Phys 7221 Homework #3

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September 27, 2006

1. **Derivation 2-4: Geodesics on a spherical surface**

Points on a sphere of radius R are determined by two angular coordinates, an azimuthal angle ψ and a polar angle θ:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = R(\sin\psi\cos\theta\hat{\mathbf{i}} + \sin\psi\sin\theta\hat{\mathbf{j}} + \cos\psi\hat{\mathbf{k}})$$

When moving on the sphere, the differential arc length ds is

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= R^2((\cos\psi\cos\theta d\psi - \sin\psi\sin\theta d\theta)^2 + (\cos\psi\sin\theta d\psi + \sin\psi\cos\theta d\theta)^2 + (-\sin\psi d\psi)^2) \\ &= R^2(d\psi^2 + \sin^2\psi d\theta^2) \end{aligned}$$

The distance on the sphere between two points is then

$$l = \int ds = R\int\sqrt{d\psi^2 + \sin^2\psi d\theta^2} = R\int d\theta\sqrt{\left(\frac{d\psi}{d\theta}\right)^2 + \sin^2\psi} = R\int d\theta f(\psi,\psi')$$

We can use a variational principle for finding the path with minimum length between two point. The path is described by a function $\psi(\theta)$, and the (differential) equation for ψ can be obtained from the Euler-Lagrange equation using $f(\psi,\psi') = \sqrt{\sin^2\psi + \psi'^2}$.

Back to the variational principle: the equation for ψ is

$$\begin{aligned} 0 &= \frac{d}{d\theta}\frac{\partial f}{\partial\psi'} - \frac{\partial f}{\partial\psi'} \\ &= \frac{d}{d\theta}\left(\frac{\psi'}{f}\right) - \frac{\sin\psi\cos\psi}{f} \\ &= \frac{\psi''}{f} - \frac{\psi' f'}{f^2} - \frac{\sin\psi\cos\psi}{f} \\ &= \frac{\psi''}{f} - \frac{\psi'}{f^2}\frac{\psi'\sin\psi\cos\psi + \psi'\psi''}{f} - \frac{\sin\psi\cos\psi}{f} \end{aligned}$$

$$\begin{aligned}
0 &= (\psi'' - \sin\psi\cos\psi) f^2 - \psi'^2(\psi'' + \sin\psi\cos\psi) \\
0 &= (\psi'' - \sin\psi\cos\psi)(\psi'^2 + \sin^2\psi) - \psi'^2(\psi'' + \sin\psi\cos\psi) \\
&= \psi''\sin^2\psi - 2\psi'^2\sin\psi\cos\psi - \sin^3\psi\cos\psi
\end{aligned}$$

This looks like a complicated equation to solve! It's always useful if we know the solution before we obtain it, admittedly not the most common case, but true in this case. We know that the shortest path between points in the sphere are great circles. Great circles are the intersection between the sphere and a plane. If the unit vector normal to the plane as $\hat{\mathbf{n}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, the points in the great circle are those points in the sphere that satisfy $\hat{\mathbf{n}} \cdot \mathbf{r} = 0 = R(\sin\phi(a\cos\theta + b\sin\theta) + c\cos\psi)$, or those points with coordinates ψ, θ satisfying

$$\frac{\cos\psi}{\sin\psi} = A\cos\theta + B\sin\theta$$

with $A^2 + B^2 < 1$. If we define a function $q(\theta) = \cos\psi(\theta)/\sin(\psi(\theta))$, we are looking for an equation of the form $d^2q/d^2\theta = -q$. If $q = 1/\tan\psi$, then $q' = -\psi'/\sin^2\psi$, and $q'' = -\psi''/\sin^2\psi + 2\psi'^2\cos\psi/\sin^3\psi$. Lagrange's equation can then be written as

$$\begin{aligned}
0 &= -q''\sin^4\psi - \sin^3\psi\cos\psi \\
q'' &= -\cos\psi/\sin\psi = -q
\end{aligned}$$

which is the equation we were looking for, with a general solution

$$q = \frac{\cos\psi}{\sin\psi} = A\cos\theta + B\sin\theta$$

which we know describes points on a great circle.

