a^2} \right) \delta r \approx -\frac{2g}{a} \delta r
\]

This equation has a periodic solution, with period $\tau = 2\pi/\omega$, with $\omega^2 = (2g/a)/(1+2r_0^2/a^2)$. 