Phys 7221, Fall 2006: Homework # 5

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Prob 3-11: Collapse of an orbital system

Consider two particles falling into each other due to gravitational forces, starting from rest at a distance a. The system has zero angular momentum, with the energy given by

$$E = T + V = \frac{1}{2}m\dot{r}^2 - \frac{k}{r} = -\frac{k}{a}$$

where m is the reduced mass of the system, and r is the distance between the masses. Notice that the value of the energy, $-k/a$, calculated from the initial condition $\dot{r} = 0, r = a$, is *not* that of a Kepler's orbit, $-k/2a$, because $l = 0$.

We can derive an equation for r as usual:

$$\begin{aligned}
\frac{dr}{dt} &= \sqrt{\frac{2}{m}}\sqrt{E-V} \\
&= \sqrt{\frac{2}{m}}\sqrt{\frac{k}{r} - \frac{k}{a}} \\
dt &= -\sqrt{\frac{ma}{2k}}\frac{\sqrt{r}dr}{\sqrt{a-r}} \\
&= \sqrt{\frac{2ma}{k}}\sqrt{a-u^2}du
\end{aligned}$$

where we used the substitution $u^2 = a - r$, and used the fact that $dr/dt < 0$ to add a negative sign when taking the square root of $\dot{r}^2$. We can integrate the equation from the initial time when $u = 0$, to the collapse time when $u = \sqrt{a}$, obtaining the time of the fall:

$$t_0 = \sqrt{\frac{2ma}{k}}\int_0^{\sqrt{a}}\sqrt{a-u^2}du = \sqrt{\frac{2ma}{k}}\frac{\pi a}{4} = \pi\sqrt{\frac{ma^3}{8k}}$$

If the masses were in a circular orbit of radius a, the period is $\tau = 2\pi\sqrt{ma^3/k}$, so the time of the fall can be expressed as $t_0 = \tau/4\sqrt{2}$.

Prob 3-21: A modified Kepler's potential

Consider a central potential of the form $V(r) = -k/r + h/r^2$. The orbit equation (3.34) for $u(\theta) = 1/r(\theta)$ is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V = -\frac{m}{l^2}\frac{d}{du}(-ku + hu^2) = \frac{km}{l^2} - \frac{2mh}{l^2}u$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{2mh}{l^2}\right)u = \frac{km}{l^2}$$

The solution to this equation is of the form

$$u = \frac{km}{l^2} + A\cos(\beta(\theta - \theta_0))$$

with $\beta^2 = 1 + 2mh/l^2$.

This is the equation of a Kepler orbit (parabola, ellipse or hyperbola) in a coordinate system where the angular coordinate is $\theta' = \beta\theta$.

A revolution around the origin sweeps a θ angle equal to 2π. If $\beta \gg 1$,there are many radial oscillations in one revolution; if $\beta \approx 1$, the orbit shows a slow precession. If the energy is negative and $2mh/l^2 \ll 1$, the orbit is a precessing ellipse. In a cycle of the periodic motion with period τ, the radial coordinate returns to the original value when $\beta(\theta - \theta_0) = 2\pi$, or

$$\theta - \theta_0 = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{1 + 2mh/l^2}} = 2\pi - \dot{\Omega}\tau$$

The precession speed is then

$$\dot{\Omega} = \frac{2\pi}{\tau}\left(1 - \frac{1}{\sqrt{1 + 2mh/l^2}}\right) \approx \frac{2\pi mh}{l^2\tau}$$

This means orbit precession can be used as a test of Newton's theory for the gravitational force being derived from a potential $-k/r$. Using $l^2 = mka(1 - e^2)$, we obtain an expression for $\dot{\Omega}$ in terms of the perturbation parameter of Kepler's potential $\eta = h/ka$, and orbital parameters:

$$\dot{\Omega} \approx \frac{2\pi mh}{l^2\tau} = \frac{2\pi mh}{mka(1 - e^2)\tau} = \frac{2\pi\eta}{(1 - e^2)\tau}$$

The effect is more pronounced for eccentric and long orbits. The perihelion of Mercury is observed to precess (after correcting for known planetary perturbations) by 43 arc-seconds per century:

$$\dot{\Omega} = \frac{43 \times (2\pi/360) \times (1/3600)\,\mathrm{rad}}{100\mathrm{yr}} = 2.1 \times 10^{-6}\mathrm{rad/yr}$$

and thus

$$\eta \approx \frac{(1-e^2)\tau\dot{\Omega}}{2\pi} \approx 7.6 \times 10^{-8}$$

This discrepancy is also (and better) explained by General Relativity.