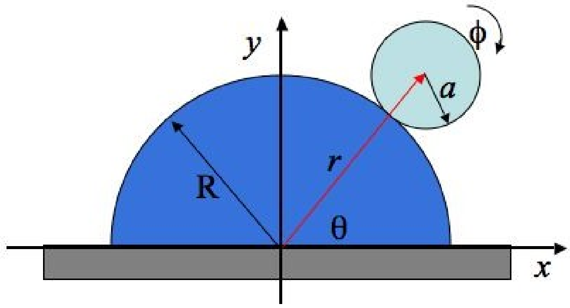
2. **Exercise 2-14: A hoop rolling on a cylinder**

   We can find out the angle at which the hoop falls from the cylinder by obtaining an expression for the normal force on the hoop as a function of the position of the hoop: the hoop will fall off when the normal force vanishes.

   We set up a coordinate system with the origin at the center of the cylinder, and describe the center of mass of the hoop with polar coordinates r, θ, and an angular coordinate ϕ for the rotation about the hoop's axis, as shown in the figure.

   The kinetic energy is

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}ma^2\dot{\phi}^2$$



and the potential energy is

$$V = -m\mathbf{g}\cdot\mathbf{r} = mgr\sin\theta$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}ma^2\dot{\phi}^2 - mgr\sin\theta$$

There are two constraints while the hoop is rolling on the cylinder :

$$\begin{aligned} f_1 &= r - (R+a) = 0 & (1) \\ f_2 &= (R+a)\dot{\theta} + a\dot{\phi} = 0 & (2) \end{aligned}$$

Note that if the hoop is rolling down, $\dot{\theta} < 0$ and $\dot{\phi} > 0$, if the angles are defined like in the figure. The rolling constraint is formulated setting up the velocity of the contact point instantaneously equal to zero, and expressing it as the velocity of the center of mass $r\dot{\theta}$, plus the velocity with respect to the center of mass, $a\dot{\phi}$. The equivalent condition for the hoop rolling on a plane is $\dot{x} + a\dot{\phi} = 0$.

The first constraint f_1 is holonomic, and we'll associate with it a Lagrange multiplier λ (which will be related to the normal force of the cylinder on the hoop). The second constraint is semi-holonomic, i.e., it depends on velocities but it could be integrated into a holonomic constraint. We will associate a Lagrange multiplier μ with it, which will be related to the friction force producing the rolling.

There are three Lagrange's equations for the coordinates r, θ, ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda\frac{\partial f_1}{\partial q_j} + \mu\frac{\partial f_2}{\partial \dot{q}_j}$$

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta &= \lambda & (3) \\ 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\cos\theta &= \mu(R+a) & (4) \\ ma^2\ddot{\phi} &= \mu a & (5) \end{aligned}$$

We have then 5 equations (1)...(5) for five unknowns, $r, \theta, \phi, \lambda, \mu$.

We use the first constraint to solve for the coordinate r: $r = R + a,\ \dot{r} = \ddot{r} = 0$. We use this solution in Lagrange's equations for r, θ:

$$\begin{aligned} -m(R+a)\dot{\theta}^2 + mg\sin\theta &= \lambda & (6) \\ m(R+a)^2\ddot{\theta} + mg(R+a)\cos\theta &= \mu(R+a) & (7) \end{aligned}$$

We use the rolling constraint to find an expression for ϕ as a function of θ:

$$\phi = -\frac{a+R}{a}\theta + \phi_0 \qquad (8)$$

and use this in Lagrange's equation (5) for ϕ to obtain

$$\mu = ma\ddot{\phi} = -m(R+a)\ddot{\theta} \qquad (9)$$

We use this expression for μ in (7), and obtain an equation for $\ddot{\theta}$:

$$\ddot{\theta} = -\frac{g}{2(R+a)}\cos\theta \qquad (10)$$

We can integrate this equation by multiplying by $\dot{\theta}$:

$$\begin{aligned} \ddot{\theta} + \frac{g}{2(R+a)}\cos\theta &= 0 \\ \ddot{\theta}\dot{\theta} + \frac{g}{2(R+a)}\dot{\theta}\cos\theta &= 0 \\ \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 + \frac{g}{2(R+a)}\sin\theta\right) &= 0 \\ \frac{1}{2}\dot{\theta}^2 + \frac{g}{2(R+a)}\sin\theta &= C \end{aligned}$$

