# Phys 7221, Fall 2006: Homework # 4

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## Problem 3-10: A comet striking a planet

A planet has a very eccentric orbit about the Sun, with eccentricity $e = 1 - \alpha$ with $\alpha \ll 1$. When the planet is at the greatest distance from the Sun (aphelion), it is struck by a small comet traveling in a tangential direction (that is, the collision is head on, with the planet and comet velocities in the same direction). The collision is inelastic: the comet sticks to the planet, and momentum is conserved (but energy is not). What is the minimum kinetic energy the comet must have to make the planet's orbit parabolic (and unbound)?

The effect of the collision will be to increase the velocity of the planet, from $v_0$ to $v_f = v_0 + \delta v$. The energy $E_0$ before the collision, which was negative but close to zero, will increase and now it may be zero or positive: the motion will be unbound. The minimum change in velocity that will make the energy zero is:

$$E_f = \frac{1}{2}Mv_f^2 - \frac{k}{r} = \frac{1}{2}M(v_0 + \delta v)^2 - \frac{k}{r} \approx E_0 + Mv_0\delta v$$

$$E_f \geq 0 \Rightarrow \delta v \geq -\frac{E_0}{Mv_0}$$

We use conservation of momentum in the collision, with $v$ the comet's velocity before collision:

$$(M + m)(v_0 + \delta v) = Mv_0 + mv \Rightarrow \delta v \approx (m/M)v$$

where the apporximation is $m/M \ll 1$. The minimum comet velocity to unbind the planet from the Sun is:

$$v \geq -\frac{E_0}{mv_0}$$

which tells us the minimum kinetic energy the comet must have to unbind the planet

$$KE_c = \frac{1}{2}mv^2 \geq \frac{1}{2}\frac{E_0^2}{mv_0^2}.$$

We can find expressions for $E_0, v_0$ in terms of $\alpha$ and the semimajor axis $a$. For elliptical orbits, the energy is

$$E_0 = -\frac{k}{2a}.$$

The eccentricity is related to the angular momentum as:

$$\begin{aligned}
e &= \sqrt{1 + \frac{2E_0 l^2}{Mk^2}} \\
-\frac{2E_0 l^2}{Mk^2} &= 1 - e^2 \\
l^2 &= -\frac{Mk^2}{2E_0}(1 - e^2) \\
&= aMk(1 - e^2) \\
&= aMk(1 - (1 - \alpha)^2) \\
&\approx 2\alpha\; aMk
\end{aligned}$$

The aphelion distance is $r = a(1 + e) = a(2 - \alpha) \approx 2a$. The angular momentum of the planet before collision is $l = Mrv_0 \approx 2Mav_0$, and thus

$$l^2 \approx (2Mav_0)^2 \approx 2\alpha\; aMk \Rightarrow v_0^2 \approx \alpha \frac{k}{2Ma}.$$

The minimum kinetic energy the comet needs to have to unbind the planet is then

$$KE_c \geq \frac{1}{2}\frac{E_0^2}{mv_0^2} \approx \frac{1}{2m}\frac{(-k/2a)^2}{(\alpha k/2Ma)} = \frac{1}{2\alpha}\frac{M}{m}\frac{k}{2a}.$$

This expression says that the comet must have a kinetic energy equal to the energy of the planet $(k/2a)$, multiplied by two large factors, $1/2\alpha$ and $M/m$: quite a large energy is needed for unbinding a planet, even from a very eccentric orbit....

## Problem 3-13: Circular orbit under the influence of a central force

Assume a particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle. Use a generalized angular coordinates $\cos\alpha$ for the particle's position on the orbit, with the origin of the force at the origin, as shown in Fig.1.