Prob 3-23: Mass ratio of Sun and Earth

The period and the semi-major axis of elliptical orbits in Kepler's potential are related by $(\tau/2\pi)^2 = a^3\mu/k$ where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass of the system, and $k = Gm_1 m_2$. When $m_1 \gg m_2$, we have $(\tau/2\pi) \approx a^3/Gm_1$. For the Earth-Sun system,

$$\left(\frac{\tau_{es}}{2\pi}\right)^2 = \frac{a_{es}^3}{GM_s}.$$

For the Earth-Moon system,

$$\left(\frac{\tau_{em}}{2\pi}\right)^2 = \frac{a_{em}^3}{GM_e}.$$

Taking ratios, we obtain

$$\left(\frac{\tau_{es}}{\tau_{em}}\right)^2 = \frac{a_{em}^3}{a_{em}^3}\frac{M_e}{M_s}$$

$$\frac{M_s}{M_e} = \left(\frac{a_{es}}{a_{em}}\right)^3 \left(\frac{\tau_{em}}{\tau_{es}}\right)^2 = \left(\frac{1.5\times 10^8}{3.8\times 10^5}\right)^3 \left(\frac{27.3}{365}\right)^2 = 3.4\times 10^5$$

The measured value, from `http://ssd.jpl.nasa.gov/?constants`, is 328900.56 ± 0.02; our estimate is within 3% of this value.

Prob 3-24: Kepler's equation

The energy in Kepler's motion is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}$$

For negative energy and elliptical orbits, the energy is $E = -\frac{k}{2a}$, and the angular momentum is $l^2 = kma(1-e^2)$, thus

$$\begin{aligned}
\dot{r}^2 &= \frac{2}{m}\left(E - \frac{l^2}{2mr^2} + \frac{k}{r}\right) \\
&= \frac{2}{m}\left(-\frac{k}{2a} - \frac{ka(1-e^2)}{2r^2} + \frac{k}{r}\right) \\
&= \frac{k}{mar^2}\left(-r^2 - a^2(1-e^2) + 2ar\right) \\
&= \frac{k}{mar^2}\left(a^2e^2 - (r-a)^2\right)
\end{aligned}$$

Since the period of the motion is $\tau = 2\pi/\omega$, with $\omega = \sqrt{k/ma^3}$, we obtain

$$\begin{aligned}
\dot{r}^2 &= \frac{k}{mar^2}\left(a^2e^2 - (r-a)^2\right) \\
&= \omega^2 \frac{a^2}{r^2}\left(a^2e^2 - (r-a)^2\right) \\
\frac{dr}{dt} &= \omega \frac{a}{r}\sqrt{a^2e^2 - (r-a)^2} \\
dt &= \frac{1}{a\omega}\frac{r\,dr}{\sqrt{a^2e^2 - (r-a)^2}}
\end{aligned}$$

which we can use to integrate $t(r)$. Using the orbit equation $r = a(1 - e\cos\psi)$, and $dr = ea\sin\psi d\psi$ we obtain

$$\begin{aligned}
\omega\,dt &= \frac{1}{a}\frac{r\,dr}{\sqrt{a^2e^2 - (r-a)^2}} \\
&= \frac{1}{a}\frac{a(1-e\cos\psi)ea\sin\psi d\psi}{\sqrt{a^2e^2 - (-ae\cos\psi)^2}} \\
&= (1 - e\cos\psi)\,d\psi
\end{aligned}$$

which can be now trivially integrated into Kepler's equation:

$$\omega t = \psi - e\sin\psi$$

Prob 3-33: A particle in a paraboloid of revolution

A particle with coordinates $\mathbf{r} = (x, y, z)$ is constrained to move in a paraboloid f revolution. i.e., $z = r^2/a = (x^2 + y^2)^2/a$. We will use generalized polar coordinates r, θ to describe the motion of the particle, with $z = r^2/a$. The kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + 4\frac{r^2}{a^2}\dot{r}^2)$$

The potential energy is

$$V = -m\mathbf{g}\cdot\mathbf{r} = +mgz = \frac{mg}{a}r^2$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + 2m\frac{r^2}{a^2}\dot{r}^2 - \frac{mg}{a}r^2$$

We see that the coordinate θ is cyclic, so the z-component of the angular momentum is conserved (associated with the symmetry of rotation about the z-axis):

$$mr^2\dot{\theta} = l = \text{constant}$$

Lagrange's equation for the coordinate r is

$$\begin{aligned} \frac{d}{dt}\left(m\dot{r}+2m\frac{r^2}{a^2}\dot{r}\right)-mr\dot{\theta}^2-4m\frac{r\dot{r}^2}{a^2}+\frac{2mg}{a}r &= 0 \\ \left(1+2\frac{r^2}{a^2}\right)\ddot{r}-\frac{l^2}{m^2r^3}+\frac{2g}{a}r &= 0 \end{aligned}$$

There are solutions for circular orbits, with $r^4 = r_0^4 = l^2a/(2gm^2)$. If the orbit is circular with radius r_0, the angular momentum is related to the radius and the speed, $l = mr_0v$. We can then find the condition between the speed and the radius for circular orbits: $v^2 = 2gr_0^2/a$. If the orbit is only approximately circular, we find an approximate equation for the perturbation $\delta r = r - r_0$:

$$\begin{aligned} \left(1+2\frac{(r_0+\delta r)^2}{a^2}\right)\ddot{\delta r} &= \frac{l^2}{m^2r_0^3}\frac{1}{1+(\delta r/r_0)^3}-\frac{2g}{a}(r_0+\delta r) \\ \left(1+2\frac{r_0^2}{a^2}\right)\ddot{\delta r} &\approx -\frac{2g}{a}\delta r \end{aligned}$$

This equation has a periodic solution, with period $\tau = 2\pi/\omega$, with $\omega^2 = (2g/a)/(1+2r_0^2/a^2)$.