If the hoop starts from rest at the top, then $\dot{\theta} = 0$ when $\theta = \pi/2$, which tells us the value of the constant of integration C:

$$\dot{\theta}^2 = \frac{g}{R+a}(1 - \sin\theta) \qquad (11)$$

We now use this in Eq.(6), to get an expression for the normal force as a function of the angle θ:

$$\begin{aligned} \lambda &= -m(R+a)\dot{\theta}^2 + mg\sin\theta \\ &= -mg(1-\sin\theta) + mg\sin\theta \\ \lambda &= mg(2\sin\theta - 1) & (12) \end{aligned}$$

At the top, when $\theta = \pi/2$, we obtain $\lambda = mg$, as expected for the normal force. If we try to apply this equation at the bottom, when $\theta = 0$, we obtain a negative value for λ, which tells us that the formulation of the problem cannot apply at that point, since the normal force cannot be negative. Equation (12) tells us that if $\sin\theta < 1/2$, the multiplier λ becomes negative: this is the angle at which the hoop falls from the cylinder, $\theta = 30°$.

3. **Exercise 2-18: A bead on a rotating hoop**

   A bead with mass m can slide without friction on a vertical hoop of radius a. The hoop is rotating along a vertical diameter with constant angular velocity ω.

   Take the origin of a coordinate system at the center of the hoop, with the z-axis pointing down, along the rotation axis. If we use spherical coordinates r, ψ, θ to describe the position the mass, we know that $r = a$ and $\dot{\theta} = \omega$, so the only generalized coordinate needed to describe the mass' postion is ψ. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\psi}^2 + r^2\sin^2\psi\dot{\theta}^2\right) \\ &= \frac{1}{2}ma^2\left(\dot{\psi}^2 + \omega^2\sin^2\psi\right) \end{aligned}$$

   The potential energy due to the gravitational acceleration $\mathbf{g} = g\hat{k}$ (since the z-axispoints down) is

$$\begin{aligned} V &= -m\mathbf{g}\cdot\mathbf{r} \\ &= -mgz \\ &= -mga\cos\psi \end{aligned}$$

   The Lagrangian is

$$L = T - V = \frac{1}{2}ma^2\dot{\psi}^2 + \frac{1}{2}ma^2\omega^2\sin^2\psi + mga\cos\psi$$

   Lagrange's equation of motion is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} - \frac{\partial L}{\partial\psi} \\ &= ma^2\ddot{\psi} - ma^2\omega^2\sin\psi\cos\psi + mga\sin\psi \\ a\ddot{\psi} &= -g\sin\psi\left(1 - \frac{a\omega^2}{g}\cos\psi\right) \qquad (13) \end{aligned}$$

   The Lagrangian has $\partial L/\partial\dot{\psi} \neq 0$, so the canonical momentum conjugate to ψ, proportional to the angular momentum component L_z, is not conserved.

However, the Lagrangian does not depend explicitly on time, so there is an integral of motion:

$$
\begin{aligned}
h_\psi &= \frac{\partial L}{\partial \dot\psi}\dot\psi - L \\
&= \frac{1}{2}ma^2\dot\psi^2 - \frac{1}{2}ma^2\omega^2\sin^2\psi - mga\cos\psi \\
\dot\psi^2 &= \frac{2h_\psi}{ma^2} + \omega^2\sin^2\psi + 2\frac{g}{a}\cos\psi
\end{aligned}
$$

The integral of motion is not the total energy $E = T + V$, but it is related to the energy by $h_\psi = E - ma^2\omega^2\sin^2\psi$.

For the mass to remain stationary on the hoop, we need $\psi = \psi_0$ and $\dot\psi = \ddot\psi = 0$. From Eq. **??**, we see that is possible only if $\sin\psi_0 = 0$ (top or bottom of the hoop), or $\cos\psi_0 = g/a\omega^2$, which is possible only if the hoop's velocity is high enough so that $g/a\omega^2 < 1$.