From the drawing, we see that

$$r \sin\theta = R \sin\alpha$$

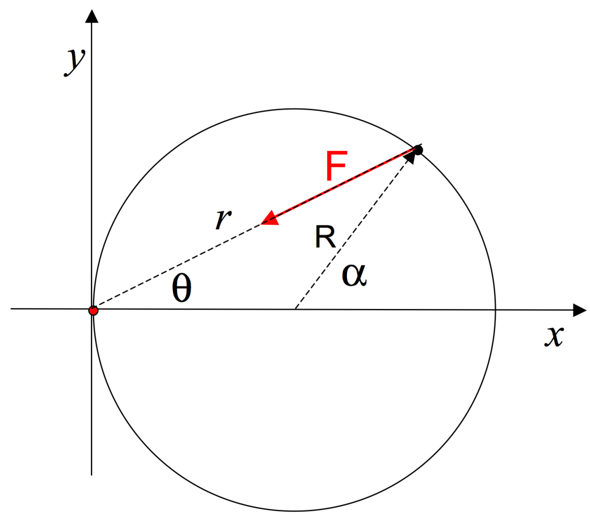


Figure 1: Problem 3-13

$$r\cos\theta = R(1+\cos\alpha)$$

from which we obtain the orbit equation:

$$r = 2R\cos\theta$$

We can then use the orbit equation to deduce the potential $V(r)$. The potential energy is

$$V(r) \quad = \quad E - T = E - \left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2}\right)$$

Taking time derivatives of the orbit equation , we get

$$\dot{r} = -2R\dot{\theta}\sin\theta.$$

The angular momentum is related to $\dot{\theta}$ as $mr^2\dot{\theta} = l$, so we then have

$$\dot{r} = -\frac{2Rl}{mr^2}\sin\theta$$

Then,

$$\begin{aligned} m\dot{r}^2 &= m\left(-\frac{2Rl}{mr^2}\sin\theta\right)^2 \\ &= \frac{4R^2l^2}{mr^4}\sin^2\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{4R^2l^2}{mr^4}\left(1-\cos^2\theta\right) \\
&= \frac{4R^2l^2}{mr^4}\left(1-\frac{r^2}{4R^2}\right) \\
&= \frac{4R^2l^2}{mr^4}-\frac{l^2}{mr^2}
\end{aligned}
$$

We can now calculate the potential:

$$
\begin{aligned}
V(r) &= E-T=E-\left(\frac{1}{2}m\dot{r}^2+\frac{1}{2}\frac{l^2}{mr^2}\right) \\
&= E-\frac{2R^2l^2}{mr^4}
\end{aligned}
$$

The force will be

$$
\mathbf{F}=-\frac{dV}{dr}\hat{\mathbf{e}}_r=-\frac{8R^2l^2}{mr^5}\hat{\mathbf{e}}_r
$$

Since we can always add any arbitrary constant to the potential, we choose it so that

$$
V(r)=-\frac{k}{r^4}
$$

where $k=2R^2l^2/m$. This is equivalent to saying that the total energy of the particle $E$ is zero.

The effective potential is given by

$$
V_{\text{eff}}=V(r)+\frac{l^2}{2mr^2}=-\frac{k}{r^4}+\frac{l^2}{2mr^2}=-\frac{l^2}{2mr^2}\left(\frac{4R^2}{r^2}-1\right)
$$

The effective potential will be $\approx -k/4r^4$ at short distances: negative, with a positive slope. At long distances, for non-zero angular momentum, the effective potential will be $\approx l^2/2mr^2$: positive, with a negative slope. There is thus a maximum value where the slope vanishes, at $r_0^2=8km/l^2=16R^2$, where $V_{max}=l^4/4km^2$. If the energy of a particle in this potential is less than $V_{max}$, then depending on the initial position, the orbit is either bounded or unbounded, in both cases with a turning point. For the particular orbit we are studying, the initial position is smaller than R, and the orbit is bounded. Moreover, the turning point is at $r=2R$, where we see again that $E=V_{\text{eff}}=0$.

We may find the period from the constant areal velocity and the total area:

$$
\frac{dA}{dt}=\frac{1}{2}r^2\dot{\theta}=\frac{l}{2m}=\frac{A}{\tau}=\frac{\pi R^2}{\tau}
$$

$$
\tau=\frac{2\pi mR^2}{l}=2\pi R^3\sqrt{\frac{m}{k}}
$$

We can compare this formula with Kepler's period, $\tau = 2\pi a^{3/2}\sqrt{m/k}$, but we should remember that the constants $k$ in each case have different units!