If the mass starts near the bottom, where $\psi \ll 1$, we can use a small angle approximation in the equation of motion, and

$$
a\ddot\psi \approx -g\psi(1 - a\omega^2/g).
$$

If the angular velocity is not larger than $\omega_0^2 = g/a$, this equation describes a harmonic oscillator with frequency $\Omega^2 = (g/a)(1 - a\omega^2/g) = g'/a$. The mass oscillates as the pendulum bob of a pendulum with length a, in a gravitational acceleration reduced by the rotation, $g' = g(1 - a\omega^2/g)$ [1].

In general, if $a\omega^2/g < 1$, the acceleration given by Eq. **??** will be always negative (since $\sin\psi > 0$), and will drive the mass to the bottom, making it oscillate about the lowest point (unless it starts there with zero velocity, when it will stay at the bottom).

If the angular velocity is larger than $\omega_0^2 = g/a$, there is angle α given by $\cos\alpha = g/a\omega^2$, where the acceleration given by Eq.**??** is zero. If the mass starts near such a point, we can define a small angle $\psi' = \psi - \alpha \ll 1$, so that $\ddot\psi' = \ddot\psi$, and use Eq.**??**:

$$
\begin{aligned}
\ddot\psi' &= -\frac{g}{a}\sin\psi\left(1 - \frac{\omega^2 a}{g}\cos\psi\right) \\
&= -\frac{g}{a}\sin(\psi' + \alpha)\left(1 - \frac{\omega^2 a}{g}\cos(\psi' + \alpha)\right)
\end{aligned}
$$

[1] Strictly speaking, the angular coordinate ψ can only be positive, so it cannot oscillate about zero, which is a singular point for the spherical coordinate system. However, we could define another coordinate system where the point at the bottom of the hopp is not singular, and we would obtain the same SHO equation of motion.

$$
\begin{aligned}
&= -\frac{g}{a}\left(\sin\psi'\cos\alpha + \cos\psi'\sin\alpha\right)\left(1 - \frac{\omega^2 a}{g}\left(\cos\psi'\cos\alpha - \sin\psi'\sin\alpha\right)\right) \\
&\approx -\frac{g}{a}\sin\alpha\left(1 - \cos\psi' + \frac{\omega^2 a}{g}\sin\alpha\sin\psi'\right) \\
&\approx -(\omega^2\sin^2\alpha)\psi'
\end{aligned}
$$

The equation for ψ' is again that of a simple harmonic oscillator, with frequency $\Omega^2 = \omega^2\sin^2\alpha$: a smaller frequency than the hoop's angular frequency. For very high rotation frequencies $\omega^2 \gg a/g$, we have $\cos\alpha \sim 0$ ($\alpha \sim \pi/2$), $\sin\alpha \sim 1$, and $\Omega \sim \omega$: the mass stays in the center of the hoop, in a constant position relative to the hoop's coordinate system.

4. **Exercise 2-19: Symmetries and conserved quantities**

   We consider the gravitational forces created on particles by different mass distributions. If the mass distribution has a particular symmetry, so will the potential associated with the force, and so will the Lagrangian. Since symmetries are associated with conserved quantities through Noether's theorem, we can find the conserved quantities: translational symmetries are associated with components of the linear momentum; rotational symmetries with components of the angular momentum, and time independence with conservation of energy. Since the potential is fixed in all cases (i.e., independent of time), the energy is conserved in all systems.

   (a) The mass is uniformly distributed in the plane $z = 0$ (an infinite, flat, Earth): the forces do not depend on the coordinates x, y, and thus the components of the linear momentum p_x, p_y will be conserved. Also, the force is invariant under a rotation about the z axis, so L_z is conserved.

   (b) The mass is uniformly distributed in the half plane $z = 0, y > 0$ (a finite, flat, Earth, like Columbus feared): there's only translational symmetry with respect to x, and no rotational symmetries: only p_x will be conserved.

   (c) The mass is uniformly distributed in a circular cylinder of infinite length, with axis along the z-axis: the configuration has translational symmetry along z, and rotational symmetry about z, so p_z and L_z are conserved.

   (d) The mass is uniformly distributed in a circular cylinder of finite length, with axis along the z-axis: there is now no translational symmetry, but there is still rotational symmetry about z: only L_z is conserved.

   (e) The mass is uniformly distributed in a right cylinder of elliptical cross section and infinite length, wit axis along the z axis: there is now no rotational symmetry, but because the cylinder is infinite along z, there is translational symmetry along z: only p_z is conserved.

(f) The mass is uniformly distributed in a dumbbell whose axis is oriented along the z axis: no translational symmetries, but there is rotational symmetry about the z axis, so L_z is conserved.

(g) The mass is the form of a uniform wire wound in the geometry of an infinite helical solenoid, with axis along the z axis. There are no pure translational or rotational symmetries, but there is a symmetry combining a z-translation of distance h (the distance between coils), and a rotation about z of 2π. Thus, although p_z or L_z are not individually conserved, $hp_z + L_z$ will be conserved.

5. **Exercise 2-20: A particle on a sliding wedge**

A mass m is sliding down without friction along a wedge with angle α and mass M. The wedge can move without friction on a smooth horizontal surface.

There are two objects in the problem, the mass m and the wedge M. The edge is a rigid body, but since it cannot rotate, it is described by the coordinates of just one point, say the top corner. We can treat the problem as a two dimensional problem so we have two coordinates for each mass. Let us choose a coordinate system with the origin at the initial position of the top of the wedge, with a horizontal x-axis and a vertical y-axis pointing down, as shown in the figure.

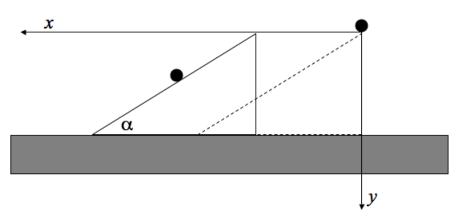


Figure 1: Problem 2-20: a mass sliding down a sliding wedge.

The coordinates of the top corner of the wedge will be $\mathbf{R} = (X, 0)$. The coordinates of the mass sliding down the wedge are $\mathbf{r} = (x, y)$. The constraint that the mass is on the wedge is

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + l(\cos\alpha, \sin\alpha) \quad \text{, or} \\ x &= X + l\cos\alpha \quad \text{and} \\ y &= l\sin\alpha \end{aligned}$$

where l is the distance the mss traveled down the wedge. This is one constraint, which we can express as a function of x, y, X as

$$f = (x - X)\sin\alpha - y\cos\alpha = 0.$$

The kinetic energy of the system is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}M\dot{X}^2.$$

The potential energy is just gravitational, with $\mathbf{g} = g\hat{\jmath}$. The gravitational potential energy of the wedge is constant, so we can ignore it. The gravitational potential energy of the mass m is

$$V = -m\mathbf{g}\cdot\mathbf{r} = -mgy$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}M\dot{X}^2 + mgy \tag{14}$$

There are three Lagrange equations, for $q_i = X, x, y$, which together with the constraint, form a system of four equations for the four variables X, x, y, λ (where λ is the Lagrange multiplier associated with the constraint).

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} &= \lambda\frac{\partial f}{\partial q_i} \\
m\ddot{x} &= \lambda\sin\alpha \\
m\ddot{y} - mg &= -\lambda\cos\alpha \\
M\ddot{X} &= -\lambda\sin\alpha
\end{aligned}$$

The equations for x, X can be added and result in

$$m\ddot{x} + M\ddot{X} = 0$$

This equation is saying that the horizontal position of the center of mass of the system has a constant velocity: we know this, since the only external force is gravity, and it is vertical (Notice that the figure does not represent the actual motion then!).

We can assume the initial position of the mass is at the top of the wedge, and the initial velocity of the center of mass is zero, and then we have a solution for X in terms of x:

$$X = -mx/M.$$

We can use the constraint to solve x in terms of y (or viceversa):

$$y = (x - X)\tan\alpha = x(1 + m/M)\tan\alpha.$$

We can also use Lagrange's equation for y to solve for λ, and use these results in Lagrange's equation for x:

$$\begin{aligned}
m\ddot{x} &= \lambda\sin\alpha = m(g - \ddot{y})\tan\alpha \\
&= m\tan\alpha(g - \ddot{x}(1 + m/M)\tan\alpha) \\
(1 + (1 + m/M)\tan^2\alpha)\ddot{x} &= g\tan\alpha \\
\frac{1 + (m/M)\sin^2\alpha}{\cos^2\alpha}\ddot{x} &= g\tan\alpha \\
\ddot{x} &= g\frac{\sin\alpha\cos\alpha}{1 + (m/M)\sin^2\alpha} \\
\ddot{x} &= a_x
\end{aligned}$$