We now want an expression for the velocity. Using the orbit equation $r = 2R\cos\theta$ and $mr^2\dot{\theta} = l$, we have

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\ &= (-2R\dot{\theta}\sin\theta)^2 + (2R\cos\theta)^2\dot{\theta}^2 \\ &= 4R^2\dot{\theta}^2 \\ &= \frac{4R^2 l}{r^2} \end{aligned}$$

We see that the velocity approaches infinity as the particle goes through the center of the force, where $r = 0$.

## Problem 3-19: Yukawa potential

Variations from Newton's law at long distances are often expressed in terms of a "Yukawa" potential,

$$V(r) = -\frac{k}{r}e^{-r/a}$$

This potential approximates Newton's potential at short distances $r \ll a$, but it approaches zero faster than any power law at long distances. (Variations from Newton's at short distances are usually expressed as $-kr(1 - be^{-r/a})$).

Since the force is a central force dependent on $r$, the equations of motion for a single particle can be reduced to a one-dimensional equation for $r$, with $\dot{\theta} = l/mr^2$, and

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}$$

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r) = \frac{l^2}{2mr^2} - \frac{k}{r}e^{-r/a} = \frac{l^2}{2m}\left(\frac{1}{r^2} - \frac{2mk}{l^2}\frac{e^{-r/a}}{r}\right) = A\left(\frac{1}{r^2} - \frac{e^{-r/a}}{br}\right)$$

where $a$ and $b = l^2/2mk$ are constants with units of distance, and $A = l^2/2m$ is a positive constant. If $l \neq 0$, at short distances, the potential is $V_{\text{eff}} \approx A/r^2$, and is large, positive, with a negative slope. At long distances, the potential is also $A/r^2$, since the exponential terms makes the "gravitational" term in the effective potential decays now faster than $1/r^2$. We plot some of the possible shapes of the potential in Fig. 2.

If the angular momentum is large, the negative term in the effective potential will never dominate, and the effective potential is always positive: the energy has to be positive, and the orbits will be unbound, with a turning point.

If the angular momentum is small, the Yukawa negative exponential term in the effective potential will dominate for some $r$values: the effective potential will have negative values,

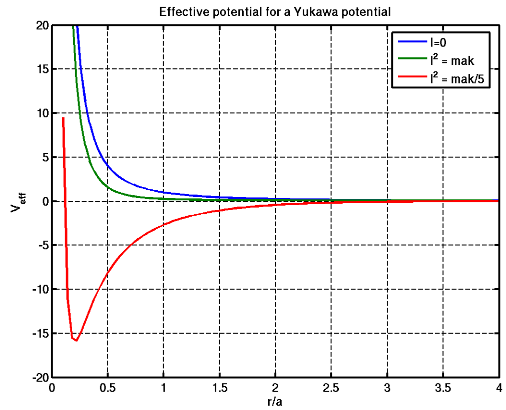


Figure 2: Effective potential for a Yukawa potential, with different values of angular momentum.

and a minimum value, similar to Kepler's effective potential. (With Kepler's potential, there is *always* a minimum value for $l \neq 0$.) In this case, Yukawa's effective potential will also have a local maximum value (small, and difficutl to see in the figure).

If the angular momentum is small, and the energy is larger than the local maximum of the effective potential, the orbits will be unbound with a turning point. There is a positive (and small) value of the energy which will allow an unstable circular orbit. Positive values of energy smaller than the local maximum allow for either bound orbits, or unbound orbits with a turning point, depending on the initial values of the system.

If the angular momentum is small, and the energy is negative, there will be bound orbits. There is also a minimum value of the energy that will allow a stable circular orbit.

The effective potential will have an extremum (local minimum or maximum) when

$$\frac{d}{dr}V_{\text{eff}} = 0 \Rightarrow e^{-r/a}\frac{r}{a}\left(1+\frac{r}{a}\right) = 2\frac{b}{a} = \frac{l^2}{mka} \tag{1}$$

This equation can be written as

$$f(x) = e^{-x}x(1+x) = C$$

with $x = r/a$ and C a dimensionless constant $C = l^2/mka$. The function $f(x) = e^{-x}x(1+x)$, with $x = r/a$, has a maximum value when $x^2 = 1+x$, or $x_0 = (1+\sqrt{5})/2 = 1.62$. The maximum value is $f(x_0) = e^{-x_0}x_0^3 = 0.84$.