The acceleration of x is constant, and the general solution is $x = x_0 + v_{0x}t + (1/2)a_x t^2$. Since we assumed the mass started at the origin, x_0=0. If the mass starts from rest, $v_{0x} = 0$. We can now use this result to obtain the equation for y:

$$\begin{aligned}
\ddot{y} &= \ddot{x}(1 + m/M)\tan\alpha \\
&= g\frac{\sin\alpha\cos\alpha}{1 + (m/M)\sin^2\alpha}\ (1 + m/M)\tan\alpha \\
&= g\sin^2\alpha\frac{1 + m/M}{1 + (m/M)\sin^2\alpha} \\
&= a_y
\end{aligned}$$

The acceleration of the mass along the direction tangent to the wedge is

$$\begin{aligned}
a_s &= a_x\cos\alpha + a_y\sin\alpha \\
&= g\frac{\sin\alpha\cos\alpha}{1 + (m/M)\sin^2\alpha}\ \cos\alpha + g\sin^2\alpha\frac{1 + m/M}{1 + (m/M)\sin^2\alpha}\ \sin\alpha \\
&= g\sin\alpha
\end{aligned}$$

The solutions for X, λ are:

$$\begin{aligned}
\ddot{X} &= -m\ddot{x}/M \\
&= -g\frac{m}{M}\frac{\sin\alpha\cos\alpha}{1 + (m/M)\sin^2\alpha} \\
&= -a_M
\end{aligned}$$

$$
\begin{aligned}
\lambda &= \frac{m}{\sin\alpha}\ddot{x} \\
&= mg\frac{\cos\alpha}{1+(m/M)\sin^2\alpha}
\end{aligned}
$$

The multiplier λ is the normal force from the wedge on the mass: it is mg if the wedge is horizontal ($\alpha = 0$), and it is smaller as the wedge is steeper $\alpha \to \pi/2$.

If the wedge has a large mass ($M \gg m$), it does not move much ($\ddot{X} \approx 0$), and the system approximates that of a mass sliding down a fixed incline. The tangential acceleration approximates $a_s \approx g\sin\alpha$, and the normal force approximates $\lambda \approx mg\cos\alpha$, as it should be well known.

The constraint force, the normal force, is always perpendicular to the wedge: $\mathbf{N} = \lambda(\sin\alpha, -\cos\alpha)$. If the wedge is fixed, this is perpendicular to the mass' motion, but if the wedge is not fixed, it will have a component along the mass' velocity: the constraint force works on the particle. The work done in time t by the force $\mathbf{N}$ on the mass m is given by:

$$
\begin{aligned}
\frac{dW_m}{dt} &= \mathbf{N}\cdot\mathbf{v} \\
&= \lambda(\dot{x}\sin\alpha - \dot{y}\cos\alpha) \\
&= \lambda t(a_x\sin\alpha - a_y\cos\alpha) \\
&= \lambda g t\sin^2\alpha\cos\alpha\frac{m/M}{1+(m/M)\sin^2\alpha}
\end{aligned}
$$

The work done by the normal force tends to zero as m/M as the wedge mass M gets large. The constraint force also works on the wedge:

$$
\begin{aligned}
\frac{dW_M}{dt} &= \mathbf{N}\cdot\mathbf{V} \\
&= \lambda\dot{X}\sin\alpha \\
&= -\lambda(m\dot{x}/M)\sin\alpha \\
&= -\lambda(m/M)a_x t\sin\alpha \\
&= -\lambda g t\sin^2\alpha\cos\alpha\frac{m/M}{1+(m/M)\sin^2\alpha}
\end{aligned}
$$

That is, the net work done by the constraint on the system is zero.