If the angular momentum is large, $l^2/mka > f(x_0) = 0.84$, there is no solution to Eq. 1: the effective potential has no extrema: it decreases monotonically from $V_{\text{eff}} = \infty$ at $r = 0$ to $V_{\text{eff}} = 0$, at $r = \infty$.

If the angular momentum is exactly given by $l^2/mka = f(x_0)$, then $V_{\rm eff}$ has an inflexion point that will allow unstable orbits, but there will be no other bound orbits.

If the angular momentum is small, $l^2/mka < f(x_0) = 0.84$, there are two solutions to Eq. 1: the effective potential has *two* extremum points (there's no closed form for these solutions, though: you find them numerically or from plots). One extremum, at $r/a < x_0$, is the absolute minimum of the effective potential; the other extremum, at $r/a > x_0$, is a local maximum. The existence of a minimum will allow bound orbits, and a circular orbit; the existence of a local maximum allows an unstable circular orbit. In the Newtonian limit when $a \to \infty$, the condition $l^2 < 0.84mka$ is always satisfied for $l \neq 0$: there is always a minimum, and the local maximum is pushed to $r > a \to \infty$.

Let's assume the angular momentum is small enough to allow bound orbits. The minimum value of the potential will happen when Eq. 1 is satisfied, for a value $r = r_0 < a$:

$$e^{-r_0/a}\frac{r_0}{a}\left(1+\frac{r_0}{a}\right) = 2\frac{b}{a} \tag{2}$$

The minimum value of the effective potential is

$$V_{\rm min} = A\left(\frac{1}{r_0^2} - \frac{e^{-r_0/a}}{br_0}\right) = A\left(\frac{1}{r_0^2} - \frac{2b/r_0}{br_0(1+r_0/a)}\right) = -\frac{A}{r_0^2}\frac{a-r_0}{a+r_0}$$

If the energy is $E = V_{\rm min}$, the orbit will be circular with a radius $r_0$. If the energy is slightly larger, the orbit will still be bound, with $r = r_0 + \delta r$, or $u = u_0 + \delta u$ for $u = 1/r$. The orbit equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V(1/u) = F(u)$$

is then an equation for $\delta u(\theta)$, if we use a Taylor expansion of right hand side.

*What follows is generic to any central potential:*

If $F(u) \approx F(u_0) + F'(u_0)\delta u$, then the equation for $\delta u$ is

$$\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= F(u)\\
\frac{d^2(u_0+\delta u)}{d\theta^2} + (u_0+\delta u) &= F(u_0+\delta u)\\
\frac{d^2\delta u)}{d\theta^2} + u_0 + \delta u &\approx F(u_0) + F'(u_0)\delta u
\end{aligned}$$

The zeroth order of this equation is an equation for $r_0 = 1/u_0$, the radius of the circular orbit:

$$u_0 = F(u_0)$$

The first order of the equation is the "orbit" equation for $\delta u$:

$$\frac{d^2\delta u}{d\theta^2} + (1 - F'(u_0))\delta u = \frac{d^2\delta u}{d\theta^2} + \beta^2 \delta u = 0$$

This equation has an oscillatory solution with frequency $\beta$, with amplitude $\Delta$ and initial phase $\theta_0$ given by initial conditions:

$$\delta u = \Delta \cos \beta(\theta - \theta_0)$$

If $\beta = 1$ ($F'(u_0) = 0$), the orbit $u = u_0 + \delta u$ is an ellipse. Since we have assumed $\delta u \ll u_0$, the amplitude must also satisy $\Delta \ll u_0$, and the "ellipse" will be a slightly distorted circle.

In general, the angular position of the turning points, with $\dot{u} = 0$, are given by $\sin \beta(\theta - \theta_0) = 0$, or $\theta - \theta_0 = 2\pi/\beta$. If $\beta$ is a rational fraction, $\beta = p/q$, the orbit will be closed: after $p$ cycles in $r$, the particle will have made $q$ turns about the origin and will return to the same radial position. If $\beta$ is not a rational fraction, the orbit will not be closed: the particle never return to the same turning point.

If $\beta = 1 + \epsilon$, with $\epsilon \ll 1$, the orbit will be precessing. A cycle in $r$ (between two minimum or two maximum radial distances) will sweep an angle $2\pi/\beta \approx= 2\pi(1 - \epsilon)$, that is, it will have precessed by an angle $-2\pi\epsilon$: if $\epsilon > 0$, the orbit precesses backwards; if $\epsilon < 0$, the orbit precesses forward.

*Back to the Yukawa problem now:* The potential is $V(r) = -\frac{k}{r}e^{-r/a}$, so the force is

$$\begin{aligned} V(u) &= -kue^{-1/au} \\ F(u) &= -\frac{m}{l^2}\frac{dV}{du} \\ &= \frac{mk}{l^2}\left(1 + \frac{1}{au}\right)e^{-1/au} = \frac{1}{2b}\left(1 + \frac{1}{au}\right)e^{-1/au} \end{aligned}$$

The equation for the circular orbit is

$$u_0 = F(u_0) = \frac{1}{2b}\left(1 + \frac{1}{au_0}\right)e^{-1/au_0}$$

which is, of course, the same equation as Eq. 2, and admits two solutions: one stable circular orbit with $r_0 < 1.62a$ for the minimum of the effective potential; another unstable circular orbit with $r_0 > 1.62a$ near the maximum of the effective potential.

To get the precession of the orbit, we need $F'(u_0)$ :

$$\begin{aligned} F(u) &= \frac{1}{2b}\left(1 + \frac{1}{au}\right)e^{-1/au} \\ F'(u_0) &= \frac{1}{2b}\left(-\frac{1}{au_0^2} + \frac{1}{au_0^2}\left(1 + \frac{1}{au_0}\right)\right)e^{-1/au_0} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2a^2bu_0^3}e^{-1/au_0} \\
&= \frac{1}{2a^2bu_0^3}\frac{2bu_0}{1+1/au_0} \\
&= \frac{1}{au_0}\frac{1}{1+au_0} \\
\beta &= \sqrt{1-F'(u_0)} \\
&= \sqrt{1-\frac{1}{au_0}\frac{1}{1+au_0}} \\
&=
\end{aligned}
$$

The stable, smaller circular orbit has $au_0 > 1.62$; the unstable, larger circular orbit has $au_0 < 1.62$.

If $au_0 \ll 1$ (orbits near the maximum in a a strong Yukawa potential) then $\beta \approx 1 - 1/2au_0$, and the orbit precesses forward by $\pi r_0/a$:

$$\theta_1 = \frac{2\pi}{\beta} \approx \frac{2\pi}{1-\frac{1}{2au_0}} \approx 2\pi\left(1+\frac{1}{2au_0}\right) = 2\pi + \pi\frac{r_0}{a}$$

## Problem 3-28: A magnetic monopole

Assume a magnetic field $\mathbf{B} = b\mathbf{r}/r^3 = b\hat{e}_r/r^2$, and a particle moving in the field of that magnetic monople, and a central force field derived from a potential $V(r) = -k/r$.

The magnetic force is $\mathbf{F}_B = q\dot{\mathbf{r}} \times \mathbf{B}$, and the central force is $\mathbf{F} = -k\mathbf{r}/r^3$.

The torque is not zero, thus the angular momentum is not conserved:

$$
\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \mathbf{N} = \mathbf{r} \times \mathbf{F} \\
&= \mathbf{r} \times \left(q\dot{\mathbf{r}} \times \mathbf{B} - \frac{k}{r^3}\mathbf{r}\right) \\
&= \mathbf{r} \times (q\dot{\mathbf{r}} \times \mathbf{B}) \\
&= q\dot{\mathbf{r}}(\mathbf{r}\cdot\mathbf{B}) - q\mathbf{B}(\mathbf{r}\cdot\dot{\mathbf{r}}) \\
&= q\dot{\mathbf{r}}\frac{b}{r} - q\frac{b\mathbf{r}}{r^3}(r\dot{r}) \\
&= qb\left(\frac{\dot{\mathbf{r}}}{r} - \dot{r}\frac{\mathbf{r}}{r^2}\right) \\
&= \frac{d}{dt}\left(qb\frac{\mathbf{r}}{r}\right)
\end{aligned}
$$

and thus the vector $\mathbf{D} = \mathbf{L} - qb\mathbf{r}/r$ is conserved, since $d\mathbf{D}/dt = 0$.