To find out conserved quantities, we need to express the Lagrangian **??** in terms of *independent* coordinates. If we use the constraint to solve for X, we get

$$
X = x - \frac{y}{\tan\alpha}
$$

$$
\begin{aligned}
L &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}M\dot{X}^2+mgy \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}M\left(\dot{x}-\frac{\dot{y}}{\tan\alpha}\right)^2+mgy
\end{aligned}
$$

The coordinate x is cyclical, so its canonical momentum is a constant of motion:

$$
p_x = \frac{\partial L}{\partial \dot{x}} = (m+M)\dot{x} - M\frac{\dot{y}}{\tan\alpha}
$$

This can also be obtained from the solutions, since $\dot{x} = a_x t$, $\dot{y} = a_y t$ and $a_y = a_x(1+m/M)/\tan\alpha$.

The Lagrangian does not depend explicitly on time, so there is a Jacobi integral, which is equal to the total energy $E = T + V$.

6. **A carriage on rotating cross-rails**

The position vector of the mass m in the inertial frame is $\mathbf{r} = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$. The coordinates x, y are related to the spring lengths R, r, where the length R of the spring with force constant K and rest length R_0, and the length r of the spring on the perpendicular rail with force constant k (and zero rest length). We can express x, y in terms of R, r:

$$
\begin{aligned}
x &= R\cos\omega t - r\sin\omega t \\
y &= R\sin\omega t + r\cos\omega t
\end{aligned}
$$

or R, r in terms of x, y:

$$
\begin{aligned}
R &= x\cos\omega t + y\sin\omega t \\
r &= -x\sin\omega t + y\cos\omega t
\end{aligned}
$$

We can choose as generalized coordinates x, y (natural coordinates in the inertial lab frame) or R, r (natural coordinates in the rotating frame).

The kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}(\dot{x}^2+\dot{y}^2) \\
&= \frac{1}{2}m((\dot{R}-r\omega)^2+(\dot{r}+\omega R)^2) \\
&= \frac{1}{2}m(\dot{R}^2+\dot{r}^2+2\omega(R\dot{r}-\dot{R}r)+\omega^2(r^2+R^2))
\end{aligned}
$$

The potential energy is

$$
\begin{aligned}
V &= \frac{1}{2}K(R-R_0)^2 + \frac{1}{2}kr^2 \\
&= \frac{1}{2}K(x\cos\omega t + y\sin\omega t - R_0)^2 + \frac{1}{2}k(-x\sin\omega t + y\cos\omega t)^2
\end{aligned}
$$

In terms of x, y, the Lagrangian is

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}K(x\cos\omega t + y\sin\omega t - R_0)^2 - \frac{1}{2}k(-x\sin\omega t + y\cos\omega t)^2
\end{aligned}
$$

Since the Lagrangian depends explicitly on time, we know the energy function is not conserved. Because the potential does not depend on time derivatives $\dot{x}, \dot{y}$, and the kinetic energy is homogeneous of second degree in $\dot{x}, \dot{y}$, we know that the energy function is the mechanical energy, and is not conserved:

$$
\begin{aligned}
h &= \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \\
&= \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} - L \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}K(x\cos\omega t + y\sin\omega t - R_0)^2 + \frac{1}{2}k(-x\sin\omega t + y\cos\omega t)^2 \\
&= T + V = E
\end{aligned}
$$

If we now instead choose R, r as our generalized coordinates, the Lagrangian is

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m(\dot{R}^2 + \dot{r}^2 + 2\omega(R\dot{r} - \dot{R}r) + \omega^2(r^2+R^2)) - \frac{1}{2}K(R-R_0)^2 - \frac{1}{2}kr^2 \\
&= \frac{1}{2}m(\dot{R}^2+\dot{r}^2) + m\omega(R\dot{r} - \dot{R}r) + \frac{1}{2}m\omega^2(r^2+R^2) - \frac{1}{2}K(R-R_0)^2 - \frac{1}{2}kr^2
\end{aligned}
$$

and is independent of time, so the “energy function” or “Jacobi integral” will be conserved. However, since the kinetic energy is not homogenous of second degree in $\dot{R}, \dot{r}$, then the energy function is *not* equal to the mechanical energy.

The Jacobi integral is

$$
h' = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L
$$

$$
\begin{aligned}
&= \dot{r}\frac{\partial T}{\partial \dot{r}} + \dot{R}\frac{\partial T}{\partial \dot{R}} - (T - V) \\
&= m\dot{r}(\dot{r} + \omega R) + m\dot{R}(\dot{R} - \omega r) - T + V \\
&= m\dot{r}^2 + m\dot{R}^2 + m\omega(R\dot{r} - r\dot{R}) - T + V \\
&= \frac{1}{2}m(\dot{R}^2 + \dot{r}^2) - \frac{1}{2}m\omega^2(r^2 + R^2) + \frac{1}{2}K(R - R_0)^2 + \frac{1}{2}kr^2 \\
&= T + V - m\omega(R\dot{r} - \dot{R}r) - \frac{1}{2}m\omega^2(r^2 + R^2)
\end{aligned}
$$

We see that if the beams are not rotating, $\omega = 0$ and $h = T + V = E$. With the rotation on, the mechanical energy is not conserved, and the rate of change of the energy is

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(h + m\omega(R\dot{r} - \dot{R}r) + \frac{1}{2}m\omega^2(r^2 + R^2)\right) \\
&= m\omega(R\ddot{r} - r\ddot{R}) + m\omega^2(r\dot{r} + R\dot{R}) \qquad (15)
\end{aligned}
$$

(The following was not asked in the homework, but it has more interesting facts about this system).

We can find expressions for the rate of change in energy from Lagrange's equations of motion for R, r, which we can find from the Lagrangian:

$$
\begin{aligned}
L &= \frac{1}{2}m(\dot{R}^2 + \dot{r}^2) + m\omega(R\dot{r} - \dot{R}r) + \frac{1}{2}m\omega^2(r^2 + R^2) - \frac{1}{2}K(R - R_0)^2 - \frac{1}{2}kr^2 \\
&= \frac{1}{2}m(\dot{R}^2 + \dot{r}^2) - U - V
\end{aligned}
$$

We have written the Lagrangian as the kinetic energy in the (non-inertial) rotating system, minus the potential energy in the inertial system, minus an extra potential U due to the rotation of the frame:

$$
U(r, R, \dot{r}, \dot{R}) = -m\omega(R\dot{r} - \dot{R}r) - \frac{1}{2}m\omega^2(r^2 + R^2).
$$

The forces derived from this potential are the "centrifugal forces", and they are not conservative (they depend on velocities).

Lagrange's equations are

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} \\
m\ddot{r} &= \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} - \frac{\partial (U + V)}{\partial r} \\
m\ddot{r} &= -m\omega\dot{R} + m\omega^2 r - kr
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} \\
m\ddot{R} &= -\frac{d}{dt}\frac{\partial U}{\partial \dot{R}} - \frac{\partial (U+V)}{\partial r} \\
m\ddot{R} &= m\omega\dot{r} + m\omega^2 R - K(R - R_0)
\end{aligned}
$$

From these equations, we can find an expression for the combinations we need for power in Eq(**??**):

$$
\frac{dE}{dt} = m\omega(R\ddot{r} - r\ddot{R}) + m\omega^2(R\dot{R} + r\dot{r}) = \omega r(-kR + K(R - R_0))
$$

The equilibrium coordinates are the solutions to $\dot{r} = \ddot{r} = \dot{R} = \ddot{R} = 0$, which are $r = 0$ and $R = R_0/(1 - m\omega^2/K)$. The equilibrium position for the K spring is not R_0, but a smaller length. If the system rotates too fast, and $\omega^2 \geq K/m$, the cross rail will be pushed against the rotation axis. Assuming small oscillations about the equilibrium length, and defining $\xi = R - R_0/(1 - m\omega^2/K)$, the equations of motion are

$$
\begin{aligned}
m\ddot{r} &= -m\omega\dot{\xi} - (k - m\omega^2)r \\
m\ddot{\xi} &= m\omega\dot{r} + m\omega^2\left(\xi + \frac{R_0}{1 - m\omega^2/K}\right) - K\left(\xi + \frac{R_0}{1 - m\omega^2/K} - R_0\right) \\
&= m\omega\dot{r} - (K - m\omega^2)\xi
\end{aligned}
$$

We see that the springs are "softened" by the rotation, and the oscillations in the different directions are coupled. We will learn how to find solutions to the equations of motion of these systems in Chapter 